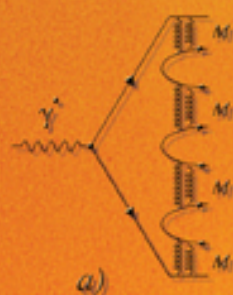


MESONS and BARYONS

Systematization and Methods of Analysis

A V Anisovich · V V Anisovich · M A Matveev
V A Nikonov · J Nyiri · A V Sarantsev



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To the memory of

Vladimir Naumovich Gribov

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Preface

The notion of quarks appeared in the early sixties just as a tool for the systematisation of the growing number of experimentally observed particles. First it was understood as a mathematical formulation of the SU(3) properties of hadrons, but soon it became clear that hadrons have to be considered as bound states of quarks (objects which we call now “constituent quarks”).

The next steps in understanding the quark–gluon structure of hadrons were made in the framework of Quantum Chromodynamics, a theory of coloured particles, as well as in the study of hard processes (*i.e.* in the study of hadron structure at small distances). We know that hadrons are, definitely, composed of large numbers of quarks, antiquarks and gluons. We have learned this from deep inelastic scattering experiments, and this picture is proven by many experiments on hard collisions and multiparticle production. At small distances quarks and gluons interact weakly, obeying the laws of QCD. An important fact is that a coloured quark or a gluon alone cannot leave the small region of the size of a hadron (*i.e.* that of the order of 10^{-23} cm): they are confined — they can fly away only in groups which are colourless.

In the fifties and sixties of the last century virtually the whole physics of “elementary particles” (at that time also hadrons were considered as such) was devoted to the consideration of these distances. With the progress of experimental physics very soon even smaller distances were reached at which hard processes were investigated, giving a strong basis to Quantum Chromodynamics – a theory in the framework of which coloured particles can be considered perturbatively. This, and the hope that the key for understanding the physics of strongly interacting quarks and gluons was hidden just here, initiated research towards smaller and smaller distances, skipping the region of strong (soft) interactions.

We accumulated a very serious amount of knowledge on the hadron structure at extremely small distances. But looking back to the region of standard hadron sizes, $10^{-24} - 10^{-23}$ cm, we realize now that, in fact, the physics at $\sim 10^{-23}$ cm in its essential domains remains unknown [1, 2]. We left behind the hadron distances without really understanding all the observed phenomena. We have learned only a small part of what could be learned from the experimental results in that region, not to mention that experiments which could be easily carried out were also abandoned. The physics community just skipped some problems of strong interactions, partly of principal importance for understanding the processes near the confinement boundary. But at the time being one can see a disenchantment in running to the smallest possible distances (the highest possible energies). There are serious arguments in favour of returning to the region of strong interactions, to problems which were missed before. Moreover, these problems became an obstacle for having a complete picture of interactions provided us by QCD.

Considering the region of soft interactions, there are, naturally, different approaches based on rather different views. Let us list here some of them.

First of all, there are attempts to get all the needed answers on a strictly theoretical basis. Maybe new experiments are not necessary, for a great deal of experimental information has been accumulated, and scientists are equipped with the fundamental theory of quarks and gluons – QCD. So the only problem is how to handle wisely this knowledge. On the other hand, new experiments of a quite different type may be helpful: this could be the lattice calculations using the most powerful computers and the most sophisticated algorithms. Lattice calculations were and are a widely used approach; still, there are also controversial opinions.

First, one should take into account the fact that field theories, QCD included, and lattice QCD are defined in the four-dimensional space over sets of different cardinalities. In lattice calculations the space is modeled by a set of points in a four-dimensional space, with the aim to decrease the distance between the points up to zero ($a \rightarrow 0$, where a is the lattice spacing) and a simultaneous strong increase of the number of points. However, a set of numerous points (a lattice) is not equivalent to a continuous set used in field theories, thus there is no mathematically correct transition to QCD. Standard mathematics, *e.g.* the theory of fractals, give us many examples when characteristics constructed on a set of numerous points are different from those obtained for a continuum set (such examples, for instance, can be found in [3]).

Nevertheless, lattice calculations are quite promising, especially if they contain ingredients of observed phenomena. Such is, *e.g.*, the use of the quench approximation (the meson consists of two, the baryon of three quarks) in the calculation of non-exotic hadrons. Another example is the calculation of the mass of the tensor glueball. For many years lattice calculations predicted its mass about 2350 MeV. But recent experiments gave a mass of the order of 2000 MeV — and as soon as lattice calculations have included the requirement of linearity of the Regge trajectories (which is the experimental observation) the result for the glueball mass became 2000 MeV. Hence, the lattice QCD may be a rather useful tool for understanding the soft interaction region, provided it is supported by experimental results.

A quite radical way to change the object of our investigations would be to return to distances of the order of 10^{-23} cm, both in experiment and theory. We know a lot about soft interactions, and this knowledge, the knowledge of the so-called quark model, though incomplete and amorphous, contains a large amount of information. Therefore the strategy, as we understand it, consists in a more fundamental study of the region $\sim 10^{-23}$ cm based on the quark model and related experimental data.

In this book we present our views on the quark model, focusing on physics of hadrons. In this sense this book is a continuation of [2] where the main topics were soft hadron collisions at high energies.

Presenting the problems of hadron spectroscopy, we underline the statements having a solid background, and discuss the points which, though missed in previous studies, are needed for the restoration of soft interaction physics.

We focus our attention on methods of obtaining information about hadrons. The inconsistency of methods which we meet frequently leads to disagreement in the results and their interpretations. To illustrate this, a simple example is that in PDG [4] up to now there is no unique definition of the mass and the widths of a resonance, though the answer here is obvious: these characteristics are to be defined by the positions of amplitude poles in the invariant energy complex plane and the residues in these poles.

We tried to write the pieces devoted to technicalities of the treatment of data and the interpretation in the form of a brief set of prescriptions, *i.e.* as a handbook. Examples, explanations complemented by relevant calculations and available fitting results are given in the Appendices. In this book we do not aim to present a complete picture of the experimental situation but we would recommend recent surveys [5, 6].

By choosing the quark model as a basis for the study of soft physics,

we understand that we do not pursue far-reaching aims but try to solve immediate problems such as the systematics of meson and baryons, the determination of effective colour particles and their characteristics (we mean constituent quarks, effective massive gluons, diquarks and possible other formations). We mention here also a more ambiguous problem: the construction of effective theories in some ways similar to those used in condensed matter physics.

One of our main purposes is the determination of amplitude singularities responsible for the confinement of colour particles.

In the final chapter we tried to review the situation related to the quark model: to what extent the recent problems have been understood and what new tasks have been pushed forward. Also, in this discussion we touch possible far perspectives.

We are deeply indebted to our friends and colleagues who are no more with us.

Since the very beginning of our investigations, we had many discussions of the problems considered here with V.N. Gribov. He always showed vivid interest in the obtained results, and his comments helped us to achieve a deeper understanding of the related physics. It was him who underlined the fundamental interconnectedness between problems of hadron spectroscopy and confinement. The book is devoted to his memory.

Many results and methods presented in this book originated from the ideas formulated in the pioneering works made in collaboration with V.M. Shekhter.

Significant progress achieved in meson spectroscopy is related to the experiments initiated and completed under the leadership of Yu.D. Prokoshkin. His contribution provided much experimental information on which this book is based.

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A.V. Anisovich, V.V. Anisovich, M.A. Matveev,
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Chapter 1

Introduction: Hadrons as Systems of Constituent Quarks

Quantum chromodynamics, QCD, the theory of coloured quarks and gluons [1, 2], has a dual face. At small hadron distances ($r \ll 1$ fm) the quark–gluon interaction is weak; QCD is realised as a perturbative theory of QCD-quarks (or current quarks) and massless gluons. At distances of the order of hadron sizes ($r \sim 1$ fm) the interaction becomes strong and the perturbative description cannot be applied.

1.1 Constituent Quarks, Effective Gluons and Hadrons

Our present understanding of the quark–gluon structure of hadrons grew out, on the one hand, of the parton hypothesis [3, 4, 5] and, naturally, it is based on the experiments such as deep inelastic scatterings, e^+e^- annihilation, the production of $\mu^+\mu^-$ pairs and hadrons with large transverse momenta in high-energy hadron collisions. On the other hand, it is the result of the progress in quark models. Our knowledge is now based on quantum chromodynamics, the microscopic theory of strong interactions, which is a non-Abelian gauge theory of Yang–Mills fields [6]. The QCD-motivated quark models play a key role in the investigation of strong interactions.

Contrary to QED, where, along with the electron, there exists one neutral photon and the main process is the emission of photons by electrons, in QCD three types of quarks (three colours) are assumed, and each of them can transform into another via the emission of eight coloured gluons. The colour charge of gluons leads to the consequence that not only quarks emit gluons (Fig. 1.1a) but gluon emission by gluons (Fig. 1.1b) and gluon–gluon scattering (Fig. 1.1c) are also taking place. The requirement of three colours determines the theory unambiguously.

Quarks and gluons are not seen as free particles. In QCD there is

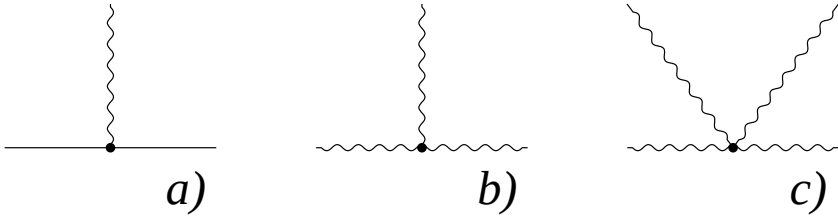


Fig. 1.1 QCD interaction vertices: gluon emission by a quark (a) or by a gluon (b); gluon–gluon scattering (c).

a confinement of coloured objects based on the increase of the effective charge at large distances. At the same time, non-Abelian gauge theories are asymptotically free [7, 8, 9], *i.e.* they are theories in which interactions at short distances are small. As a result, QCD gives a description of hard processes in a qualitative accordance with the interaction picture of the parton model.

At short distances, QCD is a well-defined renormalisable gauge theory [10]. The small value of the coupling constant at $r \rightarrow 0$ grants all the advantages of the developed technique of the Feynman diagrams in perturbation theory. The perturbative QCD (pQCD), providing a theoretical background for all the results obtained in the parton model, predicts at the same time certain deviations from the naive parton model in various hard processes. The reviews [11, 12, 13, 14, 15, 16] present a comprehensive analysis of the pQCD calculation technique and comparisons of the obtained results with experimental data.

Strong interactions change the properties of the quarks and gluons: the quark mass grows by 200 – 400 MeV, while the massless gluon turns into a massive effective gluon with $m_g \sim 700 - 1000$ MeV. Moreover, strong interactions may form new effective particles, *e.g.* composite systems of two quarks — diquarks. These can be either compact formations like constituent quarks or loosely bound systems of two quarks. Another possible class of effective particles could consist of coloured scalar mesons, which may be important in the formation of effective massive gluons.

There is one more highly important phenomenon in the region of strong interactions: the confinement of coloured particles. Coloured particles cannot occur at a distance more than 1–2 fm from each other. The only possibility to fly away (this is called deconfinement) is the formation process

of new quark–antiquark pairs followed by the production of new colourless objects: hadrons. Thus, quarks can get away from each other only as constituents of hadrons, *i.e.* if their colours are neutralised by other, newly produced quarks and gluons.

The idea that hadrons are not elementary particles is rather old: it appeared at the time when the first mesons were discovered. Fermi and Yang suggested that a pion consists of a proton and a neutron [17]. The discovery of the K -mesons gave rise to different versions of composite models. The common feature of these models was the assumption that the hadrons themselves were the constituents. In the late fifties the best known model of this kind was that of Sakata in which (p, n, Λ) are chosen as constituents, see *e.g.* [18, 19] and references in [19].

The suggestion of the quark structure of hadrons appeared first in the papers of Gell-Mann [20] and Zweig [21]. It was shown that the hadrons known at that time could be built up as composite systems of the three quarks (u, d, s) with fractional electric charges, obeying the rules of the $SU(3)$ symmetry. This was, in fact, the introduction of the constituent quarks. The quantum numbers of these three quarks (now we call them light quarks) are

<i>flavour</i>	<i>charge</i>	<i>isospin</i>	<i>baryon charge</i>	
u	$2/3$	$I = 1/2 \ I_3 = 1/2$	$1/3$	(1.1)
d	$-1/3$	$I = 1/2 \ I_3 = -1/2$	$1/3$	
s	$-1/3$	0	$1/3$	

The constituent (u, d) -quarks form an isotopic doublet and, thus, lead to the creation of hadronic isotopic multiplets.

Further, the notion of strangeness was introduced for hadrons built up from light quarks; the strangeness of the s -quark is taken to be -1 .

<i>flavour</i>	<i>strangeness</i>	
u	0	(1.2)
d	0	
s	-1	

If initially the quarks were understood just as a mathematical formulation of $SU(3)$ properties of hadrons [22, 23], soon it became clear that hadrons have to be considered as loosely bound systems of quarks. In the constituent quark picture of hadrons the meson consists of a quark–antiquark pair, while the baryons are systems of three constituent quarks:

$$M = \bar{q}q, \quad B = qq\bar{q} . \quad (1.3)$$

Let us underline that at those times only hadrons with small spins were known: mesons with $J^P = 0^-, 1^-$ and baryons with $J^P = 1/2^+, 3/2^+$. Attempts to discover free particles (quarks) with fractional electric charges failed [24]. The fact that quarks do not exist as experimentally observable particles is the phenomenon of quark confinement.

The introduction of the colour has a rather long history. Already when the quark model was constructed from constituent quarks (on the level of realisation of the $SU(3)$ symmetry), the introduction of new quark quantum numbers turned out to be necessary [25, 26, 27]. The picture of coloured quarks as we accept it now was formulated by Gell-Mann [1]. In this picture each quark possesses the quantum number of colour, which can have three values:

$$q_i \quad i = 1, 2, 3 \quad (\text{or red, green, blue}). \quad (1.4)$$

The coloured quarks realise the lowest representation of the colour group $[SU(3)]_{\text{colour}}$. It is postulated that the observable hadrons are singlets of the $[SU(3)]_{\text{colour}}$ group, *i.e.* they are white states. For the two-quark mesons and the three-quark baryons this means

$$M = \frac{1}{\sqrt{3}} \sum q_i \bar{q}_i, \quad B = \frac{1}{\sqrt{6}} \sum_{i,k,\ell} \varepsilon_{ik\ell} q_i q_k q_\ell. \quad (1.5)$$

Here the sum runs over the quark colours; $\varepsilon_{ik\ell}$ is the totally antisymmetric unit tensor.

Hence, the first historical step in understanding the quark–gluon nature of hadrons was the model of the constituent quark for the lowest hadrons, consisting of light quarks (1.1) with the new quantum number, the colour.

1.2 Naive Quark Model

The first successful steps in understanding the quark structure of hadrons were made in the framework of the non-relativistic quark model, especially when the $SU(6)$ symmetry was introduced. As time passed, it became obvious that this approach has restricted possibilities even for the lightest hadrons. Still, the simple picture given by the naive non-relativistic quark model provides us with a tool for the qualitative description of low-lying hadrons. Because of that, we present here the $SU(6)$ symmetry and its consequences in detail. In the end of the section we indicate those hadron properties which, obviously, cannot be handled in the framework of this description.

1.2.1 Spin-flavour $SU(6)$ symmetry for mesons

For the systematisation of hadrons, the $SU(6)$ symmetry was suggested in [28, 29]; this symmetry is a generalisation of $SU(4)$ which was used by Wigner for the description of nuclei [30].

Realising $SU(6)$ symmetry, the spin-flavour variables can be separated from the coordinate variables with a good accuracy. Hence, the wave functions can be written as

$$\Psi = C(\alpha(1), \bar{\alpha}(2))h(q(1), \bar{q}(2))\Phi_L(\mathbf{r}_1, \mathbf{r}_2) . \quad (1.6)$$

The colour part of the wave function $C(\alpha(1), \bar{\alpha}(2))$ is a common expression for all mesons, it is a colour singlet:

$$C(\alpha(1), \bar{\alpha}(2)) = \frac{1}{\sqrt{3}} \alpha_i(1) \bar{\alpha}_i(2) , \quad (1.7)$$

where the indices $i = 1, 2, 3$ describe the colours of the quark.

The spin-flavour part of the wave function $h(q(1)\bar{q}(2))$ realises a definite $SU(6)$ representation. In non-relativistic quark models an $SU(6)$ multiplet is characterized by the radial excitation quantum number (n) and the angular momentum (L). In the $SU(6)$ representation the standard notation for such a multiplet is $[N, L^P]_n$, where N is the total number of states in the multiplet (*i.e.* the dimension of the representation) and P is the parity of the states.

The coordinate part of the wave function $\Phi_L(\mathbf{r}_1, \mathbf{r}_2)$ is the same for all states of an $SU(6)$ multiplet. It is characterized by the total angular momentum L and its projection onto one of the axes, *e.g.* Z , *i.e.* L_Z :

$$\Phi_L(\mathbf{r}_1, \mathbf{r}_2) \longrightarrow Y_{LL_Z} \left(\frac{\mathbf{r}}{r} \right) \Phi_L(r) , \quad (1.8)$$

where $Y_{LL_Z}(\mathbf{r}/r)$ is a standard spherical function, and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $r = |\mathbf{r}|$. The non-trivial coordinate part of the wave function $\Phi_L(r)$, which describes the dynamics of the state, depends on the distance between quarks. In what follows, we shall discuss the lightest multiplet with $L = 0$ and the next multiplet with $L = 1$ in terms of the $SU(6)$ symmetry. The radial quantum numbers of the considered multiplets are $n = 1$, *i.e.* they are basic states.

$SU(6)$ symmetry for the S-wave $q\bar{q}$ states

States with $L = 0$ and $n = 1$ are described by two $SU(6)$ multiplets: by the 35-plet $[35, 0^+]$ and the singlet $[1, 0^+]$ (we skip here the index corresponding to the radial quantum number).

The $[35, 0^+]$ multiplet contains the following states:

$$\begin{aligned} h_0 &= \pi^+, \pi^0, \pi^-, \eta^{(8)}, K^+, K^0, \bar{K}^0, K^-, \\ h_1 &= \rho^+, \rho^0, \rho^-, \omega, \phi, K^{*+}, K^{*0}, \bar{K}^{*0}, K^{*-}. \end{aligned} \quad (1.9)$$

The total number of states (1.9) is $8 + 3 \cdot 9 = 35$ (each vector state contains three states with different spin projections).

We have one $[1, 0^+]$ state, namely $\eta^{(1)}$.

The spin-flavour wave function projection of the singlet state $[1, 0^+]$ equals

$$|\eta^{(1)}\rangle = \frac{1}{\sqrt{6}} (u^\uparrow \bar{u}^\downarrow - u^\downarrow \bar{u}^\uparrow + d^\uparrow \bar{d}^\downarrow - d^\downarrow \bar{d}^\uparrow + s^\uparrow \bar{s}^\downarrow - s^\downarrow \bar{s}^\uparrow). \quad (1.10)$$

This wave function is symmetrical in all flavour indices and antisymmetrical in the spin indices. It is a singlet in the flavour space and has a quark spin $S = 0$.

The wave function of the $[1, 0^+]$ state is written in a somewhat awkward form, because we use Clebsch–Gordan coefficients for constructing the spin wave function. We can see explicitly that $|\eta^{(1)}\rangle$ is an $SU(6)$ singlet, if we make use of the following spin functions for the quarks and antiquarks:

$$q_1 = \begin{pmatrix} q^\uparrow \\ 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 \\ q^\downarrow \end{pmatrix}, \quad \bar{q}^1 = \begin{pmatrix} \bar{q}^\downarrow \\ 0 \end{pmatrix}, \quad \bar{q}^2 = \begin{pmatrix} 0 \\ -\bar{q}^\uparrow \end{pmatrix}. \quad (1.11)$$

In this case (1.10) can be rewritten as

$$|\eta^{(1)}\rangle = \frac{1}{\sqrt{6}} \sum_{q,a} q_a \bar{q}^a, \quad (1.12)$$

where the summation is carried out over $q = u, d, s$ and $a = 1, 2$. Following, however, the traditions of spectroscopy, we continue to use the Clebsch–Gordan coefficients even if this causes some inconvenience in writing the wave functions. The complete wave function of the $[1, 0^+]$ state (but without including the colour part) can be written as $|\eta^{(1)}\rangle \Phi_0^{(1)}(r)$. Let us now write the wave function of the 35-plet. First of all, consider the pseudoscalar particles h_0 from Eq. (1.9).

The wave function $|\eta^{(8)}\rangle$ is orthogonal to $|\eta^{(1)}\rangle$ in the flavour indices, it equals

$$|\eta^{(8)}\rangle = \frac{1}{2\sqrt{3}} (u^\uparrow \bar{u}^\downarrow + d^\uparrow \bar{d}^\downarrow - 2s^\uparrow \bar{s}^\downarrow - u^\downarrow \bar{u}^\uparrow - d^\downarrow \bar{d}^\uparrow + 2s^\downarrow \bar{s}^\uparrow). \quad (1.13)$$

The wave functions of the π^+ - and π^0 -mesons are

$$\begin{aligned} |\pi^+\rangle &= \frac{1}{\sqrt{2}} (u^\uparrow \bar{d}^\downarrow - u^\downarrow \bar{d}^\uparrow), \\ |\pi^0\rangle &= \frac{1}{2} (u^\uparrow \bar{u}^\downarrow - d^\uparrow \bar{d}^\downarrow - u^\downarrow \bar{u}^\uparrow + d^\downarrow \bar{d}^\uparrow). \end{aligned} \quad (1.14)$$

The remaining wave functions of the pseudoscalar particles are obtained by the substitution of the indices in the π^+ -meson wave function: the wave function of the π^- -meson is the result of charge conjugation, $u \rightarrow \bar{u}$ and $\bar{d} \rightarrow d$. The wave function of the K^+ -meson can be obtained by substituting $\bar{d} \rightarrow \bar{s}$. We get the wave function of the K^0 -meson by the double substitution $u \rightarrow d$ and $\bar{d} \rightarrow \bar{s}$. The wave functions $|K^- \rangle$ and $|\bar{K}^0 \rangle$ are given by the charge conjugation of $|K^+ \rangle$ and $|K^0 \rangle$.

We denote the spin-flavour wave functions with quark spin $S = 0$ as $|h_0 \rangle$; let us repeat once more that this is $|\eta^{(8)} \rangle$, $|\pi^+ \rangle$, $|\pi^0 \rangle$, *etc.*, *i.e.* all eight wave functions of the pseudoscalar mesons. The complete wave function of the 35-plet states with $S = 0$ is written as $|h_0 \rangle \Phi_0^{(35)}(r)$. The wave functions of the h_1 -states of the 35-plet are the following. For the ρ^+ we have

$$|\rho_1^+ \rangle = u^\uparrow \bar{d}^\uparrow, \quad |\rho_0^+ \rangle = \frac{1}{\sqrt{2}} (u^\uparrow \bar{d}^\uparrow + u^\downarrow \bar{d}^\uparrow), \quad |\rho_{-1}^+ \rangle = u^\downarrow \bar{d}^\uparrow. \quad (1.15)$$

The wave function of the ρ^- -meson can be obtained by the substitutions $u \rightarrow d$, $\bar{d} \rightarrow \bar{u}$ in (1.15), while the substitution $(u^a \bar{d}^b) \rightarrow (u^a \bar{u}^b - d^a \bar{d}^b) / \sqrt{2}$ in (1.15) gives the wave function of the ρ^0 -meson.

The substitution $\bar{d} \rightarrow \bar{s}$ in (1.15) leads to the K^{*+} -meson wave function; the wave function of K^{*0} is the result of the double substitution $u \rightarrow d$, $\bar{d} \rightarrow \bar{s}$.

The wave functions of the isoscalar vector states are

$$\begin{aligned} |\omega_1(n\bar{n}) \rangle &= \frac{1}{\sqrt{2}} (u^\uparrow \bar{u}^\uparrow + d^\uparrow \bar{d}^\uparrow), \\ |\omega_0(n\bar{n}) \rangle &= \frac{1}{2} (u^\uparrow \bar{u}^\downarrow + d^\uparrow \bar{d}^\downarrow + u^\downarrow \bar{u}^\uparrow + d^\downarrow \bar{d}^\uparrow), \\ |\omega_{-1}(n\bar{n}) \rangle &= \frac{1}{\sqrt{2}} (u^\downarrow \bar{u}^\downarrow + d^\downarrow \bar{d}^\downarrow) \end{aligned} \quad (1.16)$$

and

$$|\phi_1(s\bar{s}) \rangle = s^\uparrow \bar{s}^\uparrow, \quad |\phi_0(s\bar{s}) \rangle = \frac{1}{\sqrt{2}} (s^\uparrow \bar{s}^\downarrow + s^\downarrow \bar{s}^\uparrow), \quad |\phi_{-1}(s\bar{s}) \rangle = s^\downarrow \bar{s}^\downarrow. \quad (1.17)$$

Let us remind that the wave functions of the real mesons ω and ϕ are mixtures of pure $|\omega(n\bar{n}) \rangle$ and $|\phi(s\bar{s}) \rangle$ states of Eqs. (1.16) and (1.17). As a whole, we have 27 states with quark spins $S = 1$. We denote all spin-flavour wave functions of these states (given by (1.15)–(1.17) and similar formulae) as $|h_{1S_Z} \rangle$. The complete wave functions of the 35-plet with $S = 1$ can be written as $|h_{1J_Z} \rangle \Phi_0^{(35)}(r)$. The coordinate wave function coincides with that in $|h_0 \rangle \Phi_0^{(35)}(r)$. Superpositions of $\eta^{(1)}$ and $\eta^{(8)}$ form observable η and η' mesons, they are mixed; this fact means that $\Phi_0^{(1)}(r)$ and $\Phi_0^{(35)}(r)$ are

sufficiently close to each other. That's why one speaks usually not about two multiplets, 1 and 35, but about one 36-plet.

SU(6) symmetry for the P-wave $q\bar{q}$ states

The application of SU(6) symmetry to P -wave meson $q\bar{q}$ states is not a flawless procedure since in P -wave mesons the relativistic effects cannot be small. Nevertheless, SU(6) symmetry is sometimes suitable for the description of such states. Let us, therefore, construct the wave functions.

States with $L \neq 0$ contain SU(6) multiplets $35 \otimes (2L+1)$ and $1 \otimes (2L+1)$. Hence, for $L = 1$ we have meson multiplets $[35 \otimes 3, 1^+]$ and $[1 \otimes 3, 1^+]$. The states belonging to these multiplets, the 35-plet and the axial singlet, are considerably mixed (in the same way as in the case of $L = 0$, when we observed the mixing of $\eta^{(1)}$ and $\eta^{(8)}$), and thus it is again reasonable to consider just a unique $(1 \oplus 35)$ -plet.

The spin-flavour part of the $L = 1$ meson wave functions is determined by the same functions $|\eta^{(1)}\rangle$, $|h_0\rangle$ and $|h_1\rangle$, as in the case of $L = 0$: the wave function of the $[1 \otimes 3, 1^+]$ multiplet can be written in the form

$$|\eta^{(1)}\rangle Y_{1L_Z} \left(\frac{\mathbf{r}}{r} \right) \Phi_1^{(1)}(r). \quad (1.18)$$

The wave functions of the $[35 \otimes 3, 1^+]$ -plet with spin $S = 0$ are defined with the help of $|h_0\rangle$:

$$|h_0\rangle Y_{1L_Z} \left(\frac{\mathbf{r}}{r} \right) \Phi_1^{(35)}(r). \quad (1.19)$$

We denote meson states related to this multiplet as b_1^+ , b_1^0 , b_1^- , $h_1^{(8)}(I = 0)$ and $K_1(I = 1/2)$, while for the wave functions with $S = 1$ we use $|h_{1S_Z}\rangle$:

$$\sum_{L_Z + S_Z = J_Z} C_{1L_Z 1S_Z}^{JJ_Z} |h_{1S_Z}\rangle Y_{1L_Z} \left(\frac{\mathbf{r}}{r} \right) \Phi_1^{(35)}(r). \quad (1.20)$$

The corresponding meson states are denoted as a_J^+ , a_J^0 , a_J^- , $f_J(n\bar{n})$, $f_J(s\bar{s})$, K_J with $J = 0, 1, 2$.

It is reasonable to suppose that $\Phi_1^{(1)}(r)$ and $\Phi_1^{(35)}(r)$ nearly coincide, and we can consider a unique set of states $(1 \oplus 35) \otimes 3$.

Predictions for 36 - plets with $L = 1$ and the estimations of their masses were first given in [31, 32].

1.2.2 Low-lying baryons

Low-lying baryons, octets and decuplets in the terminology of $SU(3)_{flavour}$ symmetry, may also be described qualitatively in the framework of $SU(6)$ symmetry.

We have in mind the following baryons:

(i) the octet with $J^P = 1/2^+$:

isospin	strangeness	particles
1/2	0	p, n
0	-1	Λ
1	-1	$\Sigma^+, \Sigma^0, \Sigma^-$
1/2	-2	Ξ^0, Ξ^- ;

(1.21)

(ii) the decuplet with $J^P = 3/2^+$:

isospin	strangeness	particles
3/2	0	$\Delta^{++}, \Delta^+, \Delta^0, \Delta^-$
1	-1	$\Sigma^{*+}, \Sigma^{*0}, \Sigma^{*-}$
1/2	-2	Ξ^{*0}, Ξ^{*-}
0	-3	Ω^-

(1.22)

Below, we discuss the description of the wave functions of these baryons in terms of the $SU(6)$ symmetry.

1.2.3 Spin-flavour $SU(6)$ symmetry for baryons

The baryons consist of three quarks qqq ; the colour part of the wave function is the same for all baryons

$$C(\alpha(1), \alpha(2), \alpha(3)) = \frac{1}{\sqrt{6}} \varepsilon_{ikl} \alpha_i(1) \alpha_k(2) \alpha_l(3). \quad (1.23)$$

Since the decuplet is antisymmetric with respect to any permutation of quarks, which obey Fermi statistics, the remaining part of the wave function (*i.e.* the coordinate and the spin-flavour one) should be exactly symmetric.

It seems to be natural that once the coordinate wave function $\Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ is completely symmetric for the lowest baryon states, the spin-flavour part must be also symmetric: this corresponds to the 56-plet representation of the $SU(6)$ group. If $\Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ is totally antisymmetric, the spin-flavour part has to be also antisymmetric (20-plet representation). $\Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ can be also of mixed symmetry (*i.e.* it corresponds to a mixed Young scheme): this leads to the mixed symmetry of the spin-flavour wave function, which corresponds to the 70-plet representation. All the baryons observed up to now seem to belong to either the 56-plet or the 70-plet; so far no states belonging to the 20-plet are established with certainty.

The 56-plet

Assembling the baryon wave functions, it is convenient to write the spin-flavour part $h(q(1), q(2), q(3))$ in the form of a direct product of the spin function $|SS_Z\rangle$ (where S is the total spin of three quarks, S_Z is its Z-projection) and the flavour function $|q_1 q_2 q_3\rangle$ (q_i are symbols of the u, d, s quarks). The symmetric spin functions (spin 3/2) are

$$\left| \frac{3}{2} \frac{3}{2} \right\rangle = \uparrow\uparrow\uparrow, \quad \left| \frac{3}{2} \frac{1}{2} \right\rangle = \frac{1}{\sqrt{3}} (\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow), \quad (1.24)$$

etc., for spin 1/2 (mixed symmetry) two orthogonal combinations can be written

$$\left| \frac{1}{2} \frac{1}{2} \right\rangle_\lambda = \frac{1}{\sqrt{6}} (\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow - 2\downarrow\uparrow\uparrow), \quad \left| \frac{1}{2} \frac{1}{2} \right\rangle_\rho = \frac{1}{\sqrt{2}} (\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow). \quad (1.25)$$

The SU(3) decuplet flavour function is symmetric:

$$\begin{aligned} |10\rangle &= \frac{1}{\sqrt{6}} (q_1 q_2 q_3 + q_1 q_3 q_2 \\ &\quad + q_2 q_1 q_3 + q_2 q_3 q_1 + q_3 q_1 q_2 + q_3 q_2 q_1) \text{ (three different flavours)} \\ &= \frac{1}{\sqrt{3}} (q_1 q_1 q_2 + q_1 q_2 q_1 + q_2 q_1 q_1) \text{ (two flavours coincide)}, \\ &= (q_1 q_1 q_1) \text{ (all flavours coincide)}. \end{aligned} \quad (1.26)$$

There are two orthogonal octet flavour functions with mixed symmetry (that is, at least two flavours must be different):

$$\begin{aligned} |8\rangle_\lambda &= \frac{1}{2\sqrt{3}} (q_1 q_2 q_3 + q_1 q_3 q_2 \\ &\quad + q_2 q_1 q_3 + q_2 q_3 q_1 - 2q_3 q_1 q_2 - 2q_3 q_2 q_1) \text{ (three different flavours)} \\ &= \frac{1}{\sqrt{6}} (q_1 q_1 q_2 + q_1 q_2 q_1 - 2q_2 q_1 q_1) \text{ (two flavours coincide)} \\ |8\rangle_\rho &= \frac{1}{2} (q_1 q_2 q_3 - q_1 q_3 q_2 - q_2 q_3 q_1 + q_2 q_1 q_3) \text{ (three different flavours)} \\ &= \frac{1}{\sqrt{2}} (q_1 q_1 q_2 - q_1 q_2 q_1) \text{ (two flavours coincide)}. \end{aligned} \quad (1.27)$$

Finally, the SU(3) singlet is antisymmetric; therefore, only the component with three different flavours survives:

$$|1\rangle = \frac{1}{\sqrt{6}} (q_1 q_2 q_3 + q_2 q_3 q_1 + q_3 q_1 q_2 - q_2 q_1 q_3 - q_3 q_2 q_1 - q_1 q_3 q_2). \quad (1.28)$$

All the functions (1.26–1.28) are normalised to unity.

The direct product of the spin and flavour functions forms the spin-flavour baryon wave function, *e.g.*

$$|8\rangle_\rho \left| \frac{1}{2} \frac{1}{2} \right\rangle_\lambda = \frac{1}{2\sqrt{6}} (q_1^\uparrow q_2^\uparrow q_3^\downarrow + q_1^\uparrow q_2^\downarrow q_3^\uparrow - 2q_1^\downarrow q_2^\uparrow q_3^\uparrow - q_1^\uparrow q_3^\uparrow q_2^\downarrow - q_1^\uparrow q_3^\downarrow q_2^\uparrow + 2q_1^\downarrow q_3^\uparrow q_2^\uparrow - q_2^\uparrow q_3^\uparrow q_1^\downarrow - q_2^\uparrow q_3^\downarrow q_1^\uparrow + 2q_2^\downarrow q_3^\uparrow q_1^\uparrow + q_2^\uparrow q_1^\uparrow q_3^\downarrow + q_2^\uparrow q_1^\downarrow q_3^\uparrow - 2q_2^\downarrow q_1^\uparrow q_3^\uparrow) . \quad (1.29)$$

The baryons of the lowest multiplet $[56, 0^+]_0$ have a totally symmetric coordinate part of the wave function — the orbital momentum of any quark pair equals zero. The spin-flavour part is also totally symmetric; to a symmetric flavour part (decuplet) corresponds the spin value $3/2$, to a flavour function of mixed symmetry (octet) the spin $1/2$:

$$[56, 0^+]_0 = {}^4 10_{3/2} + {}^2 8_{1/2} . \quad (1.30)$$

(We denote the SU(3) multiplets by ${}^{2s+1}H_J$, where J is the baryon spin and H stands for the number of states in the multiplet.) Hence,

$$\left| {}^4 10_{\frac{3}{2}} \right\rangle_{J_z} = |10\rangle \left| \frac{3}{2} J_z \right\rangle , \quad \left| {}^2 8_{\frac{1}{2}} \right\rangle_{J_z} = \frac{1}{\sqrt{2}} \left(|8\rangle_\lambda \left| \frac{1}{2} J_z \right\rangle_\lambda + |8\rangle_\rho \left| \frac{1}{2} J_z \right\rangle_\rho \right) . \quad (1.31)$$

The 70-plet

The coordinate part of the wave function of the multiplet with $L = 1$ is of mixed symmetry — only one quark pair is in a P -wave state. Because of that, the spin-flavour part should be also of mixed symmetry, *i.e.* we have a multiplet $[70, 1^-]_1$ [33]. The symmetric and antisymmetric flavour functions correspond here to the quark spin $1/2$, the mixed flavour function to spin $1/2$ or spin $3/2$. Combining the quark spins with the angular momenta, we can obtain the SU(3) multiplets:

$$[70, 1^-] = {}^4 8_{5/2} + {}^4 8_{3/2} + {}^4 8_{1/2} + {}^2 8_{3/2} + {}^2 8_{1/2} + {}^2 10_{3/2} + {}^2 10_{1/2} + {}^2 1_{3/2} + {}^2 1_{1/2} . \quad (1.32)$$

To describe the angular dependence of the coordinate function, it is convenient to expand it in terms of an orthonormal basis. For the P -wave 70-plet it is natural to consider functions $Y_{1\ell}(\mathbf{n}_{23})$ (P -wave between quarks with coordinates $\mathbf{r}_2$ and $\mathbf{r}_3$) and $Y_{1\ell}(\mathbf{r}_{1,23})$ ($\mathbf{n}_{1,23} \sim 2\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{r}_3$, P -wave between quark $\mathbf{r}_1$, and the S -wave pair $\mathbf{r}_2, \mathbf{r}_3$). The expansion with respect

to this basis (together with the corresponding spin-flavour functions) gives

$$\begin{aligned}
|{}^4 8_J\rangle_{J_z} &= \frac{1}{\sqrt{2}} \sum_{\ell, \sigma} C_{1\ell \frac{3}{2}\sigma}^{JJ_z} \left\{ |8\rangle_\lambda \left| \frac{3}{2}\sigma \right\rangle Y_{1\ell}(\mathbf{n}_{1,23}) + |8\rangle_\rho \left| \frac{3}{2}\sigma \right\rangle Y_{1\ell}(\mathbf{n}_{23}) \right\} , \\
|{}^2 8_J\rangle_{J_z} &= \frac{1}{\sqrt{2}} \sum_{\ell, \sigma} C_{1\ell \frac{1}{2}\sigma}^{JJ_z} \left\{ \left[-|8\rangle_\lambda \left| \frac{1}{2}\sigma \right\rangle_\lambda + |8\rangle_\rho \left| \frac{1}{2}\sigma \right\rangle_\rho \right] Y_{1\ell}(\mathbf{n}_{1,23}) \right. \\
&\quad \left. + \left[|8\rangle_\lambda \left| \frac{1}{2}\sigma \right\rangle_\rho + |8\rangle_\rho \left| \frac{1}{2}\sigma \right\rangle_\lambda \right] Y_{1\ell}(\mathbf{n}_{23}) \right\} , \quad (1.33) \\
|{}^2 0_J\rangle_{J_z} &= \frac{1}{\sqrt{2}} \sum_{\ell, \sigma} C_{1\ell \frac{1}{2}\sigma}^{JJ_z} \left\{ |10\rangle \left| \frac{1}{2}\sigma \right\rangle_\lambda Y_{1\ell}(\mathbf{n}_{1,23}) + |10\rangle \left| \frac{1}{2}\sigma \right\rangle_\rho Y_{1\ell}(\mathbf{n}_{23}) \right\} , \\
|{}^2 1_J\rangle_{J_z} &= \frac{1}{\sqrt{2}} \sum_{\ell, \sigma} C_{1\ell \frac{1}{2}\sigma}^{JJ_z} \left\{ -|1\rangle \left| \frac{1}{2}\sigma \right\rangle_\rho Y_{1\ell}(\mathbf{n}_{1,23}) + |1\rangle \left| \frac{1}{2}\sigma \right\rangle_\lambda Y_{1\ell}(\mathbf{n}_{23}) \right\} .
\end{aligned}$$

1.3 Estimation of Masses of the Constituent Quarks in the Quark Model

There exists a set of predictions of the quark model, which show clearly and unambiguously that even the simple, naive quark model gives an adequate (though qualitative) description of the hadron structure. We consider these predictions in the present section.

1.3.1 Magnetic moments of baryons

If the constituent quarks can be handled as quasiparticles, they have to be virtually the same in different hadrons. It is convenient to test this by the investigation of the magnetic moments of baryons (this, in fact, was historically the first serious success of the model). For definiteness, let us consider the proton magnetic moment; according to the quark model, it has to be the sum of magnetic moments of the constituent quarks:

$$\frac{e}{2m_p} \mu_p = \sum_{i=1,2,3} \left\langle p_{\frac{1}{2}} \left| \frac{e_q(i) \sigma_Z(i)}{2m_q(i)} \right| p_{\frac{1}{2}} \right\rangle , \quad (1.34)$$

where σ_Z (or σ_3) is the Pauli matrix (see Appendix 1.A). In the framework of the naive quark model we assume $m_u = m_d = m_p/3$, *i.e.* the masses of light non-strange quarks are just one third of the nucleon mass. The matrix element in the right-hand side of (1.34) is determined just by the

spin-flavour part of the proton wave function (it is explicitly given in the previous section). Owing to the normalisation, the coordinate part is unity. The magnetic moment μ_p is expressed in $e/2m_p$ units (*i.e.* in nuclear magnetons). The baryon magnetic momenta calculated this way are given in Table 1.1, where the notation

$$\xi = \frac{m_s - m_u}{m_u}$$

is used. Here $\xi \simeq 1/2$ corresponds to $m_s - m_u \simeq 150$ MeV, which is a rather fundamental quantity for both the quark model and chiral perturbation theory based on QCD [34].

The agreement between calculation and experiment is quite satisfactory (and typical for the naive quark model): the deviations are within 20–25%. However, if one tries to treat these deviations literally, the result will be distressing. For example, calculating the quark masses on the basis of data on μ_{Ξ^0} and μ_{Ξ^-} , one gets $m_u > m_s$. One has to remember that the non-relativistic quark model is a rough approach, and such discrepancies are more or less natural. Small variations of the magnetic moments (in comparison with the calculated values) can be, for instance, consequences of either relativistic corrections, or the structure of the dressed quarks themselves. Introducing, *e.g.* a relatively small anomalous magnetic moment for the u, d and s quarks [35] (see also [36]), one can get a better agreement with the data.

Table 1.1 Magnetic moments of baryons in nuclear magnetons.

Particle	Quark model prediction ($\xi=1/2$)	Experiment
p	3	2.79
n	-2	-1.91
Λ	$-1 + \xi = -0.5$	-0.61
Σ^+	$3 - \frac{1}{3}\xi = 2.84$	2.46
Σ^-	$-1 - \frac{1}{3}\xi = -1.16$	-1.16 ± 0.03
Ξ^0	$-2 + \frac{4}{3}\xi = -1.33$	-1.25 ± 0.01
Ξ^-	$-1 + \frac{4}{3}\xi = -0.33$	-0.65 ± 0.04

1.3.2 Radiative meson decays $V \rightarrow P + \gamma$

The radiative decay of a vector meson V with the production of a pseudoscalar P (reaction $V \rightarrow \gamma P$) is determined by the magnetic moments of

Table 1.2 Values of $\sqrt{\Gamma(V \rightarrow P + \gamma)}$, $\text{keV}^{1/2}$ for vector meson decays.

Decay mode	$\sqrt{\Gamma(V \rightarrow P + \gamma)}$, $\text{keV}^{1/2}$	
	Quark model prediction	Experiment
$\omega \rightarrow \pi^0 \gamma$	34.6	26.9 ± 0.9
$\rho^- \rightarrow \pi^- \gamma$	11.0	8.2 ± 0.4
$\rho^0 \rightarrow \eta \gamma$	8.4	8.1 ± 0.9
$\phi \rightarrow \eta \gamma$	10.4	7.6 ± 0.1
$K^{*\pm} \rightarrow K^- \gamma$	7.0	7.1 ± 0.3
$K^{*0} \rightarrow K^0 \gamma$	13.7	10.8 ± 0.5

the constituent quarks:

$$A_{V \rightarrow \gamma P} \sim \sum_{i=q,\bar{q}} \left\langle V \left| \frac{e_i \sigma_Z(i)}{2m_i} \right| P \right\rangle. \quad (1.35)$$

These processes are transitions of the type of $\omega \rightarrow \gamma \pi^0$, $\phi \rightarrow \gamma \eta$, *etc.* If the idea of the constituent quarks is correct, these transitions must be determined by the same quark masses (and, respectively, magnetic moments), which gave us the magnetic moments of the baryons.

In Table 1.2 we present the calculated values and the experimental data. We use here $\sqrt{\Gamma(V \rightarrow P + \gamma)}$, since this quantity is proportional to the quark magnetic moment, and is, therefore, suitable for comparison with the calculated magnetic moment. The predictions for the radiative widths satisfy the experimental data within the same accuracy of 20–25%. It is a rather impressive fact that the quark magnetic moments are the same in mesons and baryons; this shows that the dressed quarks appear in hadrons as somewhat independent objects — quasiparticles.

In Chapters 6 and 7 we give a detailed discussion of radiative decays in the framework of the quark model.

1.3.3 Empirical mass formulae

It was understood already relatively long ago [37] that the mass splitting of light hadrons can be well described in the framework of the non-relativistic quark model by the spin–spin quark interaction. The next step was made by de Rújula, Georgi and Glashow: according to [38], the hadron mass splitting is due only to the short-range part (the spin–spin part) of the interaction, which is connected to the gluon exchange. The obtained effective potential for the interaction of two quarks (i and j) is supposed to be

$$V_{ij} = \pm \alpha_s \left(\frac{\lambda(i)}{2} \frac{\lambda(j)}{2} \right) \left(-\frac{2\pi}{3} \cdot \frac{\sigma(i)\sigma(j)}{m_q(i)m_q(j)} \delta(\mathbf{r}_{ij}) \right), \quad (1.36)$$

where α_s is the gluon-quark coupling constant squared, λ are the Gell-Mann matrices (see Appendix 1.A), acting on the colour indices of the i th and j th quarks, and the signs $\pm$ stand for the interactions of two quarks or a quark and an antiquark, respectively. It is assumed that the remaining part of the interaction, which is due to the gluon exchange, is averaged, and gives a contribution to the potential which confines the quarks. The interaction (1.36) leads in Born approximation to the following mass splitting:

$$\Delta M_{meson} = \frac{8\pi}{9} \alpha_s |\Psi_M(0)|^2 \left\langle h_M \left| \frac{\sigma(1)\sigma(2)}{m_q(1)m_q(2)} \right| h_M \right\rangle, \quad (1.37)$$

$$\Delta M_{baryon} = \frac{4\pi}{9} \alpha_s \sum_{i \neq j} \int d^3 r_k |\Psi_B(\mathbf{r}_{ij} = 0, \mathbf{r}_k)|^2 \left\langle h_B \left| \frac{\sigma(i)\sigma(j)}{m_q(i)m_q(j)} \right| h_B \right\rangle.$$

The spin-flavour part of matrix elements is calculated exactly; however, in such an approach it is impossible to define the coordinate part of the wave function. Because of that, the expressions $\alpha_s |\Phi_M(0)|^2$ and $\alpha_s \int d^3 r_k |\Phi_B(0, \mathbf{r}_k)|^2$ should be considered as phenomenological constants, which can be obtained from the comparison of masses in the meson and baryon multiplets. The result of the comparison of formulae (1.37) with experiment is demonstrated in Table 1.3. Note that in the calculations we take $|\Phi_M(0)|^2 = \int d^3 r |\Phi_B(0, \mathbf{r})|^2$. This also shows that it is roughly equiprobable to find two quarks or a quark-antiquark pair on a relatively small distance in a hadron. The relations (1.37) are valid also in the case of charmed particles (D and D^* are states of $c\bar{q}$, where $q = u, d$, with $J^P = 1^-$ and 0^- ; D_s^* and D_s — states of $c\bar{s}$ with $J^P = 1^-$ and 0^-). The constant $\alpha_s |\Phi_M(0)|^2$ is the same as for light hadrons (see Table 1.3).

Table 1.3 Baryon mass splitting values calculated in the model of de Rújula-Georgi-Glashow. It is assumed that $m_u = m_d = 360$ MeV, $m_s/m_u = 3/2$, $m_c = 1440$ MeV, $|\Phi_M(0)|^2 = \int d^3 r |\Phi_B(0, \mathbf{r})|^2$.

ΔM	Calculated (MeV)	Exp. (MeV)	ΔM	Calculated (MeV)	Exp. (MeV)
$m_\Delta - m_N$	300	295	$m_\rho - m_\pi$	600	630
$m_\Sigma - m_\Lambda$	68	77	$m_{K^*} - m_K$	400	398
$m_{\Sigma^*} - m_\Lambda$	267	274	$m_{D^*} - m_D$	150	140
$m_{\Xi^*} - m_\Xi$	200	217	$m_{D_s^*} - m_{D_s}$	100	120

The de Rújula-Georgi-Glashow approach allows us to understand and write an explicit expression for baryon masses on a rather elementary level of the quark model. This possibility was discussed in [39], where, for the

masses of the S-wave 56-plet baryons, the expression

$$m_B = \sum_i m_q(i) + b \sum_{i \neq k} \frac{\sigma(i)\sigma(j)}{m_q(i)m_q(j)} \quad (1.38)$$

was suggested. The phenomenological parameter was found from the experiment, and there is an astonishingly good description of the baryon masses (see Table 1.4). The discrepancies between predictions and mea-

Table 1.4 Baryon masses calculated in terms of Eqs. (1.38, 1.39).

mass (MeV)			mass (MeV)		
Baryon			Baryon		
	Prediction	Exp.		Prediction	Exp.
N	930	937	Σ^*	1377	1384
Δ	1230	1232	Ξ	1329	1318
Σ	1178	1193	Ξ^*	1529	1533
Λ	1110	1116	Ω	1675	1672

sured data are about 5–6 MeV. However, in trying to write a similar formula for mesons, one fails: the systematic deviations between calculation and experiment are of the order of 100 MeV (the calculated mass values for the ρ and π mesons are $m_\rho = 875$ MeV, $m_\pi = 275$ MeV). The reason for this discrepancy becomes obvious when one calculates the average quark mass in a meson and in a baryon:

$$\begin{aligned} \langle m_q \rangle_M &= \frac{1}{2} \left(\frac{1}{4} m_\pi + \frac{3}{4} m_\rho \right) = 303 \text{ MeV} , \\ \langle m_q \rangle_B &= \frac{1}{3} \left(\frac{1}{2} m_N + \frac{1}{2} m_\Delta \right) = 363 \text{ MeV} . \end{aligned} \quad (1.39)$$

In these combinations of hadron masses, the contribution of the splitting interaction (1.37) cancels completely. Equation (1.39) tells us that the quark masses in mesons are “eaten” by some additional interactions.

1.4 Light Quarks and Highly Excited Hadrons

We saw that low mass hadrons can be considered, in a way, similar to light nuclei (if we substitute nucleons by constituent quarks). The highly excited hadrons open before us, however, a new and intriguing world.

In the last two decades the highly excited states were intensely studied experimentally. Not aiming at completeness, we mention here a list of experiments, partial wave analyses and collaborations and groups: PNPI-RAL [40, 41, 42], PNPI [43, 44, 45], WA102 [46], GAMS [47, 48, 49], VES

[50], Crystal Barrel [51, 52]. They gave us a lot of information for the reanalysis of our notion of the quark–gluon structure of hadrons.

1.4.1 *Hadron systematisation*

The analysis of the Crystal Barrel experiments given by the PNPI–RAL group [42] leads to the discovery of a large number of meson resonances in the mass region 1950 – 2450 MeV. This resulted in the systematisation of $q\bar{q}$ states on the (n, M^2) planes (where n is the radial quantum number of a meson with mass M). As it turned out, mesons with the same J^{PC} but different n fit well to the linear trajectories [53]:

$$M^2(J^{PC}) = M_0^2(J^{PC}) + \mu^2(n - 1). \quad (1.40)$$

Here $M_0(J^{PC})$ is the mass of the ground state ($n = 1$), while μ^2 is a universal constant $\mu^2 = 1.25 \pm 0.05 \text{ GeV}^2$. Thus it became quite easy to construct trajectories also on the (J, M^2) plane and to build not only the basic trajectories but also a large number of daughter trajectories. This systematisation made it possible to obtain meson nonets for sufficiently high orbital and radial excitations. (All this will be discussed in detail in Chapters 2 and 8.)

The systematisation (1.40) is of great significance, however, not only in this sense. As it turns out, virtually all, sufficiently well established resonances are placed on the linear $q\bar{q}$ trajectory. Thus, there is practically no room for non- $q\bar{q}$ states such as four-quark states, $q\bar{q}q\bar{q}$, and hybrids $q\bar{q}g$. Indeed, copious non- $q\bar{q}$ states should have masses above 1500 MeV (remind that the mass of the effective gluon g is of the order of 700 – 1000 MeV, the masses of the light constituent quarks u and d are about 300 – 350 MeV).

Why in the case of mesons Nature does not "imitate" light nuclei so easily, refusing to produce states consisting of a large number of constituents — in contrast to the case of nuclei? We do not have an answer to this question, but it is definitely very important for understanding the character of forces between coloured objects at large distances.

The construction of baryon trajectories in the (n, M^2) plane exposes one more puzzle. Indeed, these trajectories are in accordance with the linear trajectories of the (1.40) type, with the same $\mu^2 \simeq 1.25 \text{ GeV}^2$ value. Does this mean the universality of forces at large distances, acting between the quark and a two-quark system called diquark?

1.4.2 Diquarks

It is an old idea that a qq -system inside a baryon can be separated as a specific object and the quark interactions can be considered as interactions of a quark with a qq -system $q + (qq)$. Such a hypothesis was used in [54] for the description of hadron–hadron collisions. In [55] baryons were described as quark–diquark systems. In hard processes on nuclei, the coherent qq -state (composite diquark) can be responsible for the interaction in the region of large Bjorken x -values, at $x \sim 2/3$; deep inelastic scatterings were considered in the framework of such an approach in [56]. A more detailed picture of the diquark and its applications can be found in [57, 58, 59].

There are two diquark states which have to be taken into account when considering the baryons, namely: qq -states with an orbital momentum $\ell = 0$, a pseudovector diquark and a scalar diquark:

$$J = 1^+ \quad d_1, \quad J = 0^+ \quad d_0. \quad (1.41)$$

If highly excited baryon states are formed in Nature as states of a quark–diquark system, with two possible types (1.41) of diquarks, the variety of highly excited states is seriously reduced, while the classification of the lowest baryons remains unchanged.

There is one more important consequence of the quark–diquark structure of highly excited states: the radial and angular excitations of the qd and $q\bar{q}$ systems must be similar, since the diquark and the antiquark have the same colour charge.

In the recently considered quark models (*e.g.*, see [60, 61, 62]), the baryon states are described by forces of the same structure in the qq and the $q\bar{q}$ sectors (with the obvious replacement of charges when changing from a quark to an antiquark). The cited works contain different hypotheses about the quark–quark (or quark–antiquark) interactions. Still, all they lead to the same specific result for the spectra: the calculated number of highly excited states turns out to be much larger than that of the observed resonances.

This is quite natural for the three-quark models. Indeed, three-quark systems are characterised by two coordinates: the relative distance $\mathbf{r}_{12}$ between quarks 1 and 2, and the coordinate of the third quark, $\mathbf{r}_3$. Accordingly, qqq -states can be determined by two orbital momenta ℓ_{12} and ℓ_3 , and by two radial excitations n_{12} and n_3 . There are also many spin states: $s_{12} = 0, 1$ and $S = |s_{12} + s_3| = 1/2, 1/2, 3/2$. Naturally, this variety is restricted by the imposed requirement of complete antisymmetry, but even

so, the number of remaining states is rather large. And it is just this large number of three-quark states which is not confirmed experimentally.

Experimental data on baryon states are unfortunately scarce compared to meson data. So a possible attitude is to wait and see, not drawing any conclusions before having more baryon data. We can, however, take seriously the information we have so far, as an indication that the number of highly excited baryon states is much smaller than expected. If so, we have to reconsider our view on the character of interactions in the $q\bar{q}$ and qq channels and to take into account that interactions in these channels may be quite different.

1.5 Scalar and Tensor Glueballs

Experimentally, we do not observe many mesons with masses higher than 1500 MeV, which could not be placed on the $q\bar{q}$ trajectories in (n, M^2) planes. This is, from our point of view, the main argument against the existence of exotic $q\bar{q}g$ and $qq\bar{q}\bar{q}$ states. As was mentioned above, if $q\bar{q}g$ and $qq\bar{q}\bar{q}$ states existed, we should observe a large number of highly excited states with both exotic and non-exotic quantum numbers, which, as we saw already, is not the case.

This does not mean, of course, that announcements of the observation of exotic mesons would not appear regularly. The reason is not the absence of sufficiently reliable experiments but rather the lack of really qualified analysis of the data. (Reviews about the search for $q\bar{q}g$, $qq\bar{q}\bar{q}$ and other states, such as, *e.g.*, the pentaquarks, can be found in [63, 64]).

To handle this problem, we devote Chapters 3, 4, 5 and 6 to the technique of investigating experimental spectra in the framework of partial wave analysis. In Chapter 3 we consider the scattering of spinless particles, and elements of the K-matrix technique and of the dispersion N/D method are presented. In Chapter 4 collisions of fermions, NN and $N\bar{N}$, are described; expressions for the amplitudes of the production of large spin particles are given. Chapter 5 is devoted to πN and γN collisions.

The analysis of mesonic spectra allowed us to discover two broad isoscalar states in the channels $J^{PC} = 0^{++}$ (see Chapter 3) and $J^{PC} = 2^{++}$ (Chapter 4).

- (i) They are superfluous from the point of view of the (n, M^2) systematisation;

- (ii) the constants of their decays into pseudoscalar mesons satisfy relations corresponding to glueballs; the decays are nearly flavour blind.

The masses and widths of these glueballs are as follows.

Scalar glueball [65, 66, 67, 68, 69, 70]:

$$0^{++} - \text{glueball} : \quad M \simeq 1200 - 1600 \text{ MeV} \quad \Gamma \simeq 500 - 900 \text{ MeV} , \quad (1.42)$$

tensor glueball [76, 74, 75]:

$$2^{++} - \text{glueball} : \quad M = 2000 \pm 30 \text{ MeV}, \quad \Gamma = 500 \pm 50 \text{ MeV} . \quad (1.43)$$

The status of the tensor glueball is rather well defined: it was seen in several experiments [71, 72, 73], and the decay couplings tell us that $f_2(2000)$ is nearly flavour blind [74, 75]. Besides, the $f_2(2000)$ is an extra state in (n, M^2) trajectories [76].

More ambiguous is the existence of the scalar glueball: its mass and width are determined with large errors. However, the ratios of the couplings of the $f_0(1200 - 1600)$ decays into different channels of two pseudoscalar particles, $f_0(1200 - 1600) \rightarrow \pi\pi, K\bar{K}, \eta\eta, \eta\eta'$ are comparatively well defined. These couplings show us that $f_0(1200 - 1600)$ is very close to a flavour singlet (so this state is flavour blind with a good accuracy). Moreover, from the point of view of the $q\bar{q}$ -systematics this state turned out to be superfluous (see Chapter 2). Hence, it is natural to identify it with a scalar glueball.

The mass $f_0(1200 - 1600)$ is twice the mass of the soft effective gluon ($m_g \simeq 700 - 1000 \text{ MeV}$), so, seemingly, this state could be considered also as a gluonium, gg . Still, this would be a rather conditional notation for $f_0(1200 - 1600)$. Indeed, it was produced as a result of a strong mixing with its neighbouring resonances: the evidence for that is both the large width of the resonance and the fact that the gluonium mixes easily with $q\bar{q}$ states (the latter will be discussed in Chapter 2). So it is reasonable to call the $f_0(1200 - 1600)$ state a gluonium descendant. In fact, its wave function is a Fock column

$$f_0(1200 - 1600) = \begin{pmatrix} gg \\ q\bar{q} \\ \pi\pi, K\bar{K}, \eta\eta, \eta\eta' \\ \pi\pi\pi\pi \\ qq\bar{q}\bar{q} \\ \dots \end{pmatrix} \quad (1.44)$$

and it is not certain at all that the gluonium component gg strongly dominates. Thus, to follow the tradition, we call $f_0(1200 - 1600)$ (though rather

conditionally) a glueball, having in mind that it is, probably, a mixture of states of the type shown in (1.44).

Similarly, the tensor glueball $f_2(2000)$ is a mixture of different states. In this case, however, the components with vector particles may be significant as well:

$$f_2(2000) = \begin{pmatrix} gg \\ q\bar{q} \\ \pi\pi, K\bar{K}, \eta\eta, \eta\eta' \\ \rho\rho, \omega\omega, \phi\phi, \omega\phi \\ \dots \end{pmatrix}. \quad (1.45)$$

The tensor glueball lies on the pomeron trajectory

$$\alpha_{\mathbf{P}}(M^2) = \alpha_{\mathbf{P}}(0) + \alpha'_{\mathbf{P}}(0)M^2, \quad (1.46)$$

where $\alpha_{\mathbf{P}}(0) \simeq 1.1 - 1.3$, $\alpha'_{\mathbf{P}} \simeq 0.15 - 0.25 \text{ GeV}^{-2}$. The scalar glueball has to be placed on a daughter trajectory. Assuming that the daughter trajectory is also linear and is characterized by the same slope as the basic trajectory, we have

$$\alpha_{\mathbf{P}(\text{daughter})}(M^2) = \alpha_{\mathbf{P}(\text{daughter})}(0) + \alpha'_{\mathbf{P}}(0)M^2; \quad (1.47)$$

here $\alpha_{\mathbf{P}(\text{daughter})}(0) \simeq -0.5$. This means that the next tensor state lying on this trajectory must be near 3500 MeV (see Fig. 1.2).

The scalar glueball was detected as a result of a set of subsequent K-matrix analyses [66, 67, 68, 69, 70]. In the course of these investigations the energy (or the invariant mass) of $\pi\pi$ was successively increased and more and more channels ($K\bar{K}$, $\eta\eta$, $\eta\eta'$, $\pi\pi\pi\pi$) were included. In the beginning, when the invariant mass of the considered spectra was small ($\sqrt{s} < 1500 \text{ MeV}$ [66, 67]), the status of the broad resonance was questionable, since its mass was on the verge of the spectra. In the subsequent investigations [68, 69] the mass interval was increased up to 2000 MeV, and the position of the broad resonance was stabilised in the region of 1400 MeV (although with a large error, of the order of $\pm 200 \text{ MeV}$). There is an essential difference between the quark contents of the scalar and the tensor resonances. The scalar resonances are the mixtures of non-strange ($n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$) and strange ($s\bar{s}$) quarkonia, while the tensor resonances are either dominant $n\bar{n}$ -states, or the $s\bar{s}$ is dominating. We have to remember that, while the $q\bar{q}$ component may be large in a glueball, the gluonium component cannot be large in a $q\bar{q}$ state owing to the fact that gg is smeared over a number of neighbouring states.

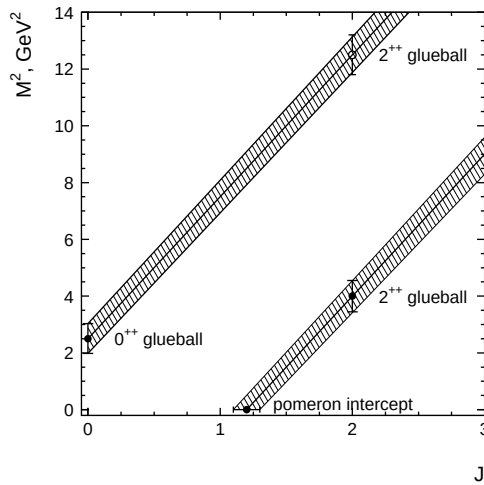


Fig. 1.2 Glueball states on the pomeron trajectories (full circles) and the predicted second tensor glueball (open circle).

The $f_0(450)$ called the σ -meson is a particular state. Strictly speaking, we are not sure that the σ -meson exists at all. However, if it exists, it could be a rather remarkable particle: the visible "remnant" of the white component of the scalar confinement forces.

1.5.1 Low-lying σ -meson

The K -matrix analysis of the $(0, 0^{++})$ wave does not give a definite answer to the question whether the σ -meson exists. Indeed, the applicability of the K -matrix analysis is restricted in the small $\sqrt{s}$ region, since the K -matrix amplitude cannot give an adequate description of the left cut of the partial amplitude at $s \leq 0$. In [77], the analysis of the $\pi\pi$ amplitude at $280 \leq \sqrt{s} \leq 900$ MeV was carried out in the framework of the dispersion N/D method. Performing the N/D -fit, we have used there, on the one hand, experimental data on the scattering phase in the region 280–500 MeV, and, on the other hand, the K -matrix amplitude [69] in the 450–900 MeV region. As a result, we got a resonance pole near the $\pi\pi$ threshold denoted as $f_0(450)$ (see Chapter 2 for more detail).

The light σ -meson is a possible manifestation a component (the white one) of the singular colour forces responsible for confinement. The scalar confinement potential describing the $q\bar{q}$ state spectrum in the 1500 - 2500

MeV region behaves at large hadron distances as $V(r) \sim r$, in the momentum representation this leads to a $1/q^2$ -type singularity in the $q\bar{q}$ amplitude. In the white channel, the transition

$$\textit{white singular term} \longrightarrow \pi\pi \longrightarrow \textit{white singular term} \quad (1.48)$$

exists, owing to which the singularities of the white amplitude may occur on the second (unphysical) sheet of the complex- s plane. It is just this singular term which may turn out to be the object we call σ . This scenario is considered in more detail in Chapter 3, where the scalar and tensor states are discussed.

1.6 High Energies: The Manifestation of the Two- and Three-Quark Structure of Low-Lying Mesons and Baryons

We have seen (Sections 1.4.1 and 1.4.2) that the investigation of highly excited hadrons may raise a doubt in the correctness of our picture of strongly interacting quarks and gluons. There could be a challenge to act as was suggested by I.Ya. Pomeranchuk: "erase everything, let us start again". Still, the physics of high-energy collisions of low-lying hadrons (pions, kaons, nucleons) prevent us from rushing to such a conclusion. Indeed, experimental data collected in the field of high energy collisions in the last five decades show unambiguously that low-lying mesons (π , K) and baryons consist of two and three constituent quarks, respectively.

We shall recall here some of the most striking and important facts. For a detailed description, see [78].

1.6.1 Ratios of total cross sections in nucleon–nucleon and pion–nucleon collisions

At moderately high energies, at momenta $p_{lab} \sim 5 - 300$ GeV/c of the incoming particles, the ratio of the total cross sections can be described by

$$\sigma_{tot}(NN)/\sigma_{tot}(\pi N) = 3/2 \quad (1.49)$$

with quite a good accuracy (of the order of 10%).

This ratio was initiated by V.N. Gribov and I.Ya. Pomeranchuk. Later on it was considered in many papers [79, 80]. The additive quark model is based just on this relation: if the constituent quarks are separated in space,

the main process is the collision of a quark of the incident hadron with a quark of the target hadron (see Fig. 1.3).

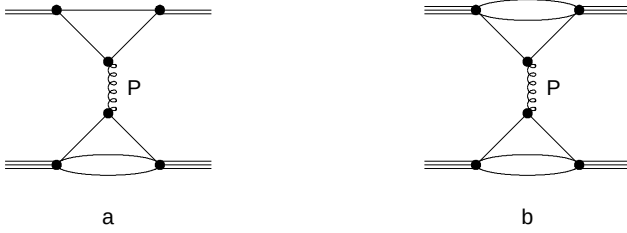


Fig. 1.3 Pion–nucleon and nucleon–nucleon scattering in the constituent quark model with pomeron exchange.

There are six meson–nucleon collisions and nine nucleon–nucleon collisions of this type. Since the total cross sections $\sigma_{tot}(NN)$ and $\sigma_{tot}(\pi N)$ are proportional to the imaginary parts of the diagrams shown in Fig. 1.3, we obtain the relation (1.49).

1.6.2 Diffraction cone slopes in elastic nucleon–nucleon and pion–nucleon diffraction cross sections

The elastic diffraction cross sections determined by the diagrams Fig. 1.3 read (see [78, 79, 80]):

$$\begin{aligned} \frac{d\sigma}{d|t|}(NN \rightarrow NN) &\sim F_N^4(t) |A_{qq}(t)|^2, \\ \frac{d\sigma}{d|t|}(\pi N \rightarrow \pi N) &\sim F_\pi^2(t) F_N^2(t) |A_{qq}(t)|^2, \end{aligned} \quad (1.50)$$

where $F_\pi(t)$ and $F_N(t)$ are triangle quark blocks, and $A_{qq} \simeq A_{q\bar{q}}$ at high energies.

On the other hand, the charge form factors of the pion $f_\pi(t)$ and of the nucleon $f_p(t)$ are determined by the processes in Fig. 1.4, i.e. by triangle diagrams of the same type as those defining the diffraction cone in (1.50). Hence,

$$f_\pi(t) = F_\pi(t) f_q(t), \quad f_p(t) = F_p(t) f_q(t). \quad (1.51)$$

Here $f_q(t)$ is the form factor of the constituent quark. Since in the model the constituent quarks are supposed to be relatively small objects compared to the hadron size,

$$\langle r_q^2 \rangle \ll \langle R_{hadron}^2 \rangle, \quad (1.52)$$

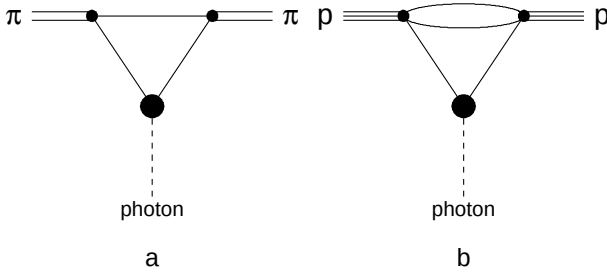


Fig. 1.4 Charge form factors of pion and proton in the additive quark model.

we can, in a rough approximation, neglect the t -dependence in both $f_q(t)$ and $A_{qq}(t)$ (though in the latter at moderately high energies only, when the pomeron size is small). Hence, considering the t -dependence at moderately high energies ($p_{lab} \sim 5 - 100$ GeV/c) we can take

$$\frac{d\sigma}{d|t|}(NN \rightarrow NN) \sim F_N^4(t), \quad \frac{d\sigma}{d|t|}(\pi N \rightarrow \pi N) \sim F_\pi^2(t)F_N^2(t), \quad (1.53)$$

where $F_\pi(t)$ and $F_N(t)$ are charge form factors of the pion and the proton. Experimental data on the slopes of diffraction cones are well described by Eq. (1.53).

1.6.3 Multiplicities of secondary hadrons in e^+e^- and hadron-hadron collisions

The multiplicity of the secondary (*i.e.* newly produced) hadrons in e^+e^- collisions is determined by the process shown in Fig. 1.5a: the virtual photon produces a high energy $q\bar{q}$ pair; in their turn the quarks, flying away, give rise to a jet (or comb) of hadrons. Similar processes take place also in hadron-hadron collisions [81], they are shown in Figs. 1.5b (pion-nucleon collision) and 1.5c (nucleon-nucleon collision). In the central region, the multiplicities of the newly produced particles are equal for all these three processes, if only the energies of e^+e^- , $q\bar{q}$ and qq are equal.

Such an equality of the multiplicities is confirmed by experiment (see [78] and references therein).

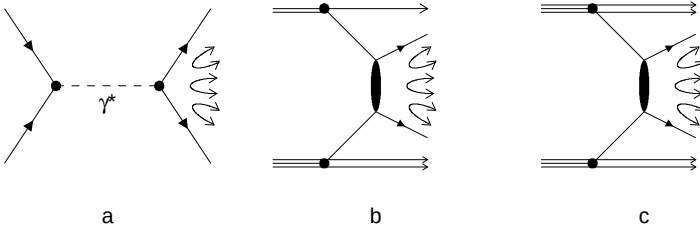


Fig. 1.5 Multiple production of hadrons in e^+e^- collisions and in πN and NN collisions where $q\bar{q} \rightarrow \text{hadrons}$ and $q\bar{q} \rightarrow \text{hadrons}$ transitions are dominating.

1.6.4 Multiplicities of secondary hadrons in πA and pA collisions

The two quarks of a pion or the three quarks of a nucleon are not able to pass a very heavy nucleus without interacting (see Fig. 1.6). If so, in πA and NA processes the multiplicities have to be related as [82]:

$$\left[\frac{\langle n \rangle_{NA}}{\langle n \rangle_{\pi A}} \right]_{A \rightarrow \infty} = \frac{3}{2}. \quad (1.54)$$

Real nuclei are not massive enough to produce this ratio explicitly. But, on the basis of experimental data, one can write $\langle n_{ch} \rangle_{pA} / \langle n_{ch} \rangle_{\pi A}$ as a function of A . In this case, it can be clearly seen that this relation goes to $3/2$ as A is growing (for details, see [78]).

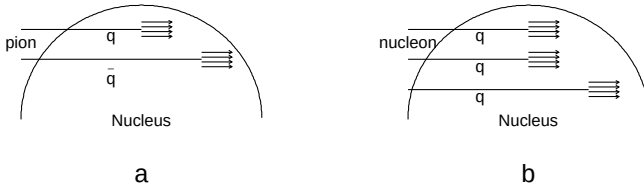


Fig. 1.6 Multiple production of hadrons in πA and NA collisions with heavy nuclei: in this case all quarks of the incoming particles interact with the nuclear matter.

1.6.5 Momentum fraction carried by quarks at moderately high energies

It is obvious from Figs. 1.5b and 1.5c that the colliding quark of the meson carries $\sim 1/2$ of the meson momentum, while the colliding quark of a nucleon carries $\sim 1/3$. These facts have to manifest themselves in the spectra of secondary particles formed by colliding quarks, *i.e.* in the

central region of secondary particle production. Experimental results [83] show that this is, indeed, the case (Fig. 1.7). We see that in the c.m. frame of the colliding hadrons in πp collisions the spectrum of secondary hadrons in the central region is shifted in the direction of the pion motion (Fig. 1.7a). In the centre-of-mass frame of the colliding quarks (Fig. 1.7b), however, the spectrum becomes symmetrical. This proves that the meson consists of two, the baryon of three quarks.

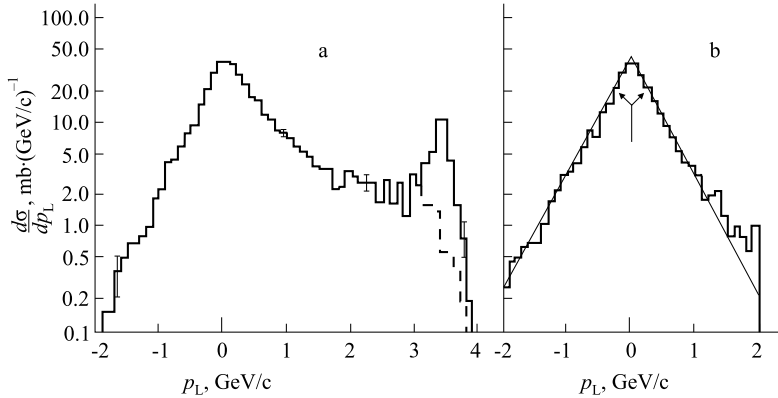


Fig. 1.7 The cross section of the $\pi^-p \rightarrow \pi^\pm$ process at 25 GeV/c in the c.m. system of the colliding particles (a) and in the centre-of-mass frame of the colliding quarks (b). To exclude the effect of the leading particle, the cross section of the $\pi^-p \rightarrow \pi^+$ process (which is close to $\pi^-p \rightarrow \pi^-$ for small x values) is drawn at $p_L > 0$ in Fig. 1.9b. Data are taken from [83].

1.7 Constituent Quarks, QCD-Quarks, QCD-Gluons and the Parton Structure of Hadrons

Attempts to combine the structure of constituent quarks with the results of deep inelastic scatterings were made relatively long ago [84].

1.7.1 Moderately high energies and constituent quarks

The constituent quarks are “dressed quarks” — indeed, from the point of view of the parton picture they consist of QCD-quarks and QCD-gluons. Each of these quark–gluon clusters (*i.e.* constituent quarks) consists of a valence QCD quark (or current quark) surrounded by quark–antiquark pairs and QCD gluons (see Fig. 1.8).

Since the quantum numbers of the constituent quarks and the valence

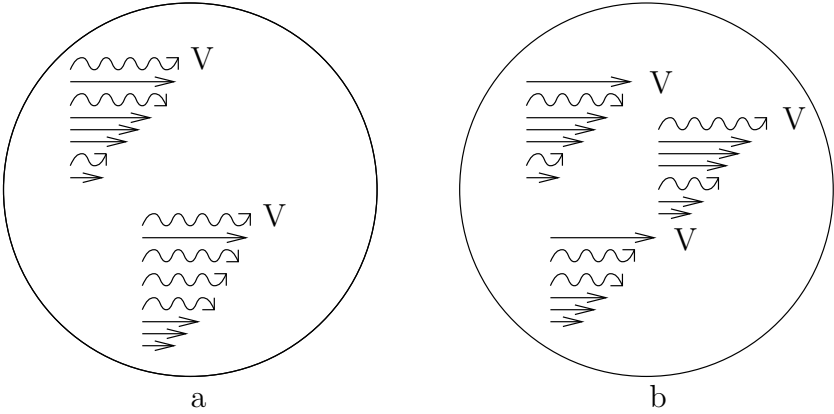


Fig. 1.8 Parton structure of a meson (a) and of a baryon (b). The baryon consists of three (the meson of two) dressed quarks; each dressed quark (antiquark) consists of a valence quark-parton (straight arrow, marked by the index V), sea partons (wavy arrows for gluons and straight arrows for quarks or antiquarks).

quarks coincide, the sea of the quark-antiquark pairs and QCD-gluons is neutral.

Let us note that the picture of spatially separated quarks is true only up to moderately high energies; only then we have three (nucleon) or two (meson) quark-parton clouds (Fig. 1.8). With the growth of energy the transverse dimensions of these clouds increase and we arrive at an essentially new picture of overlapping clouds.

1.7.2 Hadron collisions at superhigh energies

The changes which the clouds of colliding quarks go through while the moderately high energies grow to superhigh ones can be demonstrated in the impact parameter space (see Fig. 1.9).

Figure 1.9a shows the “picture” of a meson, while Fig. 1.9d is that of a nucleon in the impact parameter space (*i.e.* what the incoming hadrons look like from the point of view of the target). In the impact parameter space quarks are black at moderately high energies: this follows from investigations of the proportions of truly inelastic and quasi-inelastic processes [85]. Accordingly, in Figs. 1.9a and 1.9d two (for a meson) and three (for a baryon) black discs are drawn. But, as we just mentioned, the transverse sizes of the discs increase, and at intermediate energies ($p_{lab} \sim 500 - 1000$ GeV/c) the quarks partially overlap (Figs. 1.9b, 1.9e). In this energy region

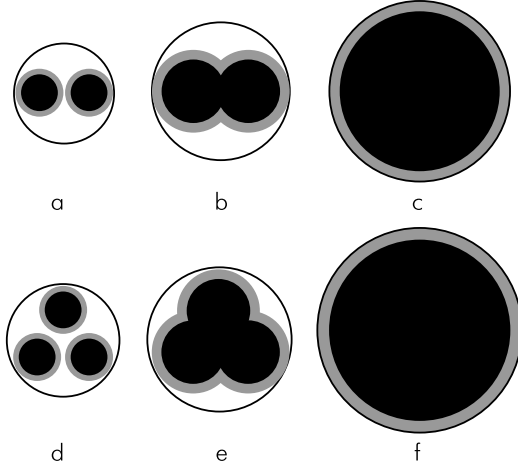


Fig. 1.9 Quark structure of a meson (a–c) and a baryon (d–e) in the constituent quark model. At moderately high energies (a,c) constituent quarks inside the hadron are spatially separated. With the energy increase, quarks become partially overlapped (b,e); at superhigh energies (c,f) quarks are completely overlapped, and hadron–hadron collisions lose the property of additivity.

the additivity may already be broken in the collision processes. Further, there is a total overlap of the clouds (Figs. 1.9c, 1.9f) and, in principle, the meson cross sections cannot be distinguished from the baryon cross sections any more. Indeed, both are just products of the collisions of black discs.

According to estimates given in [86], in this energy region

$$\sigma_{tot}(p\bar{p}) \simeq \sigma_{tot}(\pi p) \simeq 2\sigma_{el}(p\bar{p}) \simeq 2\sigma_{el}(\pi p) \simeq 0.32 \ln^2 s \text{ mb} \quad (1.55)$$

– but this is true only for energies higher than what can be reached at LHC. For energies $0.5 \text{ TeV} \leq \sqrt{s} \leq 20 \text{ TeV}$ the cross sections have to behave as [86]:

$$\begin{aligned} \sigma_{tot}(p\bar{p}) &= 49.80 + 8.16 \ln \frac{s}{9s_0} + 0.32 \ln^2 \frac{s}{9s_0} , \\ \sigma_{tot}(\pi p) &= 30.31 + 5.70 \ln \frac{s}{6s_0} + 0.32 \ln^2 \frac{s}{6s_0} . \end{aligned} \quad (1.56)$$

In (1.56) the numerical coefficients are given in mb, and $s_0 = 10^4 \text{ GeV}^2$. In the region of LHC energies ($\sqrt{s} = 16 \text{ TeV}$) we have

$$\sigma_{tot}(p\bar{p}) = 131 \text{ mb}, \quad \sigma_{el}(p\bar{p}) = 41 \text{ mb}. \quad (1.57)$$

As we see, at LHC energies the asymptotic value $\sigma_{tot} \simeq 1/2\sigma_{el}$ is not reached yet. However, already at these energies another consequence of the

quark overlap reveals itself: the scaling of proton spectra in the fragmentation region is broken at $x = p/p_{max} \sim 2/3$. The spectra of the protons have to decrease sharply in this region [87].

* * *

As was seen above, the hypothesis of hadrons being composite systems of two (mesons) or three (low-lying baryons) constituent quarks works well. But it is a question whether this hypothesis works for highly excited states, namely, whether certain highly excited states consist of a larger number of constituent quarks or contain effective gluons — this question should be answered by further experimental investigations. To avoid misleading conclusions, we should deal with advanced and refined methods for fixing pole singularities of the amplitudes.

Our further presentation is devoted mainly to the techniques used for the study of analytical structure of the amplitudes in hadron collisions.

1.8 Appendix 1.A: Metrics and $SU(N)$ Groups

There are different ways of writing the four-dimensional metric tensor, the γ -matrices, the amplitudes, *etc.*; we present here our choice for them. In addition, we give some useful relations for reference.

1.8.1 Metrics

We use the metric tensor

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1) . \quad (1.58)$$

We do not distinguish between covariant and contravariant vectors, and adopt the notation

$$A_\mu B_\mu = A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3 . \quad (1.59)$$

Summation over doubled subscripts is assumed wherever the opposite is not specified.

1.8.2 $SU(N)$ groups

The fundamental representation space for an $SU(N)$ group is formed by N -component spinors Ψ (columns of N complex numbers or field operators). The transformation

$$\Psi \rightarrow \Psi' = S\Psi \quad (1.60)$$

of the fundamental representation is carried out by $N \times N$ complex matrices which satisfy the unitarity and unimodularity conditions

$$SS^+ = I, \quad \det S = 1. \quad (1.61)$$

Every matrix S has $N^2 - 1$ real independent parameters ω_a ($a = 1, 2, \dots, N^2 - 1$) and can be represented in the form

$$S = \exp(i\omega_a t_a), \quad (1.62)$$

where $\mathbf{t} = (t_1, t_2, \dots, t_{N^2-1})$ is a fixed set of $(N^2 - 1)$ $N \times N$ matrices. According to (1.61), t_a are Hermitian and traceless:

$$t_a^+ = t_a, \quad \text{Sp}(t_a) = 0. \quad (1.63)$$

Here the matrices t_a are generators of the fundamental representation of the $SU(N)$ group. They are normalised according to the condition

$$\text{Sp}(t_a t_b) = \frac{1}{2} \delta_{ab}. \quad (1.64)$$

Every traceless Hermitian $N \times N$ matrix can be presented as a linear superposition of t_a . The commutator of two t_a matrices is a traceless anti-Hermitian matrix; thus

$$[t_a, t_b] = i f_{abc} t_c. \quad (1.65)$$

The structure constants f_{abc} are real and completely antisymmetric. The matrices $\mathbf{t}$ satisfy the Fierz identities

$$\begin{aligned} I_{\alpha\beta} I_{\gamma\delta} &= \frac{1}{N} I_{\alpha\delta} I_{\gamma\beta} + 2 \mathbf{t}_{\alpha\delta} \mathbf{t}_{\gamma\beta}, \\ \mathbf{t}_{\alpha\beta} \mathbf{t}_{\gamma\delta} &= \left(\frac{1}{2} - \frac{1}{2} N^2\right) I_{\alpha\delta} I_{\gamma\beta} - \frac{1}{N} \mathbf{t}_{\alpha\delta} \mathbf{t}_{\gamma\beta}, \end{aligned} \quad (1.66)$$

where $I_{\alpha\beta}$ is a unit $N \times N$ matrix. Below, we present the generators t_a and the structure constants f_{abc} for the simplest groups explicitly.

SU(2)-group:

$$\mathbf{t} = \frac{1}{2} \boldsymbol{\sigma}, \quad (1.67)$$

where $\boldsymbol{\sigma}$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.68)$$

The structure constants form a completely antisymmetric unit tensor ε_{abc} :

$$f_{abc} = \varepsilon_{abc}, \quad \varepsilon_{123} = 1. \quad (1.69)$$

$SU(3)$ -group:

$$\mathbf{t} = \frac{1}{2} \boldsymbol{\lambda}, \quad (1.70)$$

where $\boldsymbol{\lambda}$'s are the Gell-Mann matrices

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \quad (1.71)$$

The independent non-zero coefficients f_{abc} are

$$\begin{aligned} f_{123} &= 1, & f_{458} &= f_{678} = \sqrt{3}/2, \\ f_{147} &= f_{516} = f_{246} = f_{257} = f_{345} = f_{637} = 1/2. \end{aligned} \quad (1.72)$$

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Chapter 2

Systematics of Mesons and Baryons

In this chapter we present the quark systematics of hadrons — mesons and baryons. The systematisation of mesons and baryons was the starting point for establishing the quark structure of hadrons.

We begin with the systematics of $q\bar{q}$ meson states in (n, M^2) and (J, M^2) planes, where n and J are the radial quantum number and total angular momentum of the bound $q\bar{q}$ state with mass M , respectively. Furthermore, we discuss the meson classification with respect to $SU(3)_{flavour}$ multiplets; owing to significant mixing between the singlet and the isoscalar octet states, we present the nonet rather than the singlet+octet classification of mesons in a broad mass interval up to $M \lesssim 2.5$ GeV.

Sections 2.4, 2.5 present available data on the systematics of baryons, which seem to give arguments in favour of the quark–diquark structure of baryons.

We consider here quark–antiquark states consisting of light quarks

$$q = u, d, s, \quad (2.1)$$

which are characterised by the following quantum numbers:

$$\begin{aligned} \text{total spin of quarks:} \quad S &= 0, 1; \\ \text{angular momentum:} \quad L &= 0, 1, 2, \dots; \\ \text{radial quantum numbers:} \quad n &= 1, 2, 3, \dots \end{aligned} \quad (2.2)$$

To characterise the $q\bar{q}$ states, we use spectroscopic notations

$$n^{2S+1}L_J, \quad (2.3)$$

where J is the total spin of the $q\bar{q}$ system, $J = |\mathbf{L} + \mathbf{S}|$.

We call states with $n = 1$ basic states: in potential models with standard potentials, *e.g.* of an oscillator type or a linearly increasing one, $V(r) \sim r^2$

or $V(r) \sim r$. The basic states are the lightest ones in their class, and the radial wave functions corresponding to these states have no zeros, while the wave functions of excited radial states contain $(n - 1)$ zeros.

The $L = 0$ states, or S -wave $q\bar{q}$ states, form two well-known nonets of pseudoscalar and vector mesons:

$$\begin{aligned} 1^1S_0 : & \quad \pi^+, \pi^0, \pi^-; \eta, \eta'; K^+, K^0, \bar{K}^0, K^-; \\ 1^3S_1 : & \quad \rho^+, \rho^0, \rho^-; \omega, \phi; K^{*+}, K^{*0}, \bar{K}^{*0}, K^{*-}. \end{aligned} \quad (2.4)$$

The isospin of pions and ρ mesons equals $I = 1$, their quark content is $(u\bar{d}, (u\bar{u} - d\bar{d})/\sqrt{2}, d\bar{u})$, while the isospin of $\eta, \eta', \omega, \phi$ is $I = 0$, and these mesons are mixtures of two components

$$n\bar{n} = \frac{u\bar{u} + d\bar{d}}{\sqrt{2}}, \quad s\bar{s}. \quad (2.5)$$

The isoscalar mesons can be characterised by another set of flavour wave functions, singlet and octet ones, in terms of the $SU(3)_{flavour}$ group:

$$\begin{aligned} \text{singlet} : & \quad \frac{u\bar{u} + d\bar{d} + s\bar{s}}{\sqrt{3}}, \\ \text{octet} : & \quad \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}}. \end{aligned} \quad (2.6)$$

The (η, η') and (ω, ϕ) pairs have different flavour contents: the η meson is close to an octet, the η' to a singlet, while the ω meson is close to $n\bar{n}$, and the ϕ meson is almost a clean $s\bar{s}$ state. Using (2.5), we can write

$$\begin{aligned} \eta &= n\bar{n} \cos \theta - s\bar{s} \sin \theta, \\ \eta' &= n\bar{n} \sin \theta + s\bar{s} \cos \theta, \end{aligned} \quad (2.7)$$

where $\cos \theta \simeq 0.8$ and $\sin \theta \simeq 0.6$. For vector particles,

$$\begin{aligned} \omega &= n\bar{n} \cos \varphi_V + s\bar{s} \sin \varphi_V, \\ \phi &= -n\bar{n} \sin \varphi_V + s\bar{s} \cos \varphi_V, \end{aligned} \quad (2.8)$$

and the mixing angle φ_V is small, $|\varphi_V| \lesssim 5^\circ$.

Literally, the classification scheme of mesons as pure $q\bar{q}$ states cannot be correct; this is clear already from the example of the pseudoscalar mesons η and η' . We know that these mesons contain admixtures of two-gluon components, this is confirmed by the sufficiently large partial decay widths $J/\psi \rightarrow \gamma\eta, \gamma\eta'$. These decays are owing to the transitions $c\bar{c} \rightarrow gg \rightarrow \eta$ and $c\bar{c} \rightarrow gg \rightarrow \eta'$, where gg is a two-gluon component. The partial widths of the decays $J/\psi \rightarrow \gamma\eta, \gamma\eta'$ allow us to estimate the probability of the

presence of gg in η and η' : according to [1], $(gg)_\eta \lesssim 3\%$ and $(gg)_{\eta'} \lesssim 15\%$. Considering the $q\bar{q}$ classification of meson states, we must always have in mind the possibility of admixtures, especially gluonic ones. The fact that resonances have hadron decay channels indicates that the $q\bar{q}$ states contain also certain admixtures of hadron components or multiquark components of the type of $qq\bar{q}\bar{q}$.

The G -parity of the π , ω and ϕ mesons is negative, that of η, η' and ρ is positive; the C -parity of π^0, η, η' is positive, that of ρ^0, ω, ϕ is negative (let us remind that $G = (-)^{S+L+I}$ and $C = (-)^{S+L}$). The K and K^* mesons contain strange quarks: kaons are just $K^+ = u\bar{s}$, $K^0 = d\bar{s}$ (with strangeness +1), antikaons are $\bar{K}^0 = s\bar{d}$, $K^- = s\bar{u}$ (with strangeness -1); the isospin of the kaons is $I = 1/2$.

Mesons with $L = 1$ form four nonets, 1^1P_J and 1^3P_J :

$$\begin{array}{lll}
 J^{PC} : & I = 1 & I = 0 & I = 1/2 \\
 1^{+-} : & b_1(1229) & h_1(1170), h_1(1440) & K_1(1270) , \\
 0^{++} : & a_0(985) & f_0(980), f_0(1300) & K_0(1425) , \\
 1^{++} : & a_1(1230) & f_1(1282), f_1(1426) & K_1(1400) , \\
 2^{++} : & a_2(1320) & f_2(1285), f_2(1525) & K_2(1430) .
 \end{array}$$

The best established nonet is the multiplet of tensor mesons. The existence of the $J = 2$ mesons gave rise to the introduction of the nonet classification of highly excited $q\bar{q}$ states [2, 3]. More uncertain is the status of the nonet of scalar mesons. Indeed, the lightest scalar glueball was found in the region of 1200–1600 MeV. The mixing of the f_0 mesons with the glueball leads to some confusion in the classification of scalars. Moreover, near the $\pi\pi$ threshold another mysterious state, the σ meson, seems to exist. Below, we consider the problem of scalar mesons in detail.

2.1 Classification of Mesons in the (n, M^2) Plane

As was already mentioned, in the last decade a considerable progress was achieved in determining highly excited meson states in the mass region 1950–2400 MeV [4, 5]. These results allowed us to systematise $q\bar{q}$ meson states on the planes (n, M^2) and (J, M^2) , where n is the radial quantum number of a $q\bar{q}$ system with mass M , and J is its spin [6].

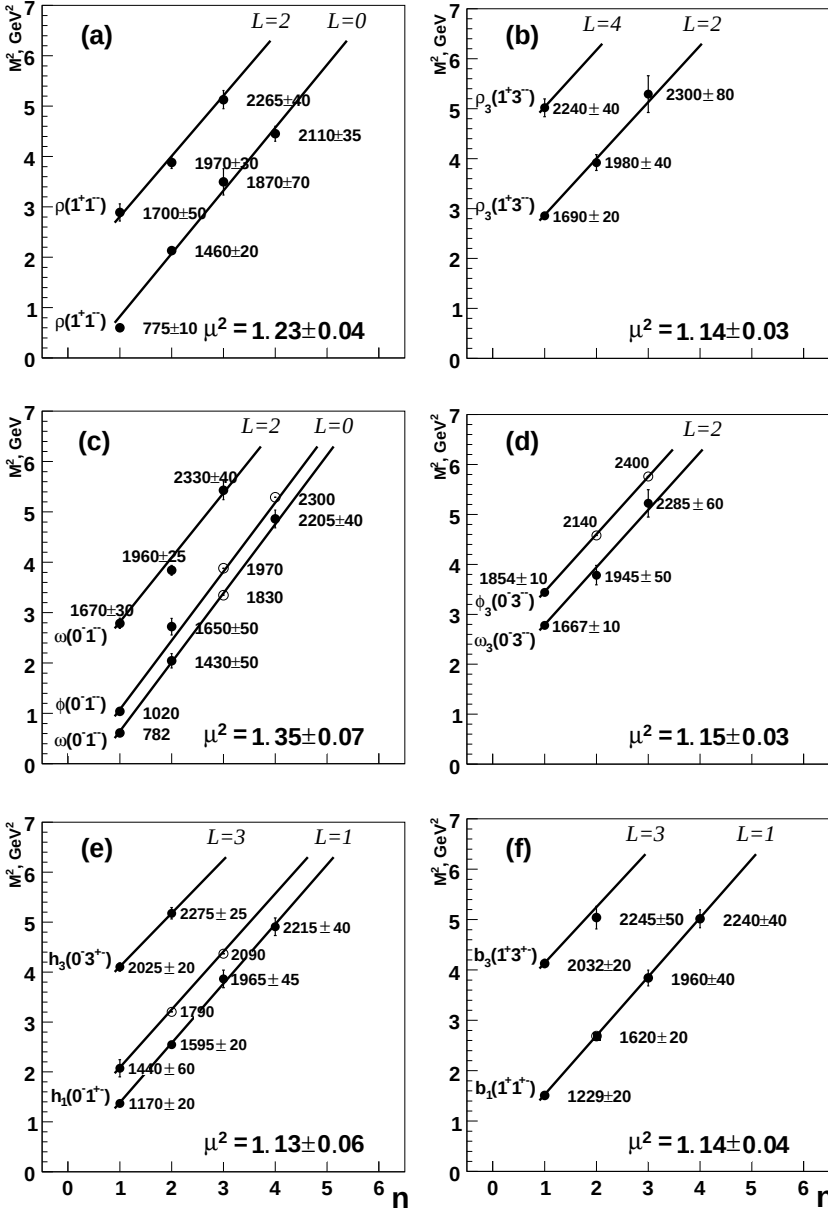


Fig. 2.1 Trajectories for $(C = -)$ meson states on the (n, M^2) plane. Open circles stand for the predicted states.

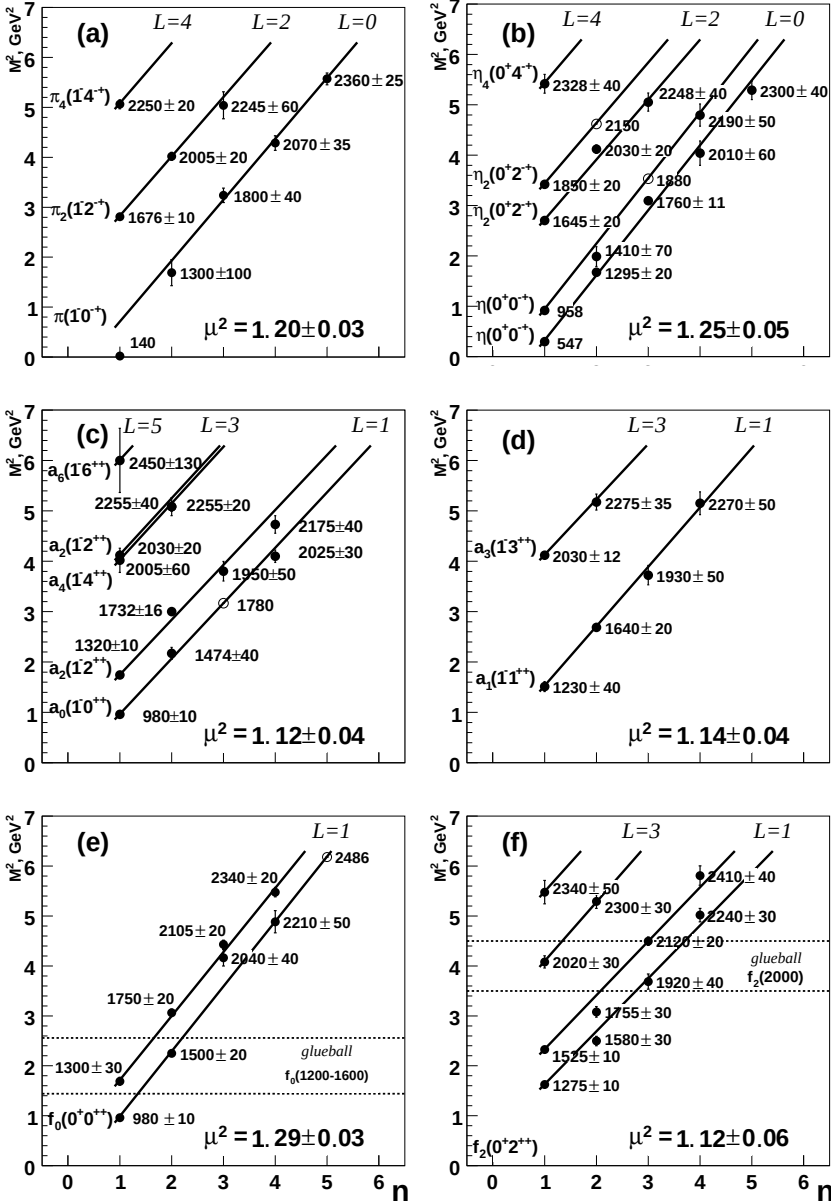


Fig. 2.2 Trajectories for ($C = +$) meson states on the (n, M^2) plane. Open circles stand for the predicted states. The bands restricted by dotted lines show mass regions of scalar and tensor glueballs.

Figures 2.1 and 2.2 show trajectories in the (n, M^2) planes for (I, J^{PC}) states with negative and positive charge parities as follows:

$$\begin{aligned}
 C = - : & \quad b_1(11^{+-}), b_3(13^{+-}), h_1(01^{+-}), \rho(11^{--}), \rho_3(13^{--}), \\
 & \quad \omega/\phi(01^{--}), \omega_3(03^{--}) ; \\
 C = + : & \quad \pi(10^{-+}), \pi_2(12^{-+}), \pi_4(14^{-+}), \eta(00^{-+}), \eta_2(02^{-+}), \\
 & \quad a_0(10^{++}), a_1(11^{++}), a_2(12^{++}), a_3(13^{++}), a_4(14^{++}), \\
 & \quad f_0(00^{++}), f_2(02^{++}) .
 \end{aligned} \tag{2.10}$$

In terms of the $q\bar{q}$ states, the mesons of the $n^{2S+1}L_J$ nonets at $M \lesssim 2400$ MeV fill in the following (n, M^2) trajectories:

$$\begin{aligned}
 {}^1S_0 & \rightarrow \pi(10^{-+}), \eta(00^{-+}) ; \\
 {}^3S_1 & \rightarrow \rho(11^{--}), \omega(01^{--})/\phi(01^{--}) ; \\
 {}^1P_1 & \rightarrow b_1(11^{+-}), h_1(01^{+-}) ; \\
 {}^3P_J & \rightarrow a_J(1J^{++}), f_J(0J^{++}), J = 0, 1, 2 ; \\
 {}^1D_2 & \rightarrow \pi_2(12^{-+}), \eta_2(02^{-+}) ; \\
 {}^3D_J & \rightarrow \rho_J(1J^{--}), \omega_J(0J^{--})/\phi_J(0J^{--}), J = 1, 2, 3 ; \\
 {}^1F_3 & \rightarrow b_3(13^{+-}), h_3(03^{+-}) ; \\
 {}^3F_J & \rightarrow a_J(1J^{++}), f_J(0J^{++}), J = 2, 3, 4 .
 \end{aligned} \tag{2.11}$$

States with $J = L \pm 1$ have, naturally, two components: at fixed J there are states with $L - 1$ and $L + 1$, so one may assert the doubling of trajectory at fixed J , for example, for $(I, 1^{--})$ and $(I, 2^{++})$. Isoscalar states have two flavour components each, $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$ and $s\bar{s}$, this again doubles the number of trajectories like $\eta(00^{-+}), f_0(00^{++})$.

Trajectories with negative charge parities, $C = -$ (Fig. 2.1), are determined virtually unambiguously (the black dots correspond to observed states [4, 7, 8], the open circles to states predicted by the trajectories). We show the observed masses of meson resonances together with errors, which are as a rule larger than those quoted by [8]. The reason is that such characteristics of resonances as mass and full width must be determined by the position of amplitude pole in the complex- M plane, while in [8] masses and full widths are often defined by averaging certain selected values found by fitting to the observed spectra. In a majority of cases these procedures lead to different results.

The trajectories are linear with a good accuracy:

$$M^2 \simeq M_0^2 + (n - 1)\mu^2 , \tag{2.12}$$

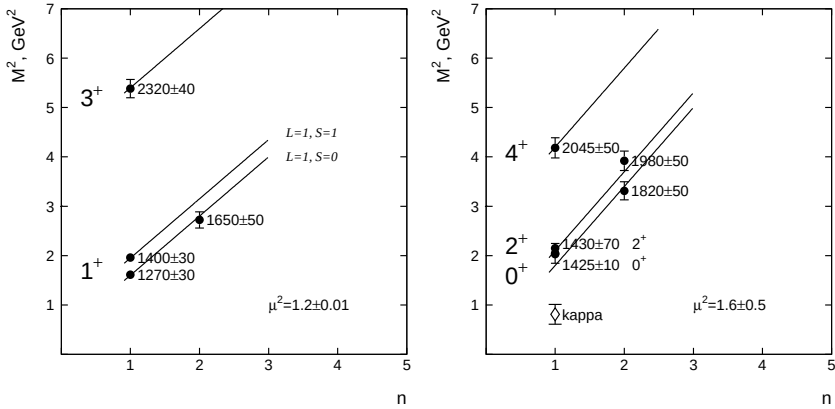


Fig. 2.3 Kaon trajectories on the (n, M^2) plane with $P = +$.

where M_0 is the mass of the basic meson $n = 1$, while the parameter of the slope is roughly equal to $\mu^2 \simeq 1.25 \pm 0.15 \text{ GeV}^2$. In the sector with $C = +$, the states π_J are definitely placed on linear trajectories with the slope $\mu^2 \simeq 1.2 \pm 0.1 \text{ GeV}^2$; the only exception is $\pi(140)$. This is not surprising, since the pion is a specific particle. The sector of a_J states with $J = 0, 1, 2, 3, 4$ demonstrates clearly a set of linear trajectories with $\mu^2 \simeq 1.10 - 1.16 \text{ GeV}^2$; the same slope is observed for the f_2 trajectories. For f_0 mesons, the slope of the trajectory is $\mu^2 \simeq 1.3 \text{ GeV}^2$.

Let us stress that two states do not appear on the linear $q\bar{q}$ trajectories: the light sigma meson, $f_0(300-500)$ [8], and the broad state $f_0(1200-1600)$, which was fixed in the K -matrix analysis [7, 9, 10, 11].

2.1.1 Kaon states

Figures 2.3 demonstrate kaon trajectories in the (n, M^2) plane with $P = +$.

It should be noted that experimental information on kaons is poor. This concerns, in particular, the $(P = -)$ kaons. Because of this, we show the (n, M^2) planes for $(P = +)$ kaons only. The present status of kaon trajectories in (n, M^2) planes is nothing but a guide for future specification and corrections.

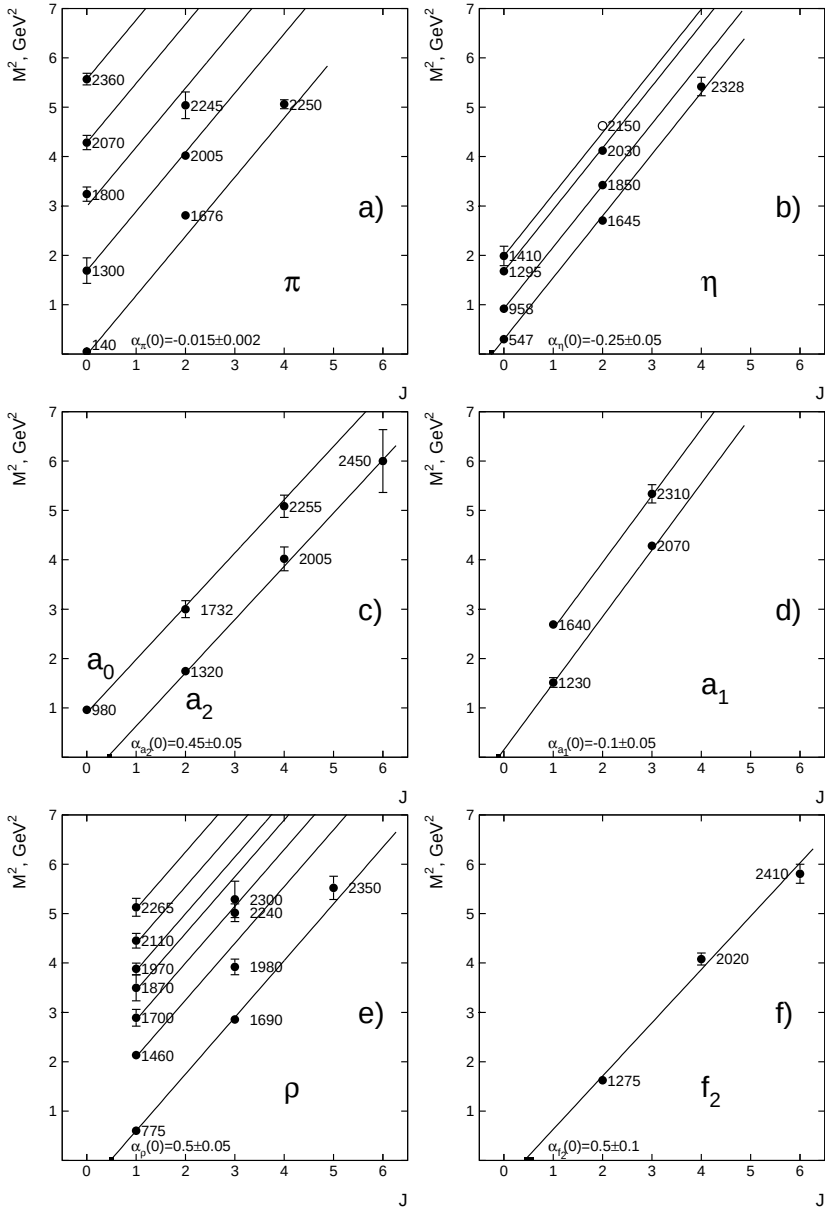


Fig. 2.4 Trajectories in the (J, M^2) plane: a) leading and daughter π -trajectories, b) leading and daughter η -trajectories, c) a_2 -trajectories, d) leading and daughter a_1 -trajectories, e) ρ -trajectories, f) P' -trajectories.

To get a complete information on the kaon sector, one needs experimental data on $\pi K \rightarrow \pi K$, ηK , $\eta' K$ over the range 800 – 2000 MeV accompanied by the combined K -matrix analysis.

2.2 Trajectories on (J, M^2) Plane

The π , η , a_2 , a_3 , ρ and P' (or f_2) trajectories on (J, M^2) planes are shown in Fig. 2.4.

Leading π and η trajectories are unambiguously determined together with their daughter trajectories, while for a_2 , a_1 , ρ and P' only the leading trajectories can be given in a definite way.

In the construction of (J, M^2) -trajectories it is essential that the leading meson trajectories (π , ρ , a_1 , a_2 and P') are well known from the analysis of the diffraction scattering of hadrons at $p_{lab} \sim 5 - 50$ GeV/c (for example, see [13] and references therein).

The pion and η trajectories are linear with a good accuracy (see Fig. 2.4). Other leading trajectories (ρ , a_1 , a_2 , P') can also be considered as linear:

$$\alpha_X(M^2) \simeq \alpha_X(0) + \alpha'_X(0)M^2. \quad (2.13)$$

The parameters of the linear trajectories, determined by the masses of the $q\bar{q}$ states, are

$$\begin{aligned} \alpha_\pi(0) &\simeq -0.015, \quad \alpha'_\pi(0) \simeq 0.83 \text{ GeV}^{-2}; \\ \alpha_\rho(0) &\simeq 0.50, \quad \alpha'_\rho(0) \simeq 0.87 \text{ GeV}^{-2}; \\ \alpha_\eta(0) &\simeq -0.25, \quad \alpha'_\eta(0) \simeq 0.80 \text{ GeV}^{-2}; \\ \alpha_{a_1}(0) &\simeq -0.10, \quad \alpha'_{a_1}(0) \simeq 0.72 \text{ GeV}^{-2}; \\ \alpha_{a_2}(0) &\simeq 0.45, \quad \alpha'_{a_2}(0) \simeq 0.93 \text{ GeV}^{-2}; \\ \alpha_{P'}(0) &\simeq 0.50, \quad \alpha'_{P'}(0) \simeq 0.93 \text{ GeV}^{-2}. \end{aligned} \quad (2.14)$$

The slopes $\alpha'_X(0)$ of the trajectories are approximately equal. The inverse slope, $1/\alpha'_X(0) \simeq 1.25 \pm 0.15 \text{ GeV}^2$, roughly equals the parameter μ^2 for trajectories on the (n, M^2) planes:

$$\frac{1}{\alpha'_X(0)} \simeq \mu^2. \quad (2.15)$$

In the subsequent chapters, considering the scattering processes, we use for the Regge trajectories the momentum transfer squared $M^2 \rightarrow t$.

2.2.1 Kaon trajectories on (J, M^2) plane

As was said above, experimental data in the kaon sector are scarce, so in Fig. 2.5 we show only the leading K -meson trajectory (the states with $J^P = 0^-, 2^-$), the K^* trajectory ($J^P = 1^-, 3^-, 5^-$) and the leading and daughter trajectories for $J^P = 0^+, 2^+, 4^+$.

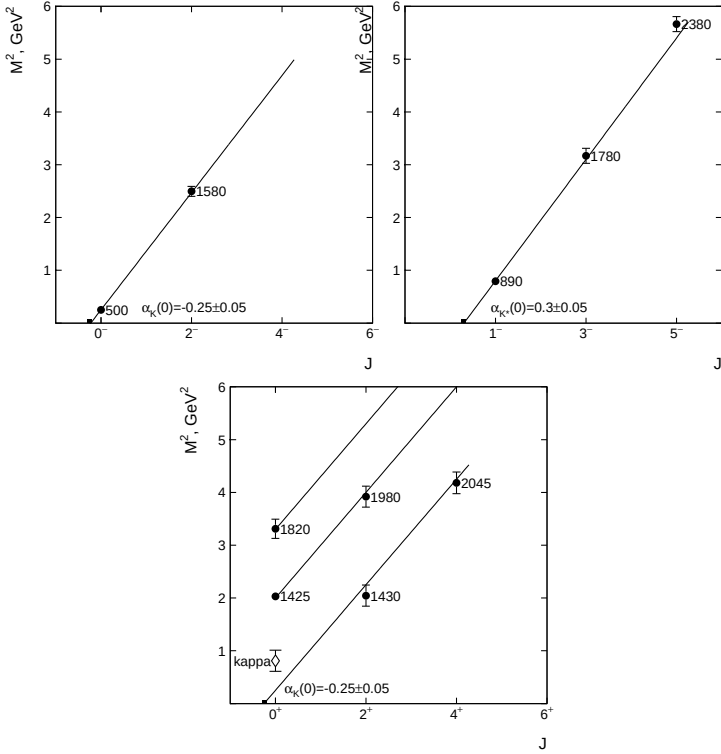


Fig. 2.5 Kaon trajectories on the (J^P, M^2) plane.

The parameters of the leading kaon trajectories are as follows:

$$\begin{aligned}
 \alpha_K(0) &\simeq -0.25, \quad \alpha'_K(0) \simeq 0.90 \text{ GeV}^{-2}; \\
 \alpha_{K^*}(0) &\simeq 0.30, \quad \alpha'_{K^*}(0) \simeq 0.85 \text{ GeV}^{-2}; \\
 \alpha_{K_{2^+}}(0) &\simeq -0.2, \quad \alpha'_{K_{2^+}}(0) \simeq 1.0 \text{ GeV}^{-2}.
 \end{aligned} \tag{2.16}$$

The trajectories with $J^P = 1^+, 3^+, 5^+$ cannot be defined unambiguously.

Table 2.1 Nonet classification ($^{2S+1}L_J$) of $q\bar{q}$ states ($n = 1$ and 2).

$q\bar{q}$ -mesons	n=1				n=2			
	I=1	I=0	I=0	I= $\frac{1}{2}$	I=1	I=0	I=0	I= $\frac{1}{2}$
$^1S_0(0^{--})$ $^3S_1(1^{--})$	$\pi(140)$ $\rho(775)$	$\eta(547)$ $\omega(782)$	$\eta'(958)$ $\phi(1020)$	$K(500)$ $K^*(890)$	$\pi(1300)$ $\rho(1460)$	$\eta(1295)$ $\omega(1430)$	$\eta(1410)$ $\phi(1650)$	$K(1460)$
$^1P_1(1^{+-})$ $^3P_0(0^{++})$ $^3P_1(1^{++})$ $^3P_2(2^{++})$	$b_1(1229)$ $a_0(980)$ $a_1(1230)$ $a_2(1320)$	$h_1(1170)$ $f_0(980)$ $f_1(1282)$ $f_2(1275)$	$h_1(1440)$ $f_0(1300)$ $f_1(1426)$ $f_2(1525)$	$K_1(1270)$ $K_0(1425)$ $K_1(1400)$ $K_2(1430)$	$b_1(1620)$ $a_0(1474)$ $a_1(1640)$ $a_2(1732)$	$h_1(1595)$ $f_0(1500)$ $f_1(1518)$ $f_2(1580)$	$h_1(1790)$ $f_0(1750)$ $f_1(1780)$ $f_2(1755)$	$K_1(1650)$ $K_0(1820)$ $K_2(1980)$
$^1D_2(2^{--})$ $^3D_1(1^{--})$ $^3D_2(2^{--})$ $^3D_3(3^{--})$	$\pi_2(1676)$ $\rho(1700)$ $\rho_2(1940)$ $\rho_3(1690)$	$\eta_2(1645)$ $\omega(1670)$ $\omega_2(1975)$ $\omega_3(1667)$	$\eta_2(1850)$ $\phi_3(1854)$	$K_2(1800)$ $K_1(1680)$ $K_2(1580)$ $K_3(1780)$	$\pi_2(2005)$ $\rho(1970)$ $\rho_2(2240)$ $\rho_3(1980)$	$\eta_2(2030)$ $\omega(1960)$ $\omega_2(2195)$ $\omega_3(1945)$	$\eta_2(2150)$ $\phi_3(2140)$	 $K_2(1773)$
$^1F_3(3^{+-})$ $^3F_2(2^{++})$ $^3F_3(3^{++})$ $^3F_4(4^{++})$	$b_3(2032)$ $a_2(2030)$ $a_3(2030)$ $a_4(2005)$	$h_3(2025)$ $f_2(2020)$ $f_3(2050)$ $f_4(2025)$	 $f_2(2340)$ $f_4(2100)$	 $K_3(2320)$ $K_4(2045)$	$b_3(2245)$ $a_2(2255)$ $a_3(2275)$ $a_4(2255)$	$h_3(2275)$ $f_2(2300)$ $f_3(2303)$ $f_4(2150)$	 $f_2(2570)$ $f_4(2300)$	
$^1G_4(4^{+-})$ $^3G_3(3^{--})$ $^3G_4(4^{--})$ $^3G_5(5^{--})$	$\pi_4(2250)$ $\rho_3(2240)$ $\rho_5(2300)$	$\eta_4(2328)$ 	 	$K_4(2500)$ $K_5(2380)$	$\rho_3(2510)$ $\rho_5(2570)$			
$^3H_6(6^{++})$	$a_6(2450)$	$f_6(2420)$						

Table 2.2 Nonet classification ($^{2S+1}L_J$) of $q\bar{q}$ states (n = 3, 4, and 5).

$q\bar{q}$ -mesons	n=3				n=4				n=5		
	I=1	I=0	I=0	I= $\frac{1}{2}$	I=1	I=0	I=0	I= $\frac{1}{2}$	I=1	I=0	I=0
$^1S_0(0^{-+})$ $^3S_1(1^{--})$	$\pi(1800)$ $\rho(1870)$	$\eta(1760)$ $\omega(\mathbf{1830})$	$\boldsymbol{\eta}(\mathbf{1880})$ $\boldsymbol{\phi}(\mathbf{1970})$	$K(1830)$	$\pi(2070)$ $\rho(2110)$	$\eta(2010)$ $\omega(2205)$	$\eta(2190)$ $\boldsymbol{\phi}(\mathbf{2300})$		$\pi(2360)$ $\boldsymbol{\rho}(\mathbf{2430})$	$\eta(2300)$	$\boldsymbol{\eta}(\mathbf{2480})$
$^1P_1(1^{+-})$ $^3P_0(0^{++})$ $^3P_1(1^{++})$ $^3P_2(2^{++})$	$b_1(1960)$ $\mathbf{a}_0(\mathbf{1780})$ $a_1(1930)$ $a_2(1950)$	$h_1(1965)$ $f_0(2040)$ $f_1(1970)$ $f_2(1920)$	$\mathbf{h}_1(\mathbf{2090})$ $f_0(2105)$ $\mathbf{f}_1(\mathbf{2060})$ $f_2(2120)$		$b_1(2240)$ $a_0(2025)$ $a_1(2270)$ $a_2(2175)$	$h_1(2215)$ $f_0(2210)$ $\mathbf{f}_1(\mathbf{2214})$ $f_2(2240)$	$f_0(2340)$ $f_1(2310)$ $f_2(2410)$		$a_1(2340)$	$\mathbf{f}_0(\mathbf{2486})$	
$^1D_2(2^{-+})$ $^3D_1(1^{--})$ $^3D_2(2^{--})$ $^3D_3(3^{--})$	$\pi_2(2245)$ $\rho(2265)$ $\rho_3(2300)$	$\eta_2(2248)$ $\omega(2330)$ $\omega_3(2285)$	$\boldsymbol{\eta}_2(\mathbf{2380})$ $K_2(2250)$ $\boldsymbol{\phi}_3(\mathbf{2400})$		$\boldsymbol{\eta}_2(\mathbf{2520})$						

2.3 Assignment of Mesons to Nonets

In Tables 2.1 and 2.2 we collected all considered meson $q\bar{q}$ states in nonets according to their $SU(3)_{flavour}$ attribution. Strictly speaking, $SU(3)_{flavour}$ has singlet and octet rather than nonet representations. However, the singlet and octet states, with the same values of the total angular momentum, mix with one another. In the lightest nonets we can determine mixing angles more or less reliably, but for the higher excitations the estimates of the mixing angles are very ambiguous. In addition, isoscalar states can contain significant glueball components. For these reasons, we give only the nonet ($9 = 1 \oplus 8$) classification of mesons. States that are predicted but not yet reliably established are shown in boldface.

2.4 Baryon Classification on (n, M^2) and (J, M^2) Planes

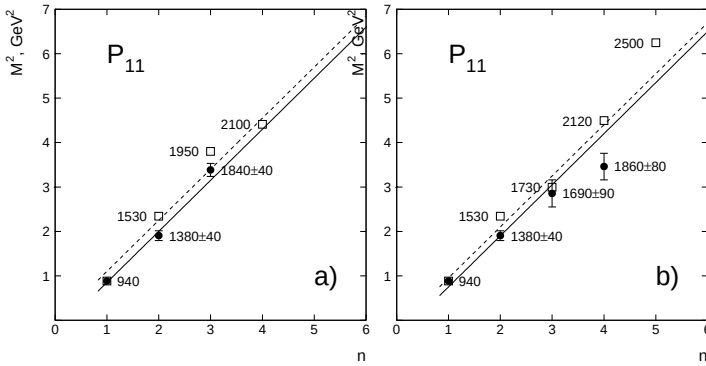


Fig. 2.6 Baryon trajectories for $1/2^+$ states on the (n, M^2) -plane according to the analysis [12]: a) K -matrix with four poles (open squares mean K -matrix poles, full squares stand for amplitude poles – physical resonances); b) results of the fit to K -matrix with five poles. In both solutions the upper pole goes beyond the fitting region, thus becoming unphysical. It is not shown in the figures.

Figures 2.6 and 2.7 show (n, M^2) trajectories for states being radial excitations of the octet ^{28} ($N(940)$, $\Lambda(1116)$, $\Sigma(1193)$, $\Xi(1320)$). We place here all the $1/2^+$ states which are known up to now. As it turns out, they all lie on one trajectory with approximately the same slope as in the meson case:

$$M^2 = M_0^2 + (n - 1)\mu^2 \quad (2.17)$$

with $\mu^2 \simeq 1.1 \text{ GeV}^2$ and $n = 1, 2, 3, \dots$; $n = 1$ corresponds to the basic states, *i.e.* M_0 is the mass of the lightest baryons, $N(940)$, $\Lambda(1116)$, $\Sigma(1193)$ or $\Xi(1320)$.

Recent data do not exhibit any increase in the number of states in the region of large masses. Such an increase would be natural for genuine three-particle states, and its absence corresponds rather to the picture of a quark–diquark system.

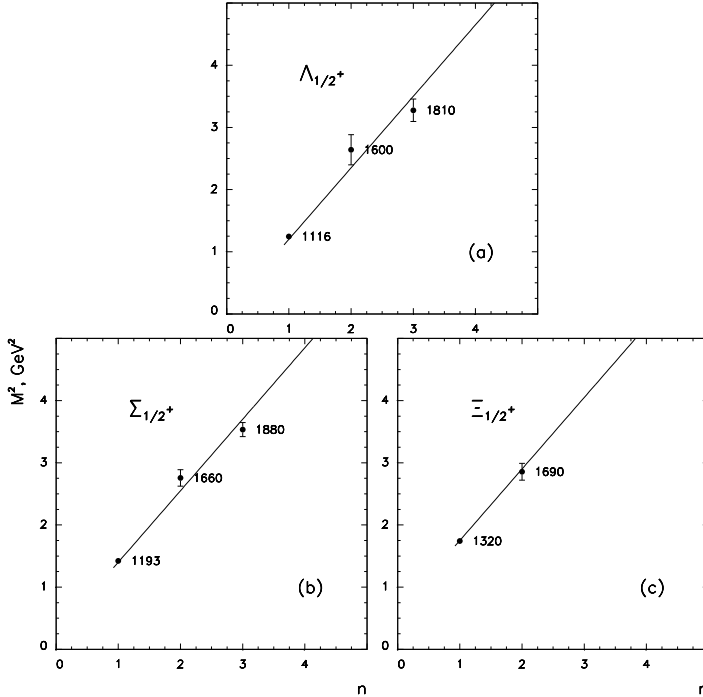


Fig. 2.7 Baryon trajectories for $1/2^+$ states on the (n, M^2) -plane.

Figure 2.8 presents (n, M^2) trajectories for the states $\Delta_{3/2^+}$ and $\Sigma_{3/2^+}$ belonging to the decuplet ^{410} . The lowest states, $\Delta(1232)$ and $\Sigma(1385)$, belong to the lowest 56-plet, like the lowest states in Fig. 2.7. Again, we have trajectories with $\mu^2 = 1.1 \text{ GeV}^2$, and again, the number of states does not grow for large masses. Hence, the picture reminds the quark–diquark structure.

In Fig. 2.9a,b we show leading and daughter nucleon and Δ -isobar trajectories on the (J, M^2) plane (for positive parity). The slopes of the tra-

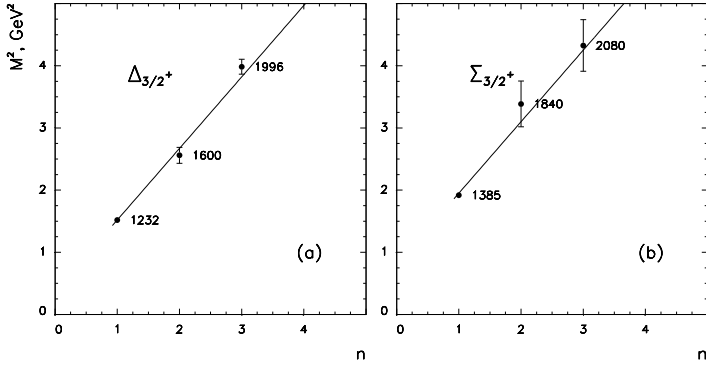


Fig. 2.8 Baryon trajectories for $3/2^+$ states on the (n, M^2) -plane.

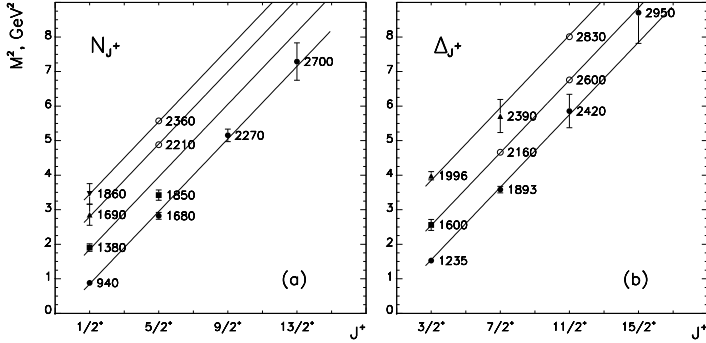


Fig. 2.9 (J, M^2) -planes: leading and daughter nucleon (a) and Δ (b) trajectories for positive parity states.

jectories coincide with each other, and they are roughly the same as the slopes in the meson sector.

Figure 2.10 displays the (J, M^2) plane for negative parity baryons N_{J^-} and Δ_{J^-} : again, the trajectories have a universal slope $1/\alpha'_R(0) \simeq 1.05 \text{ GeV}^2$.

2.5 Assignment of Baryons to Multiplets

We can now assign the baryons to the multiplets. Consider first the baryons of the 56-plets, which are expanded with respect to the $\text{SU}(6)$ multiplets as

$$56 = {}^4 10 + {}^2 8. \quad (2.18)$$

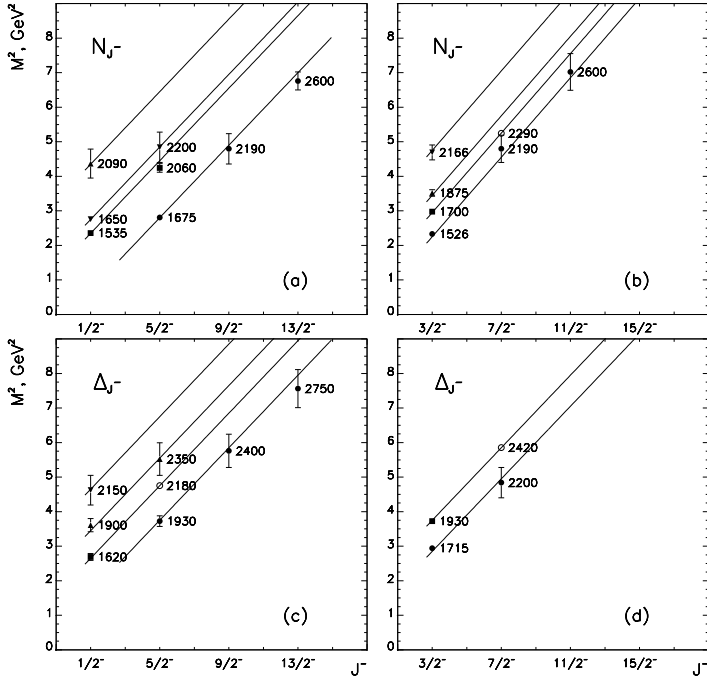


Fig. 2.10 (J, M^2) -plane: baryon trajectories for negative parity states.

The basic octet and its radial excitations form

J^P	(D, L, n)	octet members			
$1/2^+$	$(56, 0, 1)$	$N_{1/2^+}(940)$	$\Lambda_{1/2^+}(1116)$	$\Sigma_{1/2^+}(1193)$	$\Xi_{1/2^+}(1320)$
$1/2^+$	$(56, 0, 2)$	$N_{1/2^+}(1440)$	$\Lambda_{1/2^+}(1600)$	$\Sigma_{1/2^+}(1660)$	$\Xi_{1/2^+}(1690)$
$1/2^+$	$(56, 0, 3)$	$N_{1/2^+}(1840)$	$\Lambda_{1/2^+}(1812)$	$\Sigma_{1/2^+}(1880)$	$\Xi_{1/2^+}(?)$
$1/2^+$	$(56, 0, 4)$	$N_{1/2^+}(2100)$	$\Lambda_{1/2^+}(?)$	$\Sigma_{1/2^+}(?)$	$\Xi_{1/2^+}(?)$

The states marked by question marks were not seen yet, but may be predicted from Fig. 2.7.

Similarly, for decuplets we have the following set:

J^P	(D, L, n)	decuplet members			
$3/2^+$	$(56, 0, 1)$	$\Delta_{3/2^+}(1232)$	$\Sigma_{3/2^+}(1385)$	$\Xi_{3/2^+}(1530)$	$\Omega_{3/2^+}(1672)$
$3/2^+$	$(56, 0, 2)$	$\Delta_{3/2^+}(1600)$	$\Sigma_{3/2^+}(1840)$	$\Xi_{3/2^+}(?)$	$\Omega_{3/2^+}(?)$
$3/2^+$	$(56, 0, 3)$	$\Delta_{3/2^+}(1996)$	$\Sigma_{3/2^+}(2080)$	$\Xi_{3/2^+}(?)$	$\Omega_{3/2^+}(?)$

The lowest 70-plet with $L = 1$ can also be constructed more or less unam-

biguously. Its expansion in terms of SU(3) multiplets is

$$70 = {}^210 + {}^48 + {}^28 + {}^21. \quad (2.21)$$

Since we have here $L = 1$, the resulting set of states (D, J^P) is

$$\begin{aligned} & (10, 1/2^-), (10, 3/2^-); \\ & (8, 1/2^-), (8, 3/2^-), (8, 5/2^-); \\ & (8, 1/2^-), (8, 3/2^-); \\ & (1, 1/2^-), (1, 3/2^-). \end{aligned} \quad (2.22)$$

The Δ_{J^-} -states belonging to the lightest 70-plet are determined unambiguously: these are the lightest Δ_{J^-} -states in Fig. 2.10c, $\Delta_{1/2^-}$ (1620) and $\Delta_{3/2^-}$ (1715). We have for them

J^P	(D, L, n)	decuplet members
$1/2^-$	(70, 1, 1)	$\Delta_{1/2^-}$ (1620) $\Sigma_{1/2^-}$ (1770?) $\Xi_{1/2^-}$ (1920?) $\Omega_{1/2^-}$ (2070?)
$3/2^-$	(70, 1, 1)	$\Delta_{3/2^-}$ (1715) $\Sigma_{3/2^-}$ (1850?) $\Xi_{3/2^-}$ (2000?) $\Omega_{3/2^-}$ (2150?)

(2.23)

For the basic $3/2^+$ decuplet the splitting $(\Delta, \Sigma, \Xi, \Omega)$ can be well described by $\Delta M \simeq 150$ MeV. We use the same value of splitting for the members of the decuplets $1/2^-$, $3/2^-$, writing the masses of baryons Σ_{J^-} , Ξ_{J^-} , Δ_{J^-} in (2.23). Let us remind, however, that these strange baryons were not observed yet, that's why we put there question marks.

Figure 2.10c shows how to recover the $1/2^-$, $3/2^-$ decuplets being radial excitations of the multiplets (2.23): the sets of states with $n = 1, 2, 3$ are just

$$\Delta_{1/2^-} (1620), \quad \Delta_{1/2^-} (1900), \quad \Delta_{1/2^-} (2150) \quad (2.24)$$

and

$$\Delta_{3/2^-} (1715), \quad \Delta_{3/2^-} (1930). \quad (2.25)$$

Consider now the octets of the 70-plet. There are five low-lying states, $N_{1/2^-}$ (1535), $N_{1/2^-}$ (1650), $N_{3/2^-}$ (1526), $N_{3/2^-}$ (1725), $N_{5/2^-}$ (1670) shown in Figs. 2.10a,b, which are just the necessary N_{J^-} states for the octets of the 70-plet. Having them, it is easy to reconstruct the octets:

J^P	(D, L, n)	octet members
$1/2^-$	(8, 1, 1)	$N_{1/2^-}$ (1535) $\Lambda_{1/2^-}$ (1670) $\Sigma_{1/2^-}$ (1620) $\Xi_{1/2^-}$ (?)
$1/2^-$	(8, 1, 1)	$N_{1/2^-}$ (1650) $\Lambda_{1/2^-}$ (1800) $\Sigma_{1/2^-}$ (1750) $\Xi_{1/2^-}$ (?)
$3/2^-$	(8, 1, 1)	$N_{3/2^-}$ (1526) $\Lambda_{3/2^-}$ (1690) $\Sigma_{3/2^-}$ (1670) $\Xi_{3/2^-}$ (1820)
$3/2^-$	(8, 1, 1)	$N_{3/2^-}$ (1725) $\Lambda_{3/2^-}$ (?) $\Sigma_{3/2^-}$ (?) $\Xi_{3/2^-}$ (?)
$5/2^-$	(8, 1, 1)	$N_{5/2^-}$ (1670) $\Lambda_{5/2^-}$ (1830) $\Sigma_{5/2^-}$ (1775) $\Xi_{5/2^-}$ (?)

Experimental data seem to indicate that the **LS**-splitting is small (here S is the total quark spin). In this case the three-quark states are characterised by the values of the total spin, $S = 1/2, 3/2$. It is reasonable to assume that $S = 1/2$ corresponds to the lighter baryons, $N_{1/2^-}$ (1535) and $N_{3/2^-}$ (1526), while $S = 3/2$ characterises $N_{1/2^-}$ (1650), $N_{3/2^-}$ (1725) and $N_{5/2^-}$ (1670).

The two singlet states are

J^P	(D, L, n)	singlet members
$1/2^-$	$(1, 1, 1)$	$\Lambda_{1/2^-}$ (1405)
$3/2^-$	$(1, 1, 1)$	$\Lambda_{3/2^-}$ (1520).

We see that except for a few states marked by question marks in (2.26), the two lowest multiplets, the 56-plet and the 70-plet, are virtually reconstructed. Reliable states corresponding to the 20-plet

$$20 = {}^2 8 + {}^4 1 \quad (2.27)$$

are not known.

There remains an open question which is crucial for the understanding of forces acting in three-quark systems: the problem of radially and orbitally excited states. This requires the experimental knowledge of higher resonances.

2.6 Sectors of the 2^{++} and 0^{++} Mesons — Observation of Glueballs

The sectors of scalar and tensor mesons need a special discussion: here we face the low-lying glueballs. We start the discussion with tensor mesons because the situation in this sector is more transparent and it allows us to make a definite conclusion about the gluonium state $f_2(2000)$.

The situation with the scalars is more complicated: there is a strong candidate for the descendant of gluonium, the broad state $f_0(1200 - 1600)$, but the corresponding pole of the amplitude dives deeply into the complex- s plane, and the $f_0(1200 - 1600)$ is seen only by carrying out an elaborate analysis of the spectra. Besides, there are indications to an additional enigmatic state, the σ -meson, with mass ~ 450 MeV.

2.6.1 Tensor mesons

Data of the Crystal Barrel and L3 collaborations clarified essentially the situation in the 2^{++} sector in the mass region up to 2400 MeV, demonstrating the linearity of the (n, M^2) trajectories. The data show that there

exists a superfluous state for the (n, M^2) -trajectories: a broad resonance $f_2(2000)$.

The reactions $p\bar{p} \rightarrow \pi\pi, \eta\eta, \eta\eta'$ play an important role in the mass region 1990–2400 MeV in which, together with $f_2(2000)$, four relatively narrow resonances are seen: $f_2(1920)$, $f_2(2020)$, $f_2(2240)$, $f_2(2300)$. The analysis of the branching ratios of all these resonances shows that only the decay of the broad state $f_2(2000) \rightarrow \pi^0\pi^0, \eta\eta, \eta\eta'$ is nearly flavour blind that is a signature of the glueball decay.

A broad isoscalar–tensor resonance in the region of 2000 MeV was seen in various reactions [8]. Recent measurements give:

$M = 2010 \pm 25$ MeV, $\Gamma = 495 \pm 35$ MeV in $p\bar{p} \rightarrow \pi^0\pi^0, \eta\eta, \eta\eta'$ [14],

$M = 1980 \pm 20$ MeV, $\Gamma = 520 \pm 50$ MeV in $pp \rightarrow pp\pi\pi\pi\pi$ [15],

$M = 2050 \pm 30$ MeV, $\Gamma = 570 \pm 70$ MeV in $\pi^-p \rightarrow \phi\phi n$ [16];

following them, we denote the broad resonance as $f_2(2000)$.

The large width of $f_2(2000)$ arouses the suspicion that this state is a tensor glueball. In [13], Chapter 5.4, it was emphasised that a broad isoscalar 2^{++} state observed in the region ~ 2000 MeV with a width of the order of 400 – 500 MeV could well be the trace of a tensor glueball lying on the pomeron trajectory.

Another argument comes from the analysis of the mass shifts of the $q\bar{q}$ tensor mesons ([17], Section 12). It is stated there that the mass shift between $f_2(1580)$ and $a_2(1732)$ cannot be explained by the mixing of non-strange and strange components in the isoscalar sector. Both isoscalar states, $f_2(1580)$ and $f_2(1755)$, are shifted downward; this can be an indication of the presence of a tensor glueball in the mass region 1800–2000 MeV.

In [16], the following argument was put forward: a significant violation of the OZI-rule in the production of tensor mesons with dominant $s\bar{s}$ components (reactions $\pi^-p \rightarrow f_2(2120)n$, $f_2(2340)n$, $f_2(2410)n \rightarrow \phi\phi n$ [18]) is due to the presence of a broad glueball state $f_2(2000)$ in this region, resulting in a noticeable admixture of the glueball component in $f_2(2120)$, $f_2(2340)$, $f_2(2410)$.

In [19], it was emphasised that the $f_2(2000)$ is superfluous for $q\bar{q}$ systematics and can be considered as the lowest tensor glueball. The matter is that the reanalysis of the $\phi\phi$ spectra [16] in the reaction $\pi^-p \rightarrow \phi\phi n$ [18], the study of the processes $\gamma\gamma \rightarrow \pi^+\pi^-\pi^0$ [24], $\gamma\gamma \rightarrow K_S K_S$ [17] and the analysis of the $p\bar{p}$ annihilation in flight, $p\bar{p} \rightarrow \pi\pi, \eta\eta, \eta\eta'$ [14], clarifying essentially the status of the $(J^{PC} = 2^{++})$ -mesons, did not leave room for $f_2(2000)$ on the (n, M^2) -trajectories [19]. In Chapter 3 we discuss the data

[14] in detail.

The most complete quantitative analysis of the 2^{++} sector was performed in [20, 21]. Let us summarise shortly the current understanding of the situation of the tensor mesons based on these studies.

There exist various arguments in favour of the assumption that $f_2(2000)$ is a glueball. Still, it cannot be a pure gluonium $f_2(2000)$ state: as it follows from the $1/N$ expansion rules [22, 23], the quarkonium state ($q\bar{q}$) mixes with gluonium system (gg) without suppression. The problem of the mixing of (gg) and ($q\bar{q}$) systems is discussed below.

We present also the relations between decay constants of a glueball into two pseudoscalar mesons, $glueball \rightarrow PP$, and into two vector mesons, $glueball \rightarrow VV$. Precisely the relations between the decay couplings of a glueball into two pseudoscalar mesons, $glueball \rightarrow PP$, and two vector mesons, $glueball \rightarrow VV$, are decisive to reveal the glueball nature of $f_2(2000)$.

2.6.1.1 Systematisation of tensor mesons on the (n, M^2) trajectories

In Fig. 2.2c,f the present status of the (n, M^2) trajectories for tensor mesons is demonstrated, where we have used the recent data [14, 16, 17] for f_2 and [5, 24] for a_2 mesons. To avoid confusion, we list here, as before, the experimentally observed masses. First, this concerns the resonances seen in the $\phi\phi$ spectrum. In [16] the $\phi\phi$ spectra [18] were reanalysed, taking into account the existence of the broad $f_2(2000)$ resonance. As a result, the masses of three relatively narrow resonances are shifted compared to those given in the PDG compilation [8]: $f_2(2010)|_{PDG} \rightarrow f_2(2120)$ [16], $f_2(2300)|_{PDG} \rightarrow f_2(2340)$ [16], $f_2(2340)|_{PDG} \rightarrow f_2(2410)$ [16].

As was emphasised above, the trajectories for the f_2 and a_2 mesons turn out to be linear with a good accuracy: $M^2 = M_0^2 + (n-1)\mu^2$, where the value $\mu^2 = 1.15 \text{ GeV}^2$ agrees with the value of the universal slope $\mu^2 = 1.20 \pm 0.10 \text{ GeV}^2$.

The quark states with $(I = 0, J^{PC} = 2^{++})$ are determined by two flavour components $n\bar{n}$ and $s\bar{s}$ for which two states $^{2S+1}L_J = {}^3P_2, {}^3F_2$ are possible. Consequently, we have four trajectories on the (n, M^2) plane. Generally speaking, the f_2 -states are mixtures of both flavour components and $L = 1, 3$ waves. However, the real situation is such that the lowest trajectory [$f_2(1275)$, $f_2(1580)$, $f_2(1920)$, $f_2(2240)$] consists of mesons with dominant ${}^3P_2 n\bar{n}$ components (note, $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$), while the trajec-

tory $[f_2(1525), f_2(1755), f_2(2120), f_2(2410)]$ contains mesons with predominantly ${}^3P_2 s\bar{s}$ components, and the F -trajectories are represented by three resonances $[f_2(2020), f_2(2300)]$ (dominantly ${}^3F_2 n\bar{n}$) and $[f_2(2340)]$ (dominantly ${}^3F_2 s\bar{s}$ states). Following [19], we can state that the broad resonance $f_2(2000)$ is not a part of those states placed on the (n, M^2) trajectories. In the region of 2000 MeV three $n\bar{n}$ -dominant resonances, $f_2(1920)$, $f_2(2000)$ and $f_2(2020)$, are seen, while on the (n, M^2) -trajectories there are only two vacant places. This means that one state is obviously superfluous from the point of view of the $q\bar{q}$ -systematics, *i.e.* it has to be considered as exotics. The large value of the width of the $f_2(2000)$ strengthen the suspicion that, indeed, this state is an exotic one.

2.6.1.2 *Mixing of the quarkonium and gluonium states*

On the basis of the $1/N$ -expansion rules, we estimate here effects of mixing of quarkonium and gluonium states.

The rules of the $1/N$ -expansion [22, 23], where $N = N_c = N_f$ are numbers of colours and light flavours, provide a possibility to estimate the mixing of the gluonium (gg) with the neighbouring quarkonium states ($q\bar{q}$).

The admixture of the gg component in a $q\bar{q}$ -meson is small, of the order of $1/N_c$:

$$f_2(q\bar{q} - \text{meson}) = q\bar{q} \cos \alpha + gg \sin \alpha \quad (2.28)$$

$$\sin^2 \alpha \sim 1/N_c.$$

The quarkonium component in the glueball should be larger, it is of the order of N_f/N_c :

$$f_2(\text{glueball}) = gg \cos \gamma + (q\bar{q})_{\text{glueball}} \sin \gamma, \quad (2.29)$$

$$\sin^2 \gamma \sim N_f/N_c,$$

where $(q\bar{q})_{\text{glueball}}$ is a mixture of $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$ and $s\bar{s}$ components:

$$(q\bar{q})_{\text{glueball}} = n\bar{n} \cos \varphi_{\text{glueball}} + s\bar{s} \sin \varphi_{\text{glueball}}, \quad (2.30)$$

$$\sin \varphi_{\text{glueball}} = \sqrt{\lambda/(2 + \lambda)}.$$

Were the flavour SU(3) symmetry satisfied, the quarkonium component $(q\bar{q})_{\text{glueball}}$ would be a flavour singlet, $\varphi_{\text{glueball}} \rightarrow \varphi_{\text{singlet}} \simeq 37^\circ$. In reality, the probability of strange quark production in a gluon field is suppressed: $u\bar{u} : d\bar{d} : s\bar{s} = 1 : 1 : \lambda$, where $\lambda \simeq 0.5 - 0.85$. Hence, $(q\bar{q})_{\text{glueball}}$ differs slightly from the flavour singlet, being determined by the parameter λ as follows [25]:

$$(q\bar{q})_{\text{glueball}} = (u\bar{u} + d\bar{d} + \sqrt{\lambda} s\bar{s})/\sqrt{2 + \lambda}. \quad (2.31)$$

The suppression parameter λ was estimated both in multiple hadron production processes [26], and in hadronic decay processes [7, 27]. In hadronic decays of mesons with different J^{PC} the value of λ can be, in principle, different. Still, the analyses of the decays of the 2^{++} -states [27] and 0^{++} -states [7] show that the suppression parameters are of the same order, 0.5–0.85, leading to

$$\varphi_{\text{glueball}} \simeq 27^0 - 33^0. \quad (2.32)$$

Let us explain now equations (2.28)–(2.31) in detail.

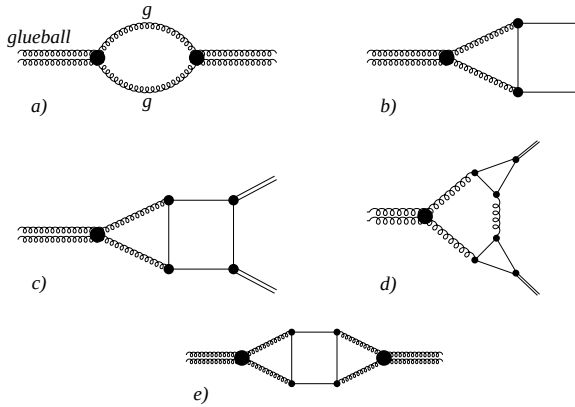


Fig. 2.11 Examples of diagrams which determine the gluonium (gg) decay.

First, let us evaluate, using the rules of $1/N$ -expansion, the decay transitions *gluonium* $\rightarrow$ two $q\bar{q}$ -mesons and *quarkonium* $\rightarrow$ two $q\bar{q}$ -mesons. For this purpose, we consider the gluon loop diagram which corresponds to the two-gluon self-energy part: *gluonium* $\rightarrow$ two *gluons* $\rightarrow$ *gluonium* (see Fig. 2.11a). This loop diagram $B(\text{gluonium} \rightarrow gg \rightarrow \text{gluonium})$ is of the order of unity, provided the gluonium is a two-gluon composite system: $B(\text{gluonium} \rightarrow gg \rightarrow \text{gluonium}) \sim g_{\text{gluonium} \rightarrow gg}^2 N_c^2 \sim 1$, where $g_{\text{gluonium} \rightarrow gg}$ is the coupling constant for the transition of a gluonium to two gluons. Therefore,

$$g_{\text{gluonium} \rightarrow gg} \sim 1/N_c. \quad (2.33)$$

The coupling constant for the *gluonium* $\rightarrow q\bar{q}$ transition is determined by the diagrams of Fig. 2.11b type. A similar evaluation gives:

$$g_{\text{gluonium} \rightarrow q\bar{q}} \sim g_{\text{gluonium} \rightarrow gg} g^2 N_c \sim 1/N_c. \quad (2.34)$$

Here g is the quark–gluon coupling constant, which is of the order of $1/\sqrt{N_c}$ [22]. The coupling constant for the *gluonium* $\rightarrow$ two $q\bar{q}$ -mesons transition is governed in the leading $1/N_c$ terms by diagrams of Fig. 2.11c type:

$$g_{\text{gluonium} \rightarrow \text{two mesons}}^L \sim g_{\text{gluonium} \rightarrow q\bar{q}} g_{\text{meson} \rightarrow q\bar{q}}^2 N_c \sim 1/N_c. \quad (2.35)$$

In (2.35), the following evaluation of the coupling for the transition $q\bar{q} - \text{meson} \rightarrow q\bar{q}$ has been used:

$$g_{\text{meson} \rightarrow q\bar{q}} \sim 1/\sqrt{N_c}, \quad (2.36)$$

which follows from the fact that the self-energy loop diagram of the $q\bar{q}$ -meson propagator (see Fig. 2.12a) is of the order of unity: $B(q\bar{q} - \text{meson} \rightarrow q\bar{q} \rightarrow \text{meson}) \sim g_{\text{meson} \rightarrow q\bar{q}}^2 N_c \sim 1$.

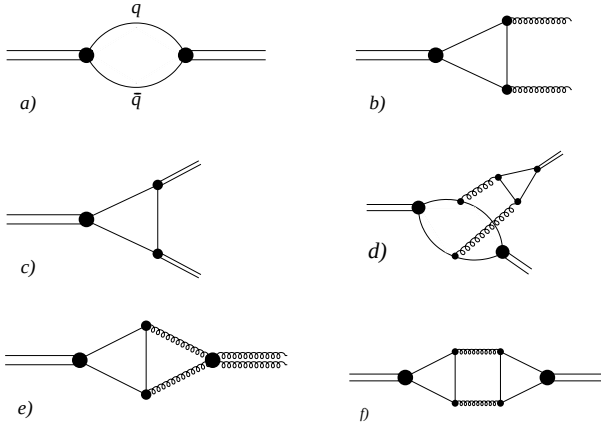


Fig. 2.12 Examples of diagrams which determine the quarkonium ($q\bar{q}$) decay.

The diagram of the type of Fig. 2.11d governs the couplings for the transition *gluonium* $\rightarrow$ two $q\bar{q}$ -mesons in the next-to-leading terms of the $1/N_c$ -expansion:

$$g_{\text{gluonium} \rightarrow \text{two mesons}}^{NL} \sim g_{\text{gluonium} \rightarrow gg} g_{\text{meson} \rightarrow gg}^2 N_c^2 \sim 1/N_c^2, \quad (2.37)$$

where the coupling $g_{\text{meson} \rightarrow gg}$ has been estimated following the diagram in Fig. 2.12b:

$$g_{\text{meson} \rightarrow gg} \sim g_{\text{meson} \rightarrow q\bar{q}} g^2 \sim 1/N_c^{3/2}. \quad (2.38)$$

Decay couplings of the $q\bar{q}$ -meson into two mesons in leading and next-to-leading terms of $1/N_c$ expansion are determined by diagrams of the type of Figs. 2.12c and 2.12d, respectively. They give:

$$g_{\text{meson} \rightarrow \text{two mesons}}^L \sim g_{\text{meson} \rightarrow q\bar{q}}^3 N_c \sim 1/\sqrt{N_c}, \quad (2.39)$$

and

$$g_{meson \rightarrow two\ mesons}^{NL} \sim g_{meson \rightarrow q\bar{q}}^2 g_{meson \rightarrow gg} g_c^2 N_c^2 \sim 1/N_c^{3/2}. \quad (2.40)$$

Now we can estimate the order of the value of $\sin^2 \gamma$ which defines the probability $(q\bar{q})_{glueball}$ in the gluonium descendant, see Eq. (2.29). This probability is determined by the self-energy part of the gluon propagator (diagrams of Fig. 2.11e type) — it is of the order of N_f/N_c , the factor N_f being the number of the light flavour quark loops. Of course, the diagram in Fig. 2.11e represents an example of the contributions of that type only: contributions of the same order are also given by similar diagrams with all possible (but planar) gluon exchanges in the quark loops.

One can evaluate $\sin^2 \gamma$ also in a different way, using the transition amplitude $gluonium \rightarrow quarkonium$ (see Fig. 2.12e), which is of the order of $1/\sqrt{N_c}$. The value $\sin^2 \gamma$ is determined by the transition amplitude of Fig. 2.12e squared, so the sum over the flavours of all quarkonia results in Eq. (2.29).

The probability of the gluonium component in the quarkonium, $\sin^2 \alpha$, is of the order of the diagram in Fig. 2.12f, $\sim 1/N_c$, giving us the estimate (2.28). In this self-energy gluonium block planar-type gluon exchanges are possible.

The flavour content of $(q\bar{q})_{glueball}$, see Eq. (2.31), can be determined by the diagram in Fig. 2.11e. As was said above, the gluon field produces light quark pairs with probabilities $u\bar{u} : d\bar{d} : s\bar{s} = 1 : 1 : \lambda$, giving (2.31). For $\lambda \sim 0.5 - 0.85$, the $(q\bar{q})_{glueball}$ is nearly a flavour singlet.

2.6.1.3 Quark combinatorial relations for decay constants

The rules of quark combinatorics lead to relations between decay couplings of mesons, which belong to the same SU(3) nonet. The violation of the flavour symmetry in the decay processes is taken into account by introducing a suppression parameter λ for the production of the strange quarks by gluons.

In the leading terms of the $1/N$ expansion, the main contribution to the decay coupling constant comes from planar diagrams. Examples of the production of new $q\bar{q}$ -pairs by intermediate gluons are shown in Figs. 2.13a and 2.13b. When an isoscalar $q\bar{q}$ -meson disintegrates, the coupling constants can be determined, up to a common factor, by two characteristics of a meson. The first is the quark mixing angle φ for the initial $q\bar{q}$ -meson, $q\bar{q} = n\bar{n} \cos \varphi + s\bar{s} \sin \varphi$, the second is the parameter λ

for the newly produced quark pair. Experimental data, as was emphasised before, provide for this λ value the interval $\lambda = 0.5 - 0.85$ [7, 26, 27].

Let us consider in more detail the production of two pseudoscalar mesons, $P_1 P_2$, by the f_J -quarkonium and the J^{++} -gluonium:

$$\begin{aligned} f_J(\text{quarkonium}) &\rightarrow \pi\pi, K\bar{K}, \eta\eta, \eta\eta', \eta'\eta', \\ J^{++}(\text{gluonium}) &\rightarrow \pi\pi, K\bar{K}, \eta\eta, \eta\eta', \eta'\eta'. \end{aligned} \quad (2.41)$$

The coupling constants for the decay into channels (2.41), which in the leading terms of the $1/N$ expansion are determined by diagrams of the type shown in Fig. 2.13a,b, may be presented as

$$\begin{aligned} g^L(q\bar{q} \rightarrow P_1 P_2) &= C_{P_1 P_2}^{q\bar{q}}(\varphi, \lambda) g_P^L, \\ g^L(gg \rightarrow P_1 P_2) &= C_{P_1 P_2}^{gg}(\lambda) G_P^L, \end{aligned} \quad (2.42)$$

where $C_{P_1 P_2}^{q\bar{q}}(\varphi, \lambda)$ and $C_{P_1 P_2}^{gg}(\lambda)$ are wholly calculable coefficients depending on the mixing angle φ and parameter λ ; g_P^L and G_P^L are common factors describing the unknown dynamics of the processes. The factor g_P^L should be common for all members of the same nonet.

Dealing with processes of the Fig. 2.13b type, one should bear in mind that they do not contain $(q\bar{q})_{\text{quarkonium}}$ components in the intermediate state but $(q\bar{q})_{\text{continuous spectrum}}$ only. The states $(q\bar{q})_{\text{quarkonium}}$ in this diagram would lead to processes of Fig. 2.13c, namely, to a diagram with the quarkonium decay vertex and the mixing block of gg and $q\bar{q}$ components. But the mixing of sub-processes is taken into account separately in (2.29).

The contributions of the diagrams of the type of Fig. 2.11d and 2.12d, which give the next-to-leading terms, $g^{NL}(q\bar{q} \rightarrow P_1 P_2)$ and $g^{NL}(gg \rightarrow P_1 P_2)$, may be presented in a form analogous to (2.42). The decay constant to the channel $P_1 P_2$ is a sum of both contributions:

$$\begin{aligned} g^L(q\bar{q} \rightarrow P_1 P_2) + g^{NL}(q\bar{q} \rightarrow P_1 P_2), \\ g^L(gg \rightarrow P_1 P_2) + g^{NL}(gg \rightarrow P_1 P_2). \end{aligned} \quad (2.43)$$

The second terms are suppressed compared to the first ones by a factor $1/N_c$; the experience in the calculation of quark diagrams teaches us that this suppression is of the order of $1/10$.

Coupling constants for gluonium decays, $g^L(gg \rightarrow P_1 P_2)$ and $g^{NL}(gg \rightarrow P_1 P_2)$, are presented in Table 2.3 while those for quarkonium decays, $g^L(q\bar{q} \rightarrow P_1 P_2)$ and $g^{NL}(q\bar{q} \rightarrow P_1 P_2)$, are given in Table 2.4.

In Table 2.5 we give the couplings for decays of the gluonium state into channels of the vector mesons: $gg \rightarrow V_1 V_2$.

Table 2.3 Coupling constants of the J^{++} -gluonium ($J = 0, 2, 4, \dots$) decaying into two pseudoscalar mesons, in the leading (G^L) and next-to-leading (G^{NL}) terms of $1/N$ expansion. Θ is the mixing angle for $\eta - \eta'$ mesons: $\eta = n\bar{n} \cos \Theta - s\bar{s} \sin \Theta$ and $\eta' = n\bar{n} \sin \Theta + s\bar{s} \cos \Theta$.

Channel	Gluonium decay couplings in the leading term of $1/N$ expansion.	Gluonium decay couplings in the next-to-leading term of $1/N$ expansion.	Identity factor
$\pi^0 \pi^0$	G^L	0	1/2
$\pi^+ \pi^-$	G^L	0	1
$K^+ K^-$	$\sqrt{\lambda} G^L$	0	1
$K^0 K^0$	$\sqrt{\lambda} G^L$	0	1
$\eta \eta$	$G^L (\cos^2 \Theta + \lambda \sin^2 \Theta)$	$2G^{NL} (\cos \Theta - \sqrt{\frac{\lambda}{2}} \sin \Theta)^2$	1/2
$\eta \eta'$	$G^L (1 - \lambda) \sin \Theta \cos \Theta$	$2G^{NL} (\cos \Theta - \sqrt{\frac{\lambda}{2}} \sin \Theta) \times (\sin \Theta + \sqrt{\frac{\lambda}{2}} \cos \Theta)$	1
$\eta' \eta'$	$G^L (\sin^2 \Theta + \lambda \cos^2 \Theta)$	$2G^{NL} \left(\sin \Theta + \sqrt{\frac{\lambda}{2}} \cos \Theta \right)^2$	1/2

2.6.1.4 Sum rules for decay couplings

In Tables 2.3 and 2.4, we present the decay constants for the *glueball* $\rightarrow$ *two pseudoscalar mesons* and $q\bar{q} = n\bar{n} \cos \varphi + s\bar{s} \sin \varphi \rightarrow$ *two pseudoscalar mesons* transitions, where $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$. The angle Θ defines the quark content of η and η' mesons assuming them to be pure $q\bar{q}$ states: $\eta = n\bar{n} \cos \Theta - s\bar{s} \sin \Theta$ and $\eta' = n\bar{n} \sin \Theta + s\bar{s} \cos \Theta$.

The leading terms of the $1/N$ expansion in Tables 2.3 and 2.4 give planar diagrams [22]; let us discuss the sum rules just for couplings determined by the leading terms.

The coupling constants given in Table 2.4 satisfy the sum rule:

$$\begin{aligned}
 \sum_{c=\pi\pi, K\bar{K}, \eta\eta, \eta\eta', \eta'\eta'} (g^L)^2 (n\bar{n} \rightarrow c) I_c + \\
 \sum_{c=K\bar{K}, \eta\eta, \eta\eta', \eta'\eta'} (g^L)^2 (s\bar{s} \rightarrow c) I_c = \frac{3}{4} (g^L)^2 (2 + \lambda) ,
 \end{aligned} \tag{2.44}$$

Table 2.4 Coupling constants of the f_2 -quarkonium decaying into two pseudoscalar mesons in the leading and next-to-leading terms of the $1/N$ expansion. The flavour content of the f_2 -gluonium is determined by the mixing angle φ as follows: $f_J(q\bar{q}) = n\bar{n} \cos \varphi + s\bar{s} \sin \varphi$, where $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$.

Channel	Decay couplings of quarkonium in leading term of $1/N$ expansion.	Decay couplings of quarkonium in next-to-leading term of $1/N$ expansion.
$\pi^0 \pi^0$	$g^L \cos \varphi / \sqrt{2}$	0
$\pi^+ \pi^-$	$g^L \cos \varphi / \sqrt{2}$	0
$K^+ K^-$	$g^L (\sqrt{2} \sin \varphi + \sqrt{\lambda} \cos \varphi) / \sqrt{8}$	0
$K^0 \bar{K}^0$	$g^L (\sqrt{2} \sin \varphi + \sqrt{\lambda} \cos \varphi) / \sqrt{8}$	0
$\eta \eta$	$g^L \left(\cos^2 \Theta \cos \varphi / \sqrt{2} + \sqrt{\lambda} \sin \varphi \sin^2 \Theta \right)$	$\sqrt{2} g^{NL} (\cos \Theta - \sqrt{\frac{\lambda}{2}} \sin \Theta) \times (\cos \varphi \cos \Theta - \sin \varphi \sin \Theta)$
$\eta \eta'$	$g^L \sin \Theta \cos \Theta \left(\cos \varphi / \sqrt{2} - \sqrt{\lambda} \sin \varphi \right)$	$\sqrt{\frac{1}{2}} g^{NL} \left[(\cos \Theta - \sqrt{\frac{\lambda}{2}} \sin \Theta) \times (\cos \varphi \sin \Theta + \sin \varphi \cos \Theta) + (\sin \Theta + \sqrt{\frac{\lambda}{2}} \cos \Theta) \times (\cos \varphi \sin \Theta - \sin \varphi \cos \Theta) \right]$
$\eta' \eta'$	$g^L \left(\sin^2 \Theta \cos \varphi / \sqrt{2} + \sqrt{\lambda} \sin \varphi \cos^2 \Theta \right)$	$\sqrt{2} g^{NL} (\sin \Theta + \sqrt{\frac{\lambda}{2}} \cos \Theta) \times (\cos \varphi \cos \Theta + \sin \varphi \sin \Theta)$

where I_c is the identity factor. The factor $(2 + \lambda)$ corresponds to the probability to produce additional $q\bar{q}$ -pairs in the decay of the $q\bar{q}$ -meson (recall, new $q\bar{q}$ -pairs are produced in the proportion $u\bar{u} : d\bar{d} : s\bar{s} = 1 : 1 : \lambda$). Equation (2.44) may be illustrated by Fig. 2.14a: the cutting of these type diagrams gives the sum of the couplings squared.

For the glueball decay the sum of squared couplings over all channels is proportional to the probability to produce two $q\bar{q}$ pairs, $\sim (2 + \lambda)^2$. Indeed, performing calculations, we have

$$\sum_{c=\pi\pi, K\bar{K}, \eta\eta, \eta\eta', \eta'\eta'} (G^L)^2(c) I(c) = \frac{1}{2} (G^L)^2 (2 + \lambda)^2. \quad (2.45)$$

Equation (2.45) is illustrated by Fig. 2.14b: the cutting of the planar diagrams with two loops gives sum of the couplings squared for gluonium.

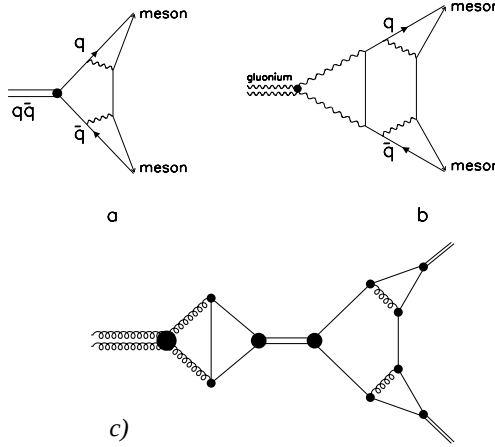


Fig. 2.13 Examples of planar diagrams responsible for the decay of the $q\bar{q}$ -state (a) and the gluonium (b) into two $q\bar{q}$ -mesons (leading terms in the $1/N$ expansion). c) Diagram for the gluonium decay with a pole in the intermediate $q\bar{q}$ -state: this process is not included into the gluonium decay vertex being actually a decay of the $q\bar{q}$ -state.

Table 2.5 Coupling constants of the glueball decay into two vector mesons in the leading (planar diagrams) and next-to-leading (non-planar diagrams) terms of $1/N$ -expansion. The mixing angle for $\omega - \phi$ mesons is defined as: $\omega = n\bar{n} \cos \varphi_V - s\bar{s} \sin \varphi_V$, $\phi = n\bar{n} \sin \varphi_V + s\bar{s} \cos \varphi_V$. Because of the small value of φ_V , we keep in the table only terms of the order of φ_V .

Channel	Couplings for glueball decays in the leading order of $1/N$ expansion	Couplings for glueball decays in next-to-leading order of $1/N$ expansion	Identity factor for decay products
$\rho^0 \rho^0$	G_V^L	0	1/2
$\rho^+ \rho^-$	G_V^L	0	1
$K^{*+} K^{*-}$	$\sqrt{\lambda} G_V^L$	0	1
$K^{*0} \bar{K}^{*0}$	$\sqrt{\lambda} G_V^L$	0	1
$\omega \omega$	G_V^L	$2G_V^{NL}$	1/2
$\omega \phi$	$G_V^L (1 - \lambda) \varphi_V$	$2G_V^{NL} \left(\sqrt{\frac{\lambda}{2}} + \varphi_V \left(1 - \frac{\lambda}{2} \right) \right)$	1
$\phi \phi$	λG_V^L	$2G_V^{NL} \left(\frac{\lambda}{2} + \sqrt{2\lambda} \varphi_V \right)$	1/2

2.6.1.5 The broad state $f_2(2000)$: the tensor glueball

In the leading terms of $1/N_c$ -expansion, we have definite ratios for the glueball decay couplings. The next-to-leading terms in the decay couplings give corrections of the order of $1/N_c$. Underline once again that, as we have seen in numerical calculations of the diagrams, the $1/N_c$ factor leads to a

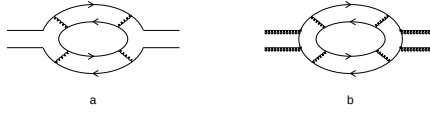


Fig. 2.14 Quark loop diagrams (a) for quarkonium and (b) gluonium. Their cutting leads to sum rules for the decay coupling squared.

smallness of the order of $1/10$, and we neglect next-to-leading terms in the analysis of the decays $f_2 \rightarrow \pi^0 \pi^0, \eta\eta, \eta\eta'$ performed below.

Considering the glueball state, which is also a mixture of the gluonium and quarkonium components, we have $\varphi \rightarrow \varphi_{\text{glueball}} = \sin^{-1} \sqrt{\lambda/(2+\lambda)}$ for the latter. So we can write

$$\frac{g^L((q\bar{q})_{\text{glueball}} \rightarrow P_1 P_2)}{g^L((q\bar{q})_{\text{glueball}} \rightarrow P'_1 P'_2)} = \frac{G^L(gg \rightarrow P_1 P_2)}{G^L(gg \rightarrow P'_1 P'_2)} . \quad (2.46)$$

Then the relations for decay couplings of the glueball in the leading terms of the $1/N$ -expansion read:

$$\begin{aligned} g_{\pi^0 \pi^0}^{\text{glueball}} &= \frac{G_{\text{glueball}}^L}{\sqrt{2+\lambda}} , \\ g_{\eta\eta}^{\text{glueball}} &= \frac{G_{\text{glueball}}^L}{\sqrt{2+\lambda}} (\cos^2 \Theta + \lambda \sin^2 \Theta) , \\ g_{\eta\eta'}^{\text{glueball}} &= \frac{G_{\text{glueball}}^L}{\sqrt{2+\lambda}} (1-\lambda) \sin \Theta \cos \Theta . \end{aligned} \quad (2.47)$$

Hence, in spite of the unknown quarkonium components in the glueball, there are definite relations between the couplings of the glueball state with the channels $\pi^0 \pi^0, \eta\eta, \eta\eta'$ which can serve as signatures to define it.

2.6.1.6 Ratios between coupling constants of

$f_2(2000) \rightarrow \pi^0 \pi^0, \eta\eta, \eta\eta'$ as

indication of the glueball nature of this state

The equation (2.47) tells us that for the glueball state the relations between the coupling constants are $1 : (\cos^2 \Theta + \lambda \sin^2 \Theta) : (1-\lambda) \cos \Theta \sin \Theta$. For $(\lambda = 0.5, \Theta = 37^\circ)$ we have $1 : 0.82 : 0.24$, and for $(\lambda = 0.85, \Theta = 37^\circ)$, respectively, $1 : 0.95 : 0.07$. Consequently, the relations between the coupling constants $g_{\pi^0 \pi^0} : g_{\eta\eta} : g_{\eta\eta'}$ for the 2^{++} -glueball have to be

$$g_{\pi^0 \pi^0}^{\text{glueball}} : g_{\eta\eta}^{\text{glueball}} : g_{\eta\eta'}^{\text{glueball}} = 1 : (0.82 - 0.95) : (0.24 - 0.07) . \quad (2.48)$$

The $\bar{p}p \rightarrow \pi^0\pi^0$, $\eta\eta$, $\eta\eta'$ amplitudes [14, 20] provide the following ratios for the f_2 resonance couplings $g_{\pi^0\pi^0} : g_{\eta\eta} : g_{\eta\eta'}$:

$$\begin{aligned} f_2(1920) & 1 : 0.56 \pm 0.08 : 0.41 \pm 0.07 , \\ f_2(2000) & 1 : 0.82 \pm 0.09 : 0.37 \pm 0.22 , \\ f_2(2020) & 1 : 0.70 \pm 0.08 : 0.54 \pm 0.18 , \\ f_2(2240) & 1 : 0.66 \pm 0.09 : 0.40 \pm 0.14 , \\ f_2(2300) & 1 : 0.59 \pm 0.09 : 0.56 \pm 0.17 . \end{aligned} \quad (2.49)$$

It follows from (2.49) that only the coupling constants of the broad $f_2(2000)$ resonance are inside the intervals (2.48): $0.82 \leq g_{\eta\eta}/g_{\pi^0\pi^0} \leq 0.95$ and $0.24 \geq g_{\eta\eta'}/g_{\pi^0\pi^0} \geq 0.07$. Hence, it is just this resonance which can be considered as a candidate for a tensor glueball, while λ is fixed in the interval $0.5 \leq \lambda \leq 0.7$. Taking into account that there is no place for $f_2(2000)$ on the (n, M^2) -trajectories (see Fig. 2.2f), it becomes evident that indeed, this resonance is the lowest tensor glueball.

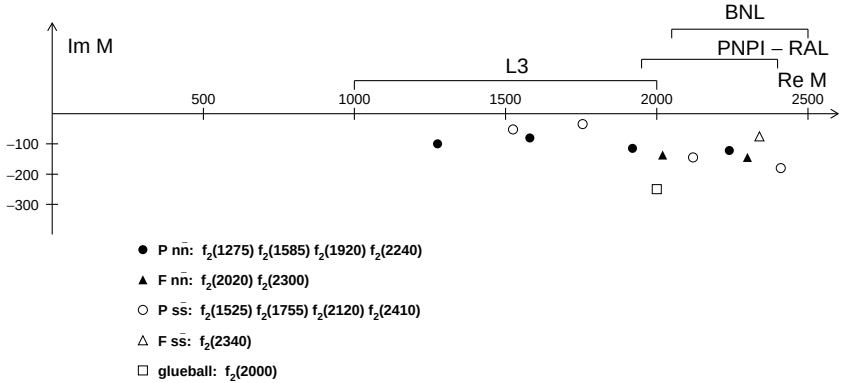


Fig. 2.15 Position of the f_2 -poles on the complex- M plane: states with dominant ${}^3P_2 n\bar{n}$ -component (full circle), ${}^3F_2 n\bar{n}$ -component (full triangle), ${}^3P_2 s\bar{s}$ -component (open circle), ${}^3F_2 s\bar{s}$ -component (open triangle); the position of the tensor glueball is shown by the open square. Mass regions studied by the groups L3 [17], PNPI-RAL [4] and BNL [16, 34] are shown.

2.6.1.7 Mixing of the glueball with neighbouring $q\bar{q}$ -resonances

The position of the f_2 -poles on the complex M -plane is shown in Fig. 2.15. We see that the glueball state $f_2(2000)$ overlaps with a large group

of $q\bar{q}$ -resonances. This means that there is a considerable mixing with the neighbouring resonances. The mixing can take place both at relatively small distances, on the quark–gluon level (processes of the type shown in Fig. 2.12e), and owing to decay processes

$$f_2(\text{glueball}) \rightarrow \text{real mesons} \rightarrow f_2(q\bar{q} - \text{meson}). \quad (2.50)$$

Examples of the processes of the type of (2.50) are shown in Fig. 2.16.

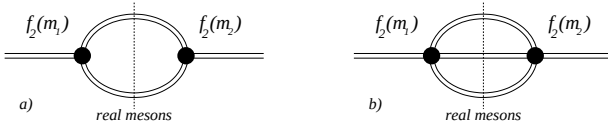


Fig. 2.16 Transitions $f_2(m_1) \rightarrow \text{real mesons} \rightarrow f_2(m_2)$, responsible for the accumulation of widths in the case of overlapping resonances.

The estimates, which were carried out above, demonstrated that even at the quark–gluon level (diagrams of the types in Fig. 2.12e) the mixing leads to a sufficiently large admixture of the quark–antiquark component in the glueball: $f_2(\text{glueball}) = gg \cos \gamma + (q\bar{q})_{\text{glueball}} \sin \gamma$, with $\sin \gamma \sim \sqrt{N_f/N_c}$. A mixing due to processes (2.50) apparently enhances the quark–antiquark component. The main effect of the processes (2.50) is, however, that in the case of overlapping resonances one of them accumulates the widths of the neighbouring resonances. The position of the f_2 -poles in Fig. 2.15 makes it obvious that such a state is the tensor glueball.

A similar situation was detected also in the sector of scalar mesons in the region 1000 – 1700 MeV: the scalar glueball, being in the neighbourhood of $q\bar{q}$ -resonances, accumulated a relevant fraction of their widths and transformed into a broad $f_0(1200 - 1600)$ state — we discuss this effect in the next section.

2.6.1.8 The $q\bar{q}$ – gg content of f_2 -mesons, observed in the reactions $p\bar{p} \rightarrow \pi^0\pi^0, \eta\eta, \eta\eta'$

Here we determine the $q\bar{q}$ – gg content of f_2 -mesons on the basis of experimentally observed relations (2.49) and of the rules of quark combinatorics taken into account in the leading terms of the $1/N$ -expansion.

For the $f_2 \rightarrow \pi^0\pi^0, \eta\eta, \eta\eta'$ transitions, when the $q\bar{q}$ -meson is a mixture of quarkonium and gluonium components, the decay vertices in the leading

$1/N$ terms (see Tables 2.3 and 2.4) read:

$$\begin{aligned}
 g_{\pi^0\pi^0}^{q\bar{q}-meson} &= g \frac{\cos \varphi}{\sqrt{2}} + \frac{G}{\sqrt{2+\lambda}} , \\
 g_{\eta\eta}^{q\bar{q}-meson} &= g \left(\cos^2 \Theta \frac{\cos \varphi}{\sqrt{2}} + \sin^2 \Theta \sqrt{\lambda} \sin \varphi \right) \\
 &\quad + \frac{G}{\sqrt{2+\lambda}} (\cos^2 \Theta + \lambda \sin^2 \Theta) , \\
 g_{\eta\eta'}^{q\bar{q}-meson} &= \sin \Theta \cos \Theta \left[g \left(\frac{\cos \varphi}{\sqrt{2}} - \sqrt{\lambda} \sin \varphi \right) + \frac{G}{\sqrt{2+\lambda}} (1 - \lambda) \right] .
 \end{aligned} \tag{2.51}$$

The terms proportional to g stand for the $q\bar{q} \rightarrow twomesons$ transitions ($g = g^L \cos \alpha$), while the terms with G represent the *gluonium* $\rightarrow twomesons$ transition ($G = G^L \sin \alpha$). Consequently, G^2 and g^2 are proportional to the probabilities for finding gluonium ($W = \sin^2 \alpha$) and quarkonium ($1 - W = \cos^2 \alpha$) components in the considered f_2 -meson. Let us remind that the mixing angle Θ stands for the $n\bar{n}$ and $s\bar{s}$ components in the η and η' mesons; we neglect the possible admixture of a gluonium component to η and η' (according to [1], the gluonium admixture to η is less than 5%, to η' — less than 20%). For the mixing angle Θ , we take $\Theta = 37^\circ$.

2.6.1.9 The analysis of the quarkonium–gluonium contents of the $\mathbf{f_2(1920)}$, $\mathbf{f_2(2020)}$, $\mathbf{f_2(2240)}$, $\mathbf{f_2(2300)}$

Making use of the data (2.49), the relations (2.51) allow us to find φ as a function of the ratio of the coupling constants, G/g . The result for the resonances $f_2(1920)$, $f_2(2020)$, $f_2(2240)$, $f_2(2300)$ is shown in Fig. 2.17. Solid curves enclose the values of $g_{\eta\eta}/g_{\pi^0\pi^0}$ for $\lambda = 0.6$ (this is the $\eta\eta$ -zone in the $(G/g, \varphi)$ plane) and dashed curves enclose $g_{\eta\eta'}/g_{\pi^0\pi^0}$ for $\lambda = 0.6$ (the $\eta\eta'$ -zone). The values of G/g and φ , lying in both zones, describe the experimental data (2.49): these regions are shadowed in Fig. 2.17.

The correlation curves in Fig. 2.17 enable us to give a qualitative estimate for the change of the angle φ (*i.e.* the relation of the $n\bar{n}$ and $s\bar{s}$ components in the f_2 meson) depending on the value of the gluonium admixture. As was said in previous section, the values g^2 and G^2 are proportional to the probabilities of having quarkonium and gluonium components in the f_2 meson, $g^2 = (g^L)^2(1 - W)$ and $G^2 = (G^L)^2W$. Since $G^L/g^L \sim 1/\sqrt{N_c}$ and $W \sim 1/N_c$, we can give a rough estimate:

$$\frac{G^2}{g^2} \sim \frac{W}{N_c(1 - W)} \rightarrow \frac{W}{10} . \tag{2.52}$$

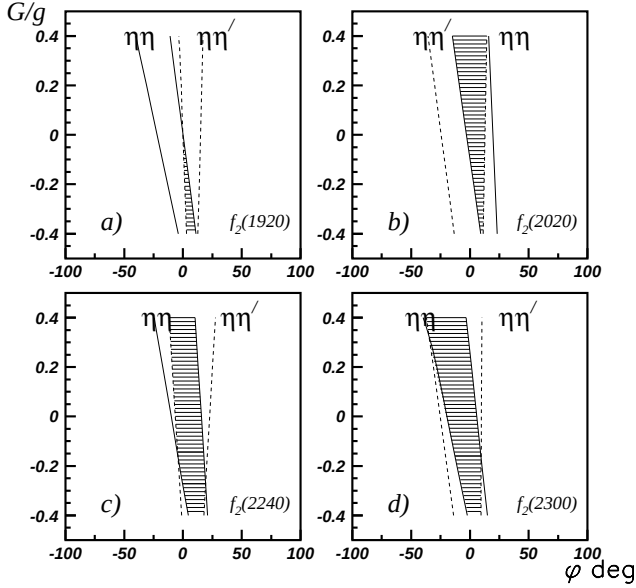


Fig. 2.17 Correlation curves $g_{\eta\eta}(\varphi, G/g)/g_{\pi^0\pi^0}(\varphi, G/g)$, $g_{\eta\eta'}(\varphi, G/g)/g_{\pi^0\pi^0}(\varphi, G/g)$ drawn according to (2.51) at $\lambda = 0.6$ for $f_2(1920)$, $f_2(2020)$, $f_2(2240)$, $f_2(2300)$. Solid and dashed curves enclose the values $g_{\eta\eta}(\varphi, G/g) / g_{\pi^0\pi^0}(\varphi, G/g)$ and $g_{\eta\eta'}(\varphi, G/g) / g_{\pi^0\pi^0}(\varphi, G/g)$ which obey (2.49) (the zones $\eta\eta$ and $\eta\eta'$ in the $(G/g, \varphi)$ plane). The values of G/g and φ , lying in both zones describe the experimental data: these are the shadowed regions.

In (2.52), we use that numerical calculations of the diagrams lead to a smallness of $1/N_c \sim 1/10$. Assuming that the gluonium components are less than 20% ($W < 0.2$) in each of the $q\bar{q}$ resonances $f_2(1920)$, $f_2(2020)$, $f_2(2240)$, $f_2(2300)$, we put roughly $W/10 \simeq G^2/g^2$, and obtain for the angles φ the following intervals:

$$\begin{aligned}
 W_{\text{gluonium}}[f_2(1920)] < 20\% : \quad & -0.8^\circ < \varphi[f_2(1920)] < 3.6^\circ , \\
 W_{\text{gluonium}}[f_2(2020)] < 20\% : \quad & -7.5^\circ < \varphi[f_2(2020)] < 13.2^\circ , \\
 W_{\text{gluonium}}[f_2(2240)] < 20\% : \quad & -8.3^\circ < \varphi[f_2(2240)] < 17.3^\circ , \\
 W_{\text{gluonium}}[f_2(2300)] < 20\% : \quad & -25.6^\circ < \varphi[f_2(2300)] < 9.3^\circ . \quad (2.53)
 \end{aligned}$$

2.6.1.10 The $n\bar{n}$ - $s\bar{s}$ content of the $q\bar{q}$ -mesons

Let us summarise what we know about the status of the ($I = 0$, $J^{PC} = 2^{++}$) $q\bar{q}$ -mesons. Estimating the $n\bar{n}$ - $s\bar{s}$ content of the f_2 -mesons, we ignore the gg admixture (remembering that it is of the order of $\sin^2 \alpha \sim 1/N_c$).

- (1) The resonances $f_2(1270)$ and $f_2'(1525)$ are well-known partners of the basic nonet with $n = 1$ and a dominant P -component, $1^3P_2q\bar{q}$. Their flavour content, obtained from the reaction $\gamma\gamma \rightarrow K_S K_S$, is

$$\begin{aligned} f_2(1270) &= \cos \varphi_{n=1} n\bar{n} + \sin \varphi_{n=1} s\bar{s}, \\ f_2(1525) &= -\sin \varphi_{n=1} n\bar{n} + \cos \varphi_{n=1} s\bar{s}, \\ \varphi_{n=1} &= -1^\circ \pm 3^\circ. \end{aligned} \quad (2.54)$$

- (2) The resonances $f_2(1560)$ and $f_2(1750)$ are partners in a nonet with $n = 2$ and a dominant P -component, $2^3P_2q\bar{q}$. Their flavour content, obtained from the reaction $\gamma\gamma \rightarrow K_S K_S$, is

$$\begin{aligned} f_2(1560) &= \cos \varphi_{n=2} n\bar{n} + \sin \varphi_{n=2} s\bar{s}, \\ f_2(1750) &= -\sin \varphi_{n=2} n\bar{n} + \cos \varphi_{n=2} s\bar{s}, \\ \varphi_{n=2} &= -12^\circ \pm 8^\circ. \end{aligned} \quad (2.55)$$

- (3) The resonances $f_2(1920)$ and $f_2(2120)$ [16] (in [8] they are denoted as $f_2(1910)$ and $f_2(2010)$) are partners in a nonet with $n = 3$ and with a dominant P -component, $3^3P_2q\bar{q}$. Ignoring the contribution of the glueball component, their flavour content, obtained from the reactions $p\bar{p} \rightarrow \pi^0\pi^0$, $\eta\eta$, $\eta\eta'$, is

$$\begin{aligned} f_2(1920) &= \cos \varphi_{n=3} n\bar{n} + \sin \varphi_{n=3} s\bar{s}, \\ f_2(2120) &= -\sin \varphi_{n=3} n\bar{n} + \cos \varphi_{n=3} s\bar{s}, \\ \varphi_{n=3} &= 0^\circ \pm 5^\circ. \end{aligned} \quad (2.56)$$

- (4) The next, predominantly $^3P_2$ states with $n = 4$ are $f_2(2240)$ and $f_2(2410)$ [16]. (By mistake, in [8] the resonance $f_2(2240)$ [14] is listed as $f_2(2300)$, while $f_2(2410)$ [16] is denoted as $f_2(2340)$). Their flavour content at $W = 0$ is determined as

$$\begin{aligned} f_2(2240) &= \cos \varphi_{n=4} n\bar{n} + \sin \varphi_{n=4} s\bar{s}, \\ f_2(2410) &= -\sin \varphi_{n=4} n\bar{n} + \cos \varphi_{n=4} s\bar{s}, \\ \varphi_{n=4} &= 5^\circ \pm 11^\circ. \end{aligned} \quad (2.57)$$

- (5) $f_2(2020)$ and $f_2(2340)$ [16] belong to the basic F -wave nonet ($n = 1$) (in [8] the $f_2(2020)$ [14] is denoted as $f_2(2000)$ and is put in the section "Other light mesons", while $f_2(2340)$ [16] is denoted as $f_2(2300)$). The flavour content of the 1^3F_2 mesons is

$$\begin{aligned} f_2(2020) &= \cos \varphi_{n(F)=1} n\bar{n} + \sin \varphi_{n(F)=1} s\bar{s}, \\ f_2(2340) &= -\sin \varphi_{n(F)=1} n\bar{n} + \cos \varphi_{n(F)=1} s\bar{s}, \\ \varphi_{n(F)=1} &= 5^\circ \pm 8^\circ. \end{aligned} \quad (2.58)$$

- (6) The resonance $f_2(2300)$ [14] has a dominant F -wave quark–antiquark component; its flavour content for $W = 0$ is defined as

$$f_2(2300) = \cos \varphi_{n(F)=2} n\bar{n} + \sin \varphi_{n(F)=2} s\bar{s}, \quad \varphi_{n(F)=2} = -8^\circ \pm 12^\circ. \quad (2.59)$$

A partner of $f_2(2300)$ in the 2^3F_2 nonet has to be a f_2 -resonance with a mass $M \simeq 2570$ MeV.

2.6.1.11 The broad $f_2(2000)$ as a glueball state

The broad $f_2(2000)$ state is the descendant of the lowest tensor glueball. This statement is favoured by estimates of parameters of the pomeron trajectory (e.g., see [13], Chapter 5.4, and references therein), according to which $M_{2^{++}glueball} \simeq 1.7 - 2.5$ GeV. Lattice calculations result in a similar value, namely, 2.2–2.4 GeV [28]. The corresponding coupling constants $f_2(2000) \rightarrow \pi^0\pi^0, \eta\eta, \eta\eta'$ satisfy the relations for the glueball, Eq.(2.48), with $\lambda \simeq 0.5 - 0.7$. The admixture of the quarkonium component $(q\bar{q})_{glueball}$ in $f_2(2000)$ cannot be determined by the ratios of the coupling constants between the hadronic channels; to define it, $f_2(2000)$ has to be observed in $\gamma\gamma$ -collisions. The value of $(q\bar{q})_{glueball}$ in $f_2(2000)$ may be rather large: the rules of $1/N$ -expansion give a value of the order of N_f/N_c . It is, probably, just the largeness of the quark–antiquark component in $f_2(2000)$ which results in its suppressed production in the radiative J/ψ decays (see discussion in [29]).

2.6.2 Scalar states

The investigation of scalar resonances was performed in a number of papers (see, for example, [8, 29, 30] and references therein), here we give a short

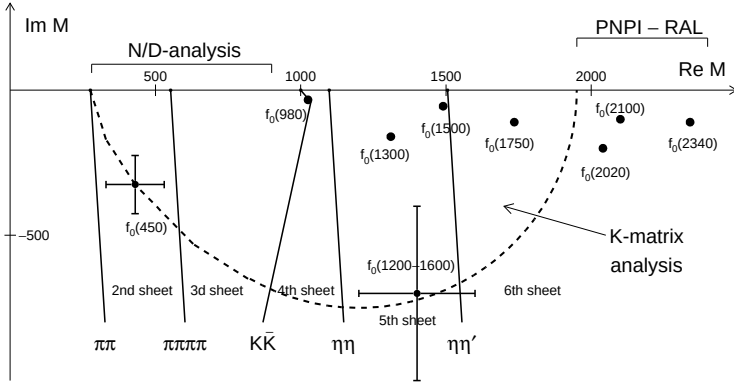


Fig. 2.18 Complex- M plane for the $(IJ^{PC} = 00^{++})$ mesons. The dashed line encircles the part of the plane where the K -matrix analysis [7] reconstructs the analytical K -matrix amplitude: in this area the poles corresponding to resonances $f_0(980)$, $f_0(1300)$, $f_0(1500)$, $f_0(1750)$ and the broad state $f_0(1200 - 1600)$ are located. Beyond this area, in the low-mass region, the pole of the light σ -meson is located (shown by the point the position of pole, $M = (430 - i320)$ MeV, corresponds to the result of N/D analysis ; the crossed bars stand for σ -meson pole found in [31]). In the high-mass region one has resonances $f_0(2030)$, $f_0(2100)$, $f_0(2340)$ [4]. Solid lines stand for the cuts related to the thresholds $\pi\pi$, $\pi\pi\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$.

review of the situation in the scalar sector based on the results of the K -matrix analysis [7, 9] in the mass region 450 - 1900 MeV, dispersion relation N/D analysis of the $\pi\pi$ scattering amplitude at $M < 500$ MeV [31] and the T -matrix study of the $p\bar{p}$ annihilation in flight at $M \simeq 1950 - 2400$ MeV [4].

In [7, 9], on the basis of experimental data of GAMS group [32], Crystal Barrel Collaboration [33] and BNL group [34], the K -matrix solution has been found for the waves 00^{++} , 10^{++} covering the mass range 450–1900 MeV. Masses and total widths of resonances have also been determined for these waves. The following states have been seen in the scalar–isoscalar sector,

$$00^{++}: f_0(980), f_0(1300), f_0(1500), f_0(1200 - 1600), f_0(1750). \quad (2.60)$$

In [8], the resonances $f_0(1300)$ and $f_0(1750)$ are referred to as $f_0(1370)$ and $f_0(1710)$.

For the states shown in (2.60), the K -matrix poles and K -matrix couplings to channels $\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$, $\pi\pi\pi\pi$ have been found in [7, 9]. Still, the K -matrix analysis [7, 9] does not supply us with partial widths of the resonances directly. To determine couplings for the transitions *resonance* $\rightarrow$ *mesons*, auxiliary calculations should be performed

to find out residues of the amplitude poles. Calculations of the residues have been carried out in [35] for the scalar–isoscalar sector, that gave us the values of partial widths for the resonances $f_0(980)$, $f_0(1300)$, $f_0(1500)$, $f_0(1750)$ and broad state $f_0(1200 - 1600)$ decaying into the channels $\pi\pi$, $\pi\pi\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$.

On the basis of the decay couplings $f_0 \rightarrow \pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$, we have analysed the quark–gluonium content of resonances $f_0(980)$, $f_0(1300)$, $f_0(1500)$, $f_0(1750)$, $f_0(1200 - 1600)$ using the quark combinatorics relations (see Tables 2.3 and 2.4).

The analytical 00^{++} -amplitude is illustrated by Fig. 2.18. The region investigated in the K -matrix analysis is shown by the dashed line: here the threshold singularities of the 00^{++} amplitude related to channels $\pi\pi$, $\pi\pi\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$ are also shown together with the corresponding cuts. The amplitude poles which correspond to the resonances (2.60) are located just in the area where the analytical structure of the amplitude 00^{++} is restored.

On the border of the mass region of the K -matrix analysis [7, 9] there is a pole related to the light σ -meson: in Fig. 2.18 its position, $M = (430 - i320)$ MeV, is shown in accordance with the results of the dispersion relation N/D -analysis [31] (the mass region covered by this analysis is also shown in Fig. 2.18). The pole related to the light σ -meson, with the mass $M \sim 450$ MeV, has been observed also in a number of papers, see [8] for details.

Above the mass region of the K -matrix analysis, there are resonances $f_0(2030)$, $f_0(2100)$, $f_0(2340)$ which were seen in $p\bar{p}$ annihilation in flight [4].

For the scalar–isovector sector, the analysis [7, 9] indicates the presence of the following resonances in the spectra:

$$10^+ : \quad a_0(980), a_0(1474) , \quad (2.61)$$

(in the compilation [8] the state $a_0(1474)$ is denoted as $a_0(1450)$).

The nonet $1^3P_0 q\bar{q}$ has been established in [36], where the K -matrix reanalysis of the $K\pi$ data [37] has been carried out. The reanalysis gives

$$\frac{1}{2}0^+ : \quad K_0(1425), K_0(1820) , \quad (2.62)$$

in agreement with previous measurements [8].

2.6.2.1 *Overlapping of f_0 -resonances in the mass region
1200–1700 MeV: accumulation of widths of the
 $q\bar{q}$ states by the glueball*

The occurrence of the broad resonance is not an accidental phenomenon at all. It originated due to a mixing of states in the decay processes, namely, transitions $f_0(m_1) \rightarrow \text{real mesons} \rightarrow f_0(m_2)$. These transitions result in a specific phenomenon, that is, when several resonances overlap one of them accumulates the widths of neighbouring resonances and transforms into the broad state.

This phenomenon had been observed in [10, 38] for scalar–isoscalar states, and the following scheme has been suggested in [39, 40]: the broad state $f_0(1200 - 1600)$ is the descendant of the pure glueball which being in the neighbourhood of $q\bar{q}$ states accumulated their widths and transformed into the mixture of gluonium and $q\bar{q}$ states. In [40] this idea had been modelled for four resonances $f_0(1300)$, $f_0(1500)$, $f_0(1200 - 1600)$ and $f_0(1750)$, by using the language of the quark–antiquark and two-gluon states, $q\bar{q}$ and gg : the decay processes were considered to be the transitions $f_0 \rightarrow q\bar{q}, gg$ and, correspondingly, the same processes realised the mixing of the resonances. In this model the gluonium component was dispersed mainly over three resonances, $f_0(1300)$, $f_0(1500)$, $f_0(1200 - 1600)$, so every state is a mixture of $q\bar{q}$ and gg components.

Accumulation of widths of overlapping resonances by one of them is a well-known effect in nuclear physics [41, 42, 43]. In meson physics this phenomenon can play a rather important role, in particular, for exotic states which are beyond the $q\bar{q}$ systematics. Indeed, being among $q\bar{q}$ resonances, the exotic state creates a group of overlapping resonances. The exotic state, which is not orthogonal to its neighbours, after accumulating the "excess" of widths, turns into a broad one. This broad resonance should be accompanied by narrow states which are the descendants of states from which the widths have been taken off. In this way, the existence of a broad resonance accompanied by narrow ones may be a signature of exotics. This possibility, in context of searching for exotic states, was discussed in [44, 45].

The broad state may be one of the components which forms the confinement barrier: the broad states after accumulating the widths of neighbouring resonances play for the latter the role of locking states. The evaluation of the mean radii squared of the broad state $f_0(1200 - 1600)$ and its neighbouring resonances argues in favour of this idea, for the radius of

$f_0(1200-1600)$ is significantly larger than that for $f_0(980)$ and $f_0(1300)$ [45, 46] thus making it possible for $f_0(1200-1600)$ to be a locking state.

2.6.2.2 *The (n, M^2) plot for scalar-isoscalar $q\bar{q}$ states and the glueball*

The systematics of $q\bar{q}$ states on the (n, M^2) plot indicates that the broad state $f_0(1200-1600)$ is beyond $q\bar{q}$ classification. In Figs. 2.2c,e and 2.3 we plotted the (n, M^2) -trajectories for f_0 , a_0 and K_0 states (remind that the doubling of f_0 trajectories is due to two flavour components, $n\bar{n}$ and $s\bar{s}$). All trajectories are roughly linear, and they clearly represent the states with dominant $q\bar{q}$ component. It is seen that one of the states, either $f_0(1200-1600)$ or $f_0(1500)$, is superfluous for $q\bar{q}$ systematics.

Lattice calculations agree with this conclusion: calculations give values for the mass of the lightest glueball in the interval 1550–1750 MeV [28].

Hadronic decays allow us to estimate of the quark-gluonium content of resonances thus indicating that the broad state $f_0(1200-1600)$, being nearly flavour blind, is the glueball.

2.6.2.3 *Hadronic decays and estimation of the quark-gluonium content of the f_0 resonances*

On the basis of the quark combinatorics for the decay coupling constants, here we analyse the quark-gluonium content of resonances $f_0(980)$, $f_0(1300)$, $f_0(1500)$, $f_0(1750)$ and $f_0(1200-1600)$. We use the decay couplings for these resonances into channels $\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$.

To extract resonance parameters from the results of the K -matrix fit, one needs additional calculations to be carried out with the obtained amplitude. The couplings for the resonance decay are extracted by calculating residues of the amplitude poles related to the resonances. In more detail, the amplitude $A_{a \rightarrow b}$, where a , b mark the channels $\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$ can be written near the pole as

$$A_{ab} \simeq \frac{g_a^{(n)} g_b^{(n)}}{\mu_n^2 - s} e^{i(\theta_a^{(n)} + \theta_b^{(n)})} + B_{ab} . \quad (2.63)$$

The first term in (2.63) represents the pole singularity and the second one, B_{ab} , is a smooth background. The pole position $s = \mu_n^2$ determines the mass of the resonance, with a total width $\mu_n = M_n - i\Gamma_n/2$. The real factors $g_a^{(n)}$ and $g_b^{(n)}$ are the decay coupling constants of the resonance to channels a and b . The couplings $g_a^{(n)}$ given in Table 2.6 stand for two

Table 2.6 Coupling constants squared g_a^2 in GeV^2 units for scalar-isoscalar resonances decaying to the hadronic channels $\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$, $\pi\pi\pi\pi$ for two K -matrix solutions [7].

Pole position (MeV)	$g_{\pi\pi}^2$	$g_{K\bar{K}}^2$	$g_{\eta\eta}^2$	$g_{\eta\eta'}^2$	$g_{\pi\pi\pi\pi}^2$	Solution
$f_0(980)$						
$1031 - i32$	0.056	0.130	0.067	—	0.004	I
$1020 - i35$	0.054	0.117	0.139	—	0.004	II
$f_0(1300)$						
$1306 - i147$	0.036	0.009	0.006	0.004	0.093	I
$1325 - i170$	0.053	0.003	0.007	0.013	0.226	II
$f_0(1500)$						
$1489 - i51$	0.014	0.006	0.003	0.001	0.038	I
$1490 - i60$	0.018	0.007	0.003	0.003	0.076	II
$f_0(1750)$						
$1732 - i72$	0.013	0.062	0.002	0.032	0.002	I
$1740 - i160$	0.089	0.002	0.009	0.035	0.168	II
$f_0(1200 - 1600)$						
$1480 - i1000$	0.364	0.265	0.150	0.052	0.524	I
$1450 - i800$	0.179	0.204	0.046	0.005	0.686	II

solutions obtained in [7]. These solutions nearly coincide, they differ for $f_0(1750)$ only, they and give in the region 1400–1600 MeV the state which is nearly flavour blind.

In the case when the f_0 state is the mixture of the quarkonium and gluonium components, the rules of quark combinatorics (see Tables 2.3 and 2.4) give us the following couplings squared for the decays $f_0 \rightarrow \pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$:

$$\begin{aligned}
g_{\pi\pi}^2 &= \frac{3}{2} \left(\frac{g}{\sqrt{2}} \cos \varphi + \frac{G}{\sqrt{2+\lambda}} \right)^2, \\
g_{K\bar{K}}^2 &= 2 \left(\frac{g}{2} (\sin \varphi + \sqrt{\frac{\lambda}{2}} \cos \varphi) + G \sqrt{\frac{\lambda}{2+\lambda}} \right)^2, \\
g_{\eta\eta}^2 &= \frac{1}{2} \left(g \left(\frac{\cos^2 \Theta}{\sqrt{2}} \cos \varphi + \sqrt{\lambda} \sin \varphi \sin^2 \Theta \right) \right. \\
&\quad \left. + \frac{G}{\sqrt{2+\lambda}} (\cos^2 \Theta + \lambda \sin^2 \Theta) \right)^2, \\
g_{\eta\eta'}^2 &= \sin^2 \Theta \cos^2 \Theta \left(g \left(\frac{1}{\sqrt{2}} \cos \varphi - \sqrt{\lambda} \sin \varphi \right) + G \frac{1-\lambda}{\sqrt{2+\lambda}} \right)^2. \quad (2.64)
\end{aligned}$$

The terms proportional to g stand for the transitions $q\bar{q} \rightarrow \text{two mesons}$, while those with G correspond to transitions $\text{glueball} \rightarrow \text{two mesons}$. Accordingly, g^2 and G^2 are proportional to the probability to find the quark-antiquark and glueball components in the considered f_0 -meson (recall that

the angle φ stands for the content of the $q\bar{q}$ -component in the decaying state, $q\bar{q} = \cos \varphi n\bar{n} + \sin \varphi s\bar{s}$, and the angle Θ for the contents of η and η' mesons: $\eta = \cos \Theta n\bar{n} - \sin \Theta s\bar{s}$ and $\eta' = \sin \Theta n\bar{n} + \cos \Theta s\bar{s}$ with $\Theta = 38^\circ$).

The glueball decay is a two-step process: initially, one $q\bar{q}$ pair is produced, then with the production of the next $q\bar{q}$ pair a fusion of quarks into mesons occurs. Therefore, at the intermediate stage of the f_0 decay, we deal with a mean quantity of the quark-antiquark component, $\langle q\bar{q} \rangle$, which later on turns into hadrons. The equation (2.64), under the condition $G = 0$, defines the content of this intermediate state $\langle q\bar{q} \rangle = n\bar{n} \cos \langle \varphi \rangle + s\bar{s} \sin \langle \varphi \rangle$.

The K -matrix analysis [7] gave us two solutions, **I** and **II**, which differ mainly by the parameters of the resonance $f_0(1750)$. Fitting to the decay couplings squared for these solutions leads to the values of $\langle \varphi \rangle$ as follows:

$$\begin{aligned} \text{Solution I : } \quad f_0(980) : \quad & \langle \varphi \rangle \simeq -69^\circ, \quad \lambda \simeq 0.5 - 1.0, \\ f_0(1300) : \quad & \langle \varphi \rangle \simeq (-3^\circ) - 4^\circ, \quad \lambda \simeq 0.5 - 0.9, \\ f_0(1200 - 1600) : \quad & \langle \varphi \rangle \simeq 27^\circ, \quad \lambda \simeq 0.54, \\ f_0(1500) : \quad & \langle \varphi \rangle \simeq 12^\circ - 19^\circ, \quad \lambda \simeq 0.5 - 1.0, \\ f_0(1750) : \quad & \langle \varphi \rangle \simeq -72^\circ, \quad \lambda \simeq 0.5 - 0.7, \quad (2.65) \end{aligned}$$

$$\begin{aligned} \text{Solution II : } \quad f_0(980) : \quad & \langle \varphi \rangle \simeq -67^\circ, \quad \lambda \simeq 0.6 - 1.0, \\ f_0(1300) : \quad & \langle \varphi \rangle \simeq (-16^\circ) - (-13^\circ), \quad \lambda \simeq 0.5 - 0.6, \\ f_0(1200 - 1600) : \quad & \langle \varphi \rangle \simeq 33^\circ, \quad \lambda \simeq 0.85, \\ f_0(1500) : \quad & \langle \varphi \rangle \simeq 2^\circ - 11^\circ, \quad \lambda \simeq 0.6 - 1.0, \\ f_0(1750) : \quad & \langle \varphi \rangle \simeq -18^\circ, \quad \lambda \simeq 0.5. \quad (2.66) \end{aligned}$$

In both solutions, the average values of the mixing angle for $f_0(980)$, are approximately the same with a good accuracy $\langle \varphi \rangle \simeq -68^\circ \pm 1^\circ$.

The values of average mixing angles for $f_0(1300)$ are small for both solutions **I** and **II**, so we may accept $\langle \varphi[f_0(1300)] \rangle = -6^\circ \pm 10^\circ$.

The mean mixing angle for the $f_0(1500)$ does not differ noticeably for solutions **I** and **II** either, so we may adopt $\langle \varphi[f_0(1500)] \rangle = 11^\circ \pm 8^\circ$.

For the $f_0(1750)$, Solutions **I** and **II** provide different mean values for the mixing angle. In Solution **I**, the resonance $f_0(1750)$ is dominantly $s\bar{s}$ system; correspondingly, $\langle \varphi[f_0(1750)] \rangle = -72^\circ \pm 5^\circ$. In solution **II**, the absolute value of the mixing angle is much less, $\langle \varphi[f_0(1750)] \rangle = -18^\circ \pm 5^\circ$.

For the broad state, both solutions give approximate values of the mixing angle, namely, $\langle \varphi[f_0(1200 - 1600)] \rangle = 30^\circ \pm 3^\circ$. This value favours the opinion that the broad state can be treated as the glueball, because such a

value of the mean mixing angle corresponds to $\varphi_{glueball} = \sin^{-1} \sqrt{\lambda/(2+\lambda)}$ at $\lambda \sim 0.50 - 0.85$, see Eq. (2.30).

Let us emphasise that the coupling values for the f_0 -resonances found in [7] do not provide us with any alternative variants for the glueball. Indeed, the value which is the closest to the $\varphi_{singlet}$ is the limit value of the mean angle for $f_0(1500)$ in Solution **I**: $\langle \varphi[f_0(1500)] \rangle = 19^\circ$. Such a quantity being used for the definition of $\varphi_{glueball}$ corresponds to $\lambda = 0.24$, but this suppression parameter is much lower than those observed in other processes: for the decaying processes we have $\lambda = 0.6 \pm 0.2$ [27, 30], while for the high-energy multiparticle production it is $\lambda \simeq 0.5$ [26]. This way, the quark combinatorics points to one candidate only, to the broad state $f_0(1200 - 1600)$; we shall return to this important statement later on.

Generally, the formulae (2.64) allow us to find φ as a function of the coupling constant ratio G/g for the $glueball \rightarrow mesons$ and $q\bar{q}$ -state $\rightarrow mesons$ decays. The results of the fit for $f_0(980)$, $f_0(1300)$, $f_0(1500)$, $f_0(1750)$ and the broad state $f_0(1200 - 1600)$ are shown in Fig. 2.19.

First, consider the results for $f_0(980)$, $f_0(1300)$, $f_0(1500)$, $f_0(1750)$ shown in Fig. 2.19a for Solution **I** and in Fig. 2.19c for Solution **II**. The bunches of curves in the $(\varphi, G/g)$ -plane demonstrate correlations between the mixing angle φ and the G/g -ratio values for which the description of couplings given in Table 2.6 is satisfactory. A vague dissipation of curves, in particular noticeable for $f_0(1300)$ and $f_0(1500)$, is due to the uncertainty of λ .

The correlation curves in Fig. 2.19a,c allow one to see, on a qualitative level, to what extent the admixture of the gluonium component in $f_0(980)$, $f_0(1300)$, $f_0(1500)$, $f_0(1750)$ affects the $n\bar{n}-s\bar{s}$ content. The values g^2 and G^2 are proportional to the probability to find, respectively, the quarkonium and gluonium components, $W_{q\bar{q}}$ and $W_{gluonium}$, in a considered resonance:

$$g^2 = g_{q\bar{q}}^2 W_{q\bar{q}} \ , \quad G^2 = G_{gluonium}^2 W_{gluonium} \ . \quad (2.67)$$

The coupling constants $g_{q\bar{q}}^2$ and $G_{gluonium}^2$ are of the same order of magnitude, therefore we accept as a qualitative estimate

$$G^2/g^2 \simeq W_{gluonium}/W_{q\bar{q}} \ . \quad (2.68)$$

The figures 2.19a,c show the following permissible scale of values φ for the resonances $f_0(980)$, $f_0(1300)$, $f_0(1500)$, $f_0(1750)$, after mixing with the gluonium component.

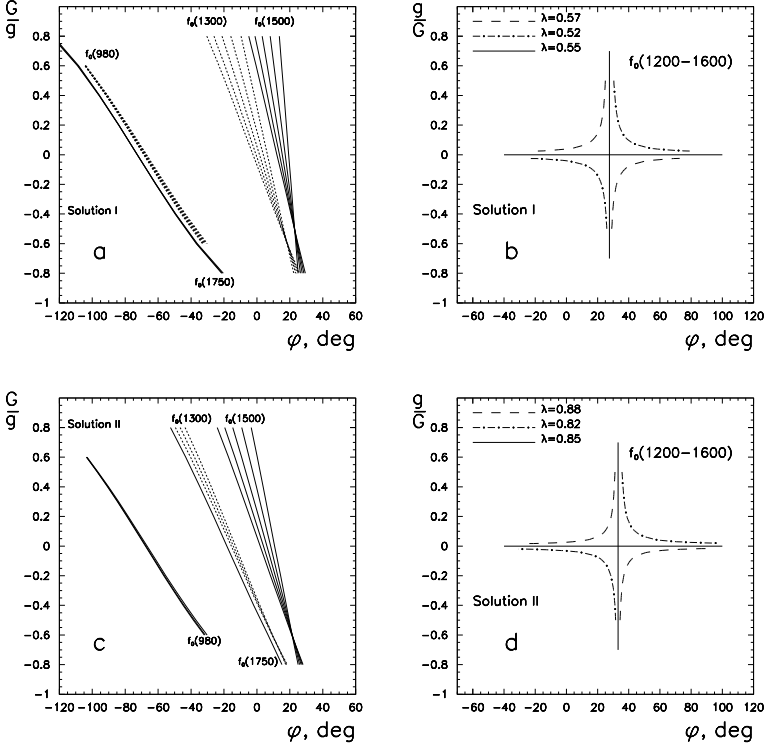


Fig. 2.19 Correlation curves on the $(\varphi, G/g)$ and $(\varphi, g/G)$ plots for the description of the decay couplings of resonances (Table 2.6) in terms of quark-combinatorics relations (38). a,c) Correlation curves for the $q\bar{q}$ -originated resonances: the curves with appropriate λ 's cover strips on the $(\varphi, G/g)$ plane. b,d) Correlation curves for the glueball descendant: the curves at appropriate λ 's form a cross on the $(\varphi, g/G)$ plane with the centre near $\varphi \sim 30^\circ$, $g/G \sim 0$.

Solution I :

$$\begin{aligned}
 W_{gluonium}[f_0(980)] &\lesssim 15\% : & -93^\circ &\lesssim \varphi[f_0(980)] \lesssim -42^\circ, \\
 W_{gluonium}[f_0(1300)] &\lesssim 30\% : & -25^\circ &\lesssim \varphi[f_0(1300)] \lesssim 25^\circ, \\
 W_{gluonium}[f_0(1500)] &\lesssim 30\% : & -2^\circ &\lesssim \varphi[f_0(1500)] \lesssim 25^\circ, \\
 W_{gluonium}[f_0(1750)] &\lesssim 30\% : & -112^\circ &\lesssim \varphi[f_0(1750)] \lesssim -32^\circ.
 \end{aligned}$$

(2.69)

Solution II :

$$\begin{aligned}
W_{gluonium}[f_0(980)] &\lesssim 15\% : & -90^\circ \lesssim \varphi[f_0(980)] &\lesssim -43^\circ, \\
W_{gluonium}[f_0(1300)] &\lesssim 30\% : & -42^\circ \lesssim \varphi[f_0(1300)] &\lesssim 10^\circ, \\
W_{gluonium}[f_0(1500)] &\lesssim 30\% : & -18^\circ \lesssim \varphi[f_0(1500)] &\lesssim 23^\circ, \\
W_{gluonium}[f_0(1750)] &\lesssim 30\% : & -46^\circ \lesssim \varphi[f_0(1750)] &\lesssim 7^\circ. \quad (2.70)
\end{aligned}$$

The φ -dependence of G/g is linear for $f_0(980)$, $f_0(1300)$, $f_0(1500)$, $f_0(1750)$. Another type of correlation takes place for the state which is the glueball descendant: the correlations curves for this case form in the $(\varphi, G/g)$ -plane a typical cross. Just this cross appeared for the broad state $f_0(1200 - 1600)$ for both Solutions I and II, see Fig. 2.19b,d.

The appearance of the glueball cross in the correlation curves on the $(\varphi, G/g)$ -plane is due to the formation mechanism of the quark–antiquark component in the gluonium state: in the transition $gg \rightarrow (q\bar{q})_{glueball}$ the state $(q\bar{q})_{glueball}$ is fixed by the value of λ . So the gluonium descendant is the quarkonium–gluonium composition as follows:

$$\begin{aligned}
&gg \cos \gamma_0 + (q\bar{q})_{glueball} \sin \gamma_0, \\
(q\bar{q})_{glueball} &= n\bar{n} \cos \varphi_{glueball} + s\bar{s} \sin \varphi_{glueball}, \quad (2.71)
\end{aligned}$$

and $\varphi_{glueball} = \tan^{-1} \sqrt{\lambda/2} \simeq 27^\circ - 33^\circ$ for $\lambda \simeq 0.50 - 0.85$. The ratios of couplings for the decay transitions of gluonium $gg \rightarrow \pi\pi, K\bar{K}, \eta\eta, \eta\eta'$ are the same as for the quarkonium $(q\bar{q})_{glueball} \rightarrow \pi\pi, K\bar{K}, \eta\eta, \eta\eta'$, so the study of hadronic decays only do not permit to fix the mixing angle γ_0 . This property – the similarity of hadronic decays for the states gg and $(q\bar{q})_{glueball}$ – implies a specific form of the correlation curve in the $(\varphi, g/G)$ -plane: the gluonium cross. The vertical component of the glueball cross means that the gluonium descendant has a considerable admixture of the quark–antiquark component $(q\bar{q})_{glueball}$. The horizontal line of the cross corresponds to the dominance of the gg component. The value of λ which affects the cross-like correlation on the $(\varphi, g/G)$ -plane is denoted from now on as $\lambda_{glueball}$. For Solution I, we have $\lambda_{glueball} = 0.55$, while for Solution II $\lambda_{glueball} = 0.85$.

If λ is not far from its mean value $\lambda_{glueball}$, the coupling constants $f_0(1200 - 1600) \rightarrow \pi\pi, K\bar{K}, \eta\eta, \eta\eta'$ can be also described, with a reasonable accuracy, by Eq. (2.64); in this case the correlation curves on the $(\varphi, g/G)$ -plane take the form of a hyperbola. Shifting the value of λ in $|\lambda - \lambda_{glueball}| \sim 0.2$ breaks the description of couplings of the broad state by formulae (2.64).

The cross-type correlation on the $(\varphi, g/G)$ -plane in the description of coupling constants $f_0 \rightarrow \pi\pi, K\bar{K}, \eta\eta, \eta\eta'$ is a characteristic signature of the

glueball or glueball descendant. And *vice versa*: the absence of the cross-type correlation should point to the quark–antiquark nature of resonance. Therefore, the K -matrix analysis gives strong arguments in favour glueball nature of $f_0(1200 - 1600)$, while $f_0(980)$, $f_0(1300)$, $f_0(1500)$, $f_0(1750)$ cannot pretend to be the glueballs.

The analysis proves that $f_0(1300)$, $f_0(1500)$ are dominantly the $n\bar{n}$ -systems. Still, in Solution **II** the $q\bar{q}$ component of the resonance $f_0(1300)$ may contain not small $s\bar{s}$ component in the case of the 30% gluonium admixture in this resonance. As to the $f_0(1500)$, the mixing angle $\langle\varphi[f_0(1500)]\rangle$ in the $q\bar{q}$ component may reach 24° at $G/g \simeq -0.6$ (Solution **I**) that is rather close to $\varphi_{glueball}$. However, in this case the description of coupling constants g_a^2 (Table 2.6) is attained as an effect of the strong destructive interference of the amplitudes $(q\bar{q}) \rightarrow \text{two pseudoscalars}$ and $gg \rightarrow \text{two pseudoscalars}$. This fact tells us that one cannot be tempted to interpret $f_0(1500)$ as the gluonium descendant.

2.6.2.4 The $f_0(980)$ and $a_0(980)$: are they the quark–antiquark states?

The nature of mesons $f_0(980)$ and $a_0(980)$ is of principal importance for the systematics of scalar states and the search for exotic mesons. This is precisely why, up to now, there is a lively discussion about the problem of whether the mesons $f_0(980)$ and $a_0(980)$ are the lightest scalar quark–antiquark particles or whether they are exotics, like four-quark ($q\bar{q}q\bar{q}$) states [47], the $K\bar{K}$ molecule [48] or minions [49]. An opposite opinion favouring the $q\bar{q}$ structure of $f_0(980)$ and $a_0(980)$ was expressed in [10, 50, 51].

The K -matrix analysis and the systematisation of scalar mesons on the (n, M^2) -plane, discussed above, give arguments favouring the opinion that $f_0(980)$ and $a_0(980)$ are dominantly $q\bar{q}$ states, with some (10 – 20%) admixture of the $K\bar{K}$ loosely bound component. There exist other arguments both qualitative and based on the calculation of certain reactions that support this idea too.

First, let us discuss qualitative arguments.

i) In hadronic reactions, the resonances $f_0(980)$ and $a_0(980)$ are produced as standard, non-exotic resonances, with compatible yields and similar distributions. This phenomenon was observed in the central meson production at high energy hadron–hadron collisions (data of GAMS [52] and Omega [53] collaborations) or hadronic decays of Z^0 mesons (OPAL collaboration [54]).

ii) The exotic nature of $f_0(980)$ and $a_0(980)$ was often discussed relying on the surprising proximity of their masses, while it would be natural to expect the variation of masses in the nonet to be of the order of 100–200 MeV. Note that the Breit–Wigner resonance pole, which determines the true mass of the state, is rather sensitive to a small admixture of hadron components, if the production threshold for these hadrons lays nearby. As to $f_0(980)$ and $a_0(980)$, it is easy to see that a small admixture of the $K\bar{K}$ component shifts the pole to the $K\bar{K}$ threshold independently of whether the pole is above or below the threshold. Besides, the peak observed in the main mode of the $f_0(980)$ and $a_0(980)$ decays, $f_0(980) \rightarrow \pi\pi$ and $a_0(980) \rightarrow \eta\pi$, is always slightly below the $K\bar{K}$ threshold: this mimics a Breit–Wigner resonance with a mass below 1000 MeV ($K\bar{K}$ threshold). This imitation of a resonance has created the legend about the "surprising proximity" of the $f_0(980)$ and $a_0(980)$ masses.

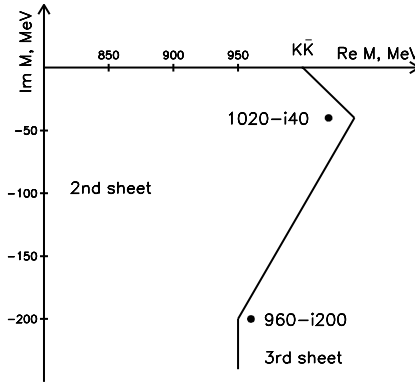


Fig. 2.20 Complex- M plane and location of two poles corresponding to $f_0(980)$; the cut related to the $K\bar{K}$ threshold is shown as a broken line.

In fact, the mesons $f_0(980)$ and $a_0(980)$ are characterised not by one pole, as in the Breit–Wigner case, but by two poles (see Fig. 2.20) as in the Flatté formula [55] or K -matrix approach; these poles are rather different for $f_0(980)$ and $a_0(980)$ [7, 9]. Note that the Flatté formula is not precise in description of spectra near these poles. So we should apply either more complicated representation of the amplitude [35, 56] or the K -matrix approach [7, 9, 10], see also [57].

In parallel with the above-mentioned qualitative considerations, there exist convincing arguments which favour the quark–antiquark nature of $f_0(980)$ and $a_0(980)$:

(I) Hadronic decays of the D_s^+ -meson make it possible to perform a combined analysis of $D_s^+ \rightarrow \pi^+ f_0(980) \rightarrow \pi^+ \pi^+ \pi^-$ and $D_s^+ \rightarrow \pi^+ \phi(1020)$. On the quark level, the dominant process in these decays is a weak transition $c \rightarrow \pi^+ s$ that leads to the following transformations $D_s^+ \rightarrow c\bar{s} \rightarrow \pi^+ s\bar{s} \rightarrow \pi^+ f_0(980)$ and $D_s^+ \rightarrow c\bar{s} \rightarrow \pi^+ s\bar{s} \rightarrow \pi^+ \phi(1020)$, see Fig. 2.21a,b. The comparison of these decays provides the possibility to estimate the $s\bar{s}$ component in $f_0(980)$. Our analysis [58] showed that $2/3$ of $s\bar{s}$ is contained in $f_0(980)$. This estimate is supported by the experimental value: $BR(\pi^+ f_0(980)) = 57\% \pm 9\%$, and $1/3$ $s\bar{s}$ is dispersed over the resonances $f_0(1300)$, $f_0(1500)$, $f_0(1200 - 1600)$. So the reaction $D_s^+ \rightarrow \pi^+ f_0$ is a measure of the $1^3P_0 s\bar{s}$ component in the f_0 mesons, it definitely tells us about the dominance of the $s\bar{s}$ component in $f_0(980)$, in accordance with results of the K -matrix analysis. The conclusion about the dominance of the $s\bar{s}$ component in $f_0(980)$ was also made in the analysis of the decay $D_s^+ \rightarrow \pi^+ \pi^+ \pi^-$ in [59, 60, 61].

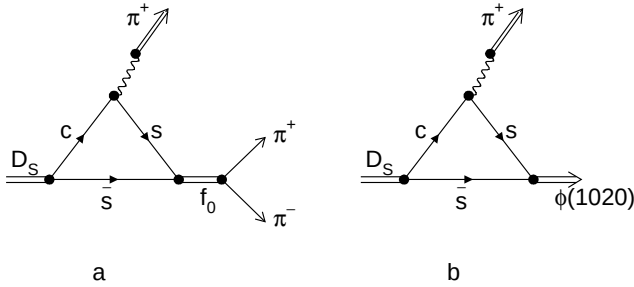


Fig. 2.21 Processes $D_s^+ \rightarrow c\bar{s} \rightarrow \pi^+ s\bar{s} \rightarrow \pi^+ f_0(980)$ and $D_s^+ \rightarrow c\bar{s} \rightarrow \pi^+ s\bar{s} \rightarrow \pi^+ \phi(1020)$ in the quark model.

(II) Radiative decays $f_0(980) \rightarrow \gamma\gamma$, $a_0(980) \rightarrow \gamma\gamma$ agree well with the calculations [62] based on the assumption of the quark-antiquark nature of these mesons. Let us emphasise again that the calculations favour the $s\bar{s}$ dominance in $f_0(980)$.

(III) The radiative decay $\phi(1020) \rightarrow \gamma f_0(980)$ was the subject of vivid discussions in the past years: there existed an opinion that the observed partial width for this decay, being too large, strongly contradicts the hypothesis of $q\bar{q}$ nature of $f_0(980)$ [63, 64, 65]. Indeed, the small value of the "visible" mass difference $(980 \text{ MeV} - M_\phi) \simeq 40 \text{ MeV}$ aroused the suspicion that

$f_0(980)$, if it is a $q\bar{q}$ state, should obey the Siegert theorem [66] for the dipole transition (the amplitude contains the factor $(M_{\phi(1020)} - M_{f_0(980)})$), and the corresponding partial width has to be small. However, as it is seen from Fig. 2.20, $f_0(980)$ is characterised by two poles in the complex- M plane that makes the theorem [66] inapplicable to the reaction $\phi(1020) \rightarrow \gamma f_0(980)$ [56].

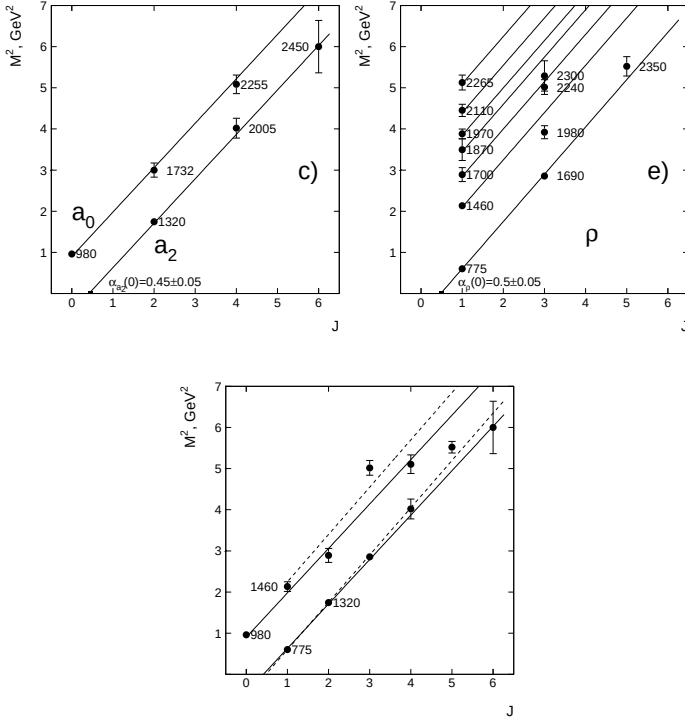


Fig. 2.22 The ρ_J and a_J trajectories on the (J, M^2) -plane; the $a_0(980)$ is on the first daughter trajectory. The right-hand plot is a combined presentation of ρ_J and a_J trajectories: if $a_0(980)$ was not a $q\bar{q}$ state, there should be another a_0 in the mass region ~ 1000 MeV.

(IV) A convincing argument in favour of the $q\bar{q}$ origin of $a_0(980)$ is given by considering (J, M^2) -planes for isovector states. In Fig. 2.22, the leading and daughter ρ_J and a_J trajectories are shown. The $a_0(980)$ is located on the first daughter trajectory. Since the ρ_J and a_J trajectories are degenerate, the right-hand side (J, M^2) -plot demonstrates the combined

presentation of low-lying trajectories: one can see that the a_0 state is definitely needed near 1000 MeV. Had $a_0(980)$ been considered as exotics and removed from the (J, M^2) -plane, the (J, M^2) -trajectories would definitely demand another a_0 state in this mass region. However, near 1000 MeV we have only one state, the $a_0(980)$.

2.6.2.5 *The light σ -meson: Is there a pole of the 00^{++} -wave amplitude?*

An effective σ -meson is needed in nuclear physics as well as in effective theories of the low-energy strong interactions — and such an object exists in the sense that there exists a rather strong interaction, which is realised by the scattering phase passing through the value $\delta_0^0 = 90^\circ$ at $M_{\pi\pi} \equiv \sqrt{s} \simeq 600 - 1000$ MeV. In the naive Breit–Wigner-resonance interpretation, this would correspond to an amplitude pole; but the low-energy $\pi\pi$ amplitude is a result of the interplay of singular contributions of different kind (left-hand cuts as well as poles located highly, $f_0(1200 - 1600)$ included), so a straightforward interpretation of the σ -meson as a pole may fail.

The question is whether the σ -meson exists as a pole of the 00^{++} -wave amplitude [31, 67]. However, until now there is no definite answer to this question, though this point is crucial for meson systematics.

The consideration of the partial S -wave $\pi\pi$ amplitude, by accounting for left singularities associated with the t - and u -channel interactions, favours the idea of the pole at $\text{Re } s \sim 4m_\pi^2$. The arguments are based on the analytical continuation of the K -matrix solution to the region $s \sim 0 - 4\mu_\pi^2$ [31].

In [31], the $\pi\pi$ -amplitude of the 00^{++} partial wave was considered in the region $\sqrt{s} < 950$ MeV. The fit was performed to the low-energy scattering phases, δ_0^0 , at $\sqrt{s} < 450$ MeV, and the scattering length, a_0^0 . In addition at $450 \leq \sqrt{s} \leq 950$ MeV the value δ_0^0 was sewn with those found in the K -matrix analysis [9]: from this point of view the solution found in [31] may be treated as an analytical continuation of the K -matrix amplitude to the region $s \sim 0 - 4m_\pi^2$. The analytical continuation of the K -matrix amplitude of such a type accompanied by the simultaneous reconstruction of the left-hand cut contribution provided us with the characteristics of the amplitude as follows. The amplitude has a pole at

$$\sqrt{s} \simeq 430 - i325 \text{ MeV}, \quad (2.72)$$

the scattering length,

$$a_0^0 \simeq 0.22 m_\pi^{-1}, \quad (2.73)$$

and the Adler zero at

$$\sqrt{s} \simeq 50 \text{ MeV} . \quad (2.74)$$

The errors in the definition of the pole in solution (2.72) are large, and unfortunately they are poorly controlled, for they are governed mainly by uncertainties when left-hand singularities are restored. As to the experimental data, the position of a pole is rather sensitive to the scattering length value, which in the fit [31] was taken in accordance with the paper [68]: $a_0^0 = (0.26 \pm 0.06) m_\pi^{-1}$. As one can see, the solution [31] requires a small scattering length value: $a_0^0 \simeq 0.22 m_\pi^{-1}$. New and much more precise measurements of the K_{e4} -decay [69] provided $a_0^0 = (0.228 \pm 0.015) m_\pi^{-1}$, that agrees completely with the value (2.73) obtained in [31]. Such a coincidence favours undoubtedly the pole position (2.72).

So, the N/D -analysis of the low-energy $\pi\pi$ -amplitude matching to the K -matrix one [9], provides us with the arguments for the existence of the light σ -meson. In a set of papers, by modeling the left-hand cut of the $\pi\pi$ -amplitude (namely, by using interaction forces or the t - and u -channel exchanges), the light σ -meson had been also obtained [70, 71, 72], but the mass values are widely dispersed, *e.g.* in [73] the pole was seen at essentially larger masses, $\sqrt{s} \sim 600 - 900 \text{ MeV}$.

Let us make an important remark about sigma-singularity. We approximate it by a single pole, and obtain its position in the complex- M region near $439 - i325 \text{ MeV}$, see (2.72). But, strictly speaking, our analysis cannot state definitely that there is a single pole in this region. It is possible that the sigma-singularity is a group of poles but the absence of precise data do not allow us to resolve these singularities.

2.6.2.6 *The σ as the white component of the confinement singularity*

It was suggested in [30] that the existence of the light σ -meson may be due to a singular behaviour of forces between the quark and the antiquark at large distances; (in quark models they are conventionally called “confinement forces”). The scalar confinement potential, which is needed for the description of the spectrum of the $q\bar{q}$ -states in the region 1000–2000 MeV, behaves at large hadronic distances as $V_{\text{confinement}}^{(c)}(r) \sim \alpha r$, where $\alpha \simeq 0.15 \text{ GeV}^2$. In the momentum representation such a growth of the potential is associated with a singular behaviour at small q :

$$V_{\text{confinement}}^{(c)}(q) \sim q^{-4} . \quad (2.75)$$

In colour space the main contribution comes from the component $c = 8$, *i.e.* the confinement forces should be the octet ones. The crucial question for the structure of the σ -meson is whether there is a component with a colour singlet $V_{confinement}^{(1)}(q)$ in the singular potential (2.75).

If the singular component with $c = 1$ exists, it must reveal itself in hadronic channels, that is, in the $\pi\pi$ -channel as well. In hadronic channels this singularity should not be exactly the same as in the colour octet ones, because the hadronic unitarisation of the amplitude (which is absent in the channel with $c = 8$) should modify somehow the low-energy amplitude. One may believe that, as a result of the unitarisation in the channel $c = 1$, *i.e.* due to the account of hadronic rescattering, the singularity of $V_{confinement}^{(1)}(q)$ may appear in the $\pi\pi$ -amplitude on the second sheet, being split into several poles. The modelling of the scalar confinement potential, taking into account the decay of unstable levels [74], confirms the pole splitting. Thus, we may think that this singularity is what we call the “light σ -meson”.

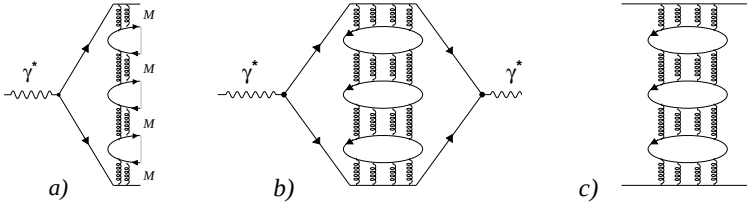


Fig. 2.23 a) Quark–gluonic comb produced by breaking a string by quarks flowing out in the process $e^+e^- \rightarrow \gamma^* \rightarrow q\bar{q} \rightarrow \text{mesons}$. b) Convolution of the quark–gluonic combs. c) Example of diagrams describing interaction forces in the $q\bar{q}$ systems.

Therefore, the main question is the following: does the $V_{confinement}^{(1)}(q^2)$ have the same singular behaviour as $V_{confinement}^{(8)}(q^2)$? The observed linearity of the (n, M^2) -trajectories, up to the large-mass region, $M \sim 2000\text{--}2500$ MeV [6], favours the idea of the universality in the behaviour of potentials $V_{confinement}^{(1)}$ and $V_{confinement}^{(8)}$ at large r , or small q . To see that (for example, in the process $\gamma^* \rightarrow q\bar{q}$, Fig. 2.23a) let us discuss the colour neutralisation mechanism of outgoing quarks as a breaking of the gluonic string by newly born $q\bar{q}$ -pairs. At large distances, which correspond to the formation of states with large masses, several new $q\bar{q}$ -pairs should be formed. It is natural to suggest that a convolution of the quark–gluon combs governs the interaction forces of quarks at large distances, see

Fig. 2.23b. The mechanism of the formation of new $q\bar{q}$ -pairs to neutralise colour charges does not have a selected colour component. In this case, all colour components $3 \otimes \bar{3} = 1 + 8$ behave similarly, that is, at small q^2 the singlet and octet components of the potential are uniformly singular, $V_{\text{confinement}}^{(1)}(q^2) \sim V_{\text{confinement}}^{(8)}(q^2) \sim 1/q^4$.

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Chapter 3

Elements of the Scattering Theory

This chapter does not aim a full description of the scattering theory. It is devoted to some key topics which are important for the analysis of the analytical structure of amplitudes. At present, the study of their analytical properties is dictated by the necessities of the experiments: the discovery and investigation of the new particles are based mainly on the study of leading singularities of the amplitudes. This is just the reason why we fix our attention mainly on the analytical properties of the amplitudes.

A systematic presentation of the problems of scattering theory which are touched here can be found in various textbooks and monographs, see, for example, refs. [1, 2, 3, 4, 5, 6]. Certain special problems considered here are based on the original articles we refer to.

3.1 Scattering in Quantum Mechanics

We start with the non-relativistic scattering theory discussed in the framework of quantum mechanics. Quantum mechanics provides a good basis for defining notions and outlining problems of the scattering theory.

3.1.1 *Schrödinger equation and the wave function of two scattering particles*

Let us consider the elastic scattering of spinless particles. In this case, particles do not change their internal states and new particles are not produced. We suppose also that interaction forces are spherically symmetric: they depend only on the distance between scattering particles.

In quantum mechanics, the problem of two-particle elastic scattering can be reformulated as a problem of scattering of one particle in the $V(r)$

field. This is done by considering two-particle scattering in the centre-of-mass frame (the centre of inertia of two particles). The Hamiltonian of the two interacting particles is:

$$\hat{H} = -\frac{1}{2m_1}\Delta_1 - \frac{1}{2m_2}\Delta_2 + V(r) . \quad (3.1)$$

Here m_i are masses of particles 1 and 2, Δ_1 and Δ_2 are Laplace operators for the coordinates, $\Delta_i = \partial^2/\partial^2 x_i + \partial^2/\partial^2 y_i + \partial^2/\partial^2 z_i$, and $V(r)$ is the interaction potential depending on the distance between particles:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 . \quad (3.2)$$

Let us introduce the coordinates of the centre of inertia for these two particles:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} , \quad (3.3)$$

The Hamiltonian written as a function of variables $\mathbf{r}$ and $\mathbf{R}$ equals

$$\hat{H} = -\frac{1}{2(m_1 + m_2)}\Delta_R - \frac{1}{2m}\Delta + V(r) , \quad (3.4)$$

where $m = m_1 m_2 / (m_1 + m_2)$. So the Hamiltonian is reduced to the sum of two independent terms, one standing for the free motion of the centre of mass, the other for the interaction of particles. The latter is equivalent to the Hamiltonian of a particle with a reduced mass m moving in the field of the potential $V(r)$. Thus, the wave function written for the two particles, $\psi(\mathbf{r}_1, \mathbf{r}_2)$, can be presented in a factorised form:

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \phi(\mathbf{R})\psi(\mathbf{r}). \quad (3.5)$$

The wave function $\phi(\mathbf{R})$ describes the centre-of-mass motion (the free motion of a particle with mass $m_1 + m_2$), while $\psi(\mathbf{r})$ describes the relative motion of particles 1 and 2 (the motion of the particle with mass m in the centrally symmetrical field $V(r)$).

The Schrödinger equation for $\psi(\mathbf{r})$ reads:

$$\left[\frac{\Delta}{2m} + V(r) \right] \psi(\mathbf{r}) = E\psi(\mathbf{r}) , \quad (3.6)$$

E is the energy of relative motion. The same equation written in spherical coordinates is

$$\begin{aligned} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2} \right] \\ + 2m [E - V(r)] \psi = 0 . \end{aligned} \quad (3.7)$$

Introducing the operators $\hat{l}^2$ and $\hat{p}_r$,

$$\begin{aligned} \frac{1}{2m} \left[-\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{\hat{l}^2}{r^2} \psi \right] + V(r) \psi &= E \psi , \\ \hat{p}_r \psi &= -i \frac{1}{r} \frac{\partial}{\partial r} (r \psi) = -i \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \psi , \end{aligned} \quad (3.8)$$

the Hamiltonian is written as

$$\hat{H} = -\frac{1}{2m} \left(\hat{p}_r^2 + \frac{\hat{l}^2}{r^2} \right) + V(r) . \quad (3.9)$$

The function $\psi(\mathbf{r})$ is a product

$$\psi = R(r) Y_{l\mu}(\theta, \varphi) , \quad (3.10)$$

where $Y_{l\mu}$ is the eigenfunction of $\hat{l}^2$ and $\hat{l}_z = -i \partial/\partial \varphi$:

$$\hat{l}^2 Y_{l\mu}(\theta, \varphi) = l(l+1) Y_{l\mu}(\theta, \varphi) . \quad (3.11)$$

The radial wave function $R(r)$ obeys the equation

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR(r)}{dr} \right) - \frac{l(l+1)}{r^2} R(r) + 2m[E - V(r)] R(r) = 0 . \quad (3.12)$$

The wave function ψ is a function of the energy E , the total angular momentum l and its projection μ . The normalisation condition for $R(r)$ is defined by the integral

$$\int_0^\infty |R(r)|^2 r^2 dr = 1 . \quad (3.13)$$

At large distance, $r \rightarrow \infty$, we can neglect $V(r)$. The equation (3.12) reads

$$\frac{1}{r} \frac{d^2(rR)}{dr^2} + k^2 R = 0 , \quad (3.14)$$

where $k = \sqrt{2mE}$. At large r , the general solution of Eq. (3.14) can be written in the form

$$R(r) \approx \sqrt{\frac{2}{\pi}} \frac{\sin(kr - l\pi/2 + \delta_l)}{r} . \quad (3.15)$$

Here δ_l is a constant which is defined by the behaviour of the radial function $R(r)$ at a comparatively small r , where $V(r)$ is not negligible; this is the so-called phase shift.

3.1.2 Scattering process

Let us reformulate now the scattering of two particles as a scattering of one particle in the stationary field $V(r)$. We do this in the c.m.s. of particles 1 and 2, where the total momentum is zero, $\mathbf{P}(1+2) = 0$. Thus, the wave function of the centre-of-mass motion can be chosen to be unity

$$\phi(\mathbf{R}) = 1. \quad (3.16)$$

In the c.m.s. the two-particle wave function is determined by $\psi(\mathbf{r})$ only:

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r}). \quad (3.17)$$

Just this wave function, $\psi(\mathbf{r})$, gives us the necessary reformulation. Let a free particle before scattering be moving along the z -axis. It is described by a plane wave

$$e^{ikz}, \quad (3.18)$$

while an outgoing particle after scattering is described, at asymptotically large distances, by the spherical outgoing wave

$$f(\theta) \frac{e^{ikr}}{r}. \quad (3.19)$$

Here $f(\theta)$ is the scattering amplitude which depends on the polar scattering angle θ . So, the wave function, being a solution of Eq. (3.6) and describing a scattering process, has the following asymptotic form at large r (see Fig. 3.1):

$$\psi(r) \simeq e^{ikz} + \frac{f(\theta)}{r} e^{ikr}. \quad (3.20)$$

The scattering amplitude $f(\theta)$ is completely determined by the phase shifts δ_ℓ .

3.1.3 Free motion: plane waves and spherical waves

The wave function

$$\psi_{\mathbf{k}}(r) = \text{const} \cdot e^{i\mathbf{k}\mathbf{r}} \quad (3.21)$$

describes the state with momentum $\mathbf{k}$ (and energy $E = k^2/2m$).

A state with orbital momentum ℓ and its projection μ is characterized by

$$\psi_{k\ell\mu} = R_{k\ell}(r)Y_{\ell\mu}(\theta, \varphi). \quad (3.22)$$

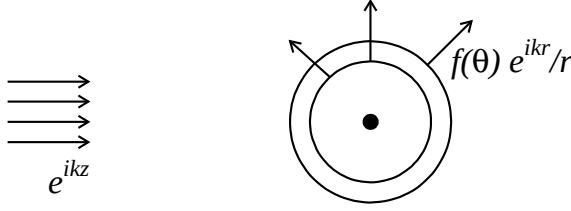


Fig. 3.1 Plane waves and outgoing waves.

The radial wave function is determined by the equation

$$R''_{k\ell} + \frac{2}{r} R'_{k\ell} + \left[k^2 - \frac{\ell(\ell+1)}{r^2} \right] R_{k\ell} = 0, \quad (3.23)$$

where $\psi_{k\ell\mu}$ and $R_{k\ell}$ obey the normalisation conditions:

$$\begin{aligned} \int dV \psi_{k'\ell'\mu'}^* \psi_{k\ell\mu} &= \delta_{\ell\ell'} \delta_{\mu\mu'} \delta(k' - k), \\ \int_0^\infty dr r^2 R_{k'\ell} R_{k\ell} &= \delta(k' - k). \end{aligned} \quad (3.24)$$

The solution of Eq. (3.23) finite at $r \rightarrow 0$ is

$$R_{k\ell}(r) = (-1)^\ell \sqrt{\frac{2}{\pi}} \frac{r^\ell}{k^\ell} \left(\frac{d}{r dr} \right)^\ell \frac{\sin kr}{r}. \quad (3.25)$$

The plane wave can be presented as a series with respect to the functions $R_{k\ell}$:

$$e^{ikz} = \sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) P_\ell(\cos \theta) R_{k\ell}(r). \quad (3.26)$$

Here $kz = kr \cos \theta$; at $r \rightarrow \infty$

$$e^{ikz} \approx \frac{1}{kr} \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) P_\ell(\cos \theta) \sin \left(kr - \frac{l\pi}{2} \right). \quad (3.27)$$

3.1.4 Scattering process: cross section, partial wave expansion and phase shifts

The asymptotic expression for the wave function

$$\psi(\mathbf{r}) = e^{ikz} + \frac{f(\theta)}{r} e^{ikr} \quad (3.28)$$

describes the flux of incoming particles with the density

$$v|\psi_{in}|^2 = v|e^{ikz}|^2 = v \quad (3.29)$$

and the flux of outgoing particles. The probability for the scattered particle to pass an element of the surface $dS = r^2 d\Omega$ in a unit of time is equal to

$$v|\psi_{out}|^2 d\Omega = v|f(\theta)|^2 d\Omega, \quad (3.30)$$

and its ratio to the flux of the incoming particles is the cross section:

$$d\sigma = |f(\theta)|^2 d\Omega. \quad (3.31)$$

If in Eq. (3.31) the integration over $d\varphi$ is performed using azimuthal symmetry, then $d\Omega = 2\pi \sin\theta d\theta$. This is the cross section for the scattering in the angular interval $(\theta, \theta + d\theta)$:

$$d\sigma = 2\pi |f(\theta)|^2 \sin\theta d\theta. \quad (3.32)$$

Let us express now the scattering amplitude $f(\theta)$ in terms of the phase shift. The wave function $\psi(\mathbf{r})$ satisfies Eq. (1.8). At large r , the solution of this equation is

$$R_\ell(r) \approx \frac{a_\ell}{r} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right), \quad (3.33)$$

(see Eq. (3.15)). So the general form of the asymptotical wave function can be written as a series in R_ℓ defined by Eq. (3.33):

$$\begin{aligned} \psi &\simeq \sum_{\ell=0}^{\infty} (2\ell+1) A_\ell P_\ell(\cos\theta) \frac{1}{kr} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right) \\ &= \sum_{\ell=0}^{\infty} (2\ell+1) A_\ell P_\ell(\cos\theta) \frac{i}{2kr} \left\{ \exp\left[-i\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)\right] \right. \\ &\quad \left. - \exp\left[i\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)\right] \right\}. \end{aligned} \quad (3.34)$$

The coefficients should be chosen in such a way that $\psi(\mathbf{r})$ has an asymptotic form given by Eq. (3.28). In other words, at large r the expression $\psi(\mathbf{r}) - e^{ikz}$ should contain outgoing waves only. Comparing Eqs. (3.27) and (3.34), one gets the following values for A_ℓ :

$$A_\ell = i^\ell e^{i\delta_\ell}. \quad (3.35)$$

Thus, the wave function of Eq. (3.34) has the following asymptotical form:

$$\psi \simeq \frac{1}{2ikr} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta) [(-1)^{\ell+1} e^{-ikr} + e^{2i\delta_\ell} e^{ikr}]. \quad (3.36)$$

For $\delta_\ell = 0$ there is no scattering, and the right-hand side of Eq. (3.36) turns into $\exp(ikz)$. The elastic scattering does not alter the probability of outgoing particles, $|e^{2i\delta_\ell}| = 1$; the outgoing wave changes its phase only.

The equation (3.36) gives us the following expression for the scattering amplitude:

$$f(\theta) = \frac{1}{2ik} \sum (2\ell + 1)(e^{2i\delta_\ell} - 1)P_\ell(\cos \theta). \quad (3.37)$$

Partial wave amplitudes are defined as

$$f_\ell = \frac{1}{2ik}(e^{2i\delta_\ell} - 1). \quad (3.38)$$

Here $e^{2i\delta_\ell}$ is an element of the S -matrix:

$$S_\ell = e^{2i\delta_\ell}. \quad (3.39)$$

$\hat{S}$ is a unitarity operator:

$$\hat{S}\hat{S}^+ = 1. \quad (3.40)$$

This unitarity condition reflects the fact that the number of particles in elastic scattering is conserved.

The unitarity condition for the partial amplitude reads as follows:

$$\text{Im } f_\ell = k f_\ell^* f_\ell. \quad (3.41)$$

In field theory another normalisation condition is used for the scattering amplitude:

$$A_\ell = \frac{1}{2i\rho}(e^{2i\delta_\ell} - 1), \quad \rho = \frac{k}{8\pi(\sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2})}; \quad (3.42)$$

ρ is the invariant two-particle phase space factor. Then

$$\text{Im } A_\ell = \rho |A_\ell|^2. \quad (3.43)$$

3.1.5 *K-matrix representation, scattering length approximation and the Breit-Wigner resonances*

The partial wave amplitude can be represented in the K -matrix form. This representation is rather useful because it reproduces correctly the singularities of the amplitude related to the two-particle rescatterings.

Let us introduce the partial wave amplitude in the following way:

$$T_\ell = \frac{1}{2i}(e^{2i\delta_\ell} - 1) = e^{i\delta_\ell} \sin \delta_\ell. \quad (3.44)$$

The K -matrix form of the T_ℓ -amplitude is

$$T_\ell = \frac{K_\ell}{1 - iK_\ell}, \quad K_\ell = \tan \delta_\ell. \quad (3.45)$$

In the physical region ($k^2 > 0$), K is a real function of k^2 , but at $k^2 = 0$ it has the threshold singularity. This singularity can be extracted by the substitution

$$K_\ell = ka_\ell(k^2). \quad (3.46)$$

The form of T_ℓ ,

$$T_\ell = \frac{ka_\ell(k^2)}{1 - ika_\ell(k^2)}, \quad (3.47)$$

is widely used in nuclear physics for the description of the nucleon–nucleon interactions at low energies. Frequently, it is just $a_\ell(k^2)$ which is called the K -matrix. Such a change in the notation is useful when considering a many-particle amplitude, where the separation of threshold singularities is important. Further, we deal with the amplitude in this way.

(i) **Scattering length approximation**

The scattering length approximation corresponds to

$$\ell = 0, \quad a_0(k^2) \equiv a = \text{Const}, \quad (3.48)$$

that means a point-like interaction. In this case, the S -wave amplitude reads

$$T_0 = \frac{ka}{1 - ika}. \quad (3.49)$$

On the first (physical) sheet of the complex- k^2 plane, we have at negative k^2 :

$$k = i|k|, \quad (3.50)$$

and the amplitude (3.49) has a pole at $a < 0$. It is a deuteron-type pole corresponding to a loosely bound composite system.

At $a > 0$ the deuteron-type pole is absent, but there exists a virtual level, and the pole is located on the second sheet of the complex- k^2 plane (we have on the second sheet $k = -i|k|$ at $k^2 < 0$).

(ii) **The Breit–Wigner resonances**

Near the elastic threshold, the partial scattering length behaves as

$$a_\ell(k^2) \sim k^{2\ell}. \quad (3.51)$$

This can be easily seen by considering the angular momentum expansion of the amplitude (this expansion is discussed in detail in Chapter 4).

The Breit–Wigner resonance corresponds to a pole of the a_ℓ -amplitude:

$$a_\ell = \frac{g_\ell^2 k^{2\ell}}{k_0^2 - k^2}, \quad (3.52)$$

where g_ℓ^2 is a constant. If so,

$$T_\ell = \frac{g_\ell^2 k^{2\ell+1}}{k_0^2 - k^2 - i g_\ell^2 k^{2\ell+1}} = \frac{\gamma_\ell k^{2\ell+1}}{E_0 - E - i \gamma_\ell k^{2\ell+1}}. \quad (3.53)$$

Here $E_0 = k_0^2/2m$, $E = k^2/2m$, and $\gamma_\ell = g_\ell^2/2m$. If the coupling constant is small with E_0 being positive, Eq. (3.53) stands for the Breit–Wigner resonance [7].

3.1.6 Scattering with absorption

Scattering without absorption is described by the wave function in Eq. (3.36): at large r the intensities of incoming and outgoing waves are the same. Absorption means that the intensity of the outgoing wave decreased. Therefore, the scattering with absorption is described by the following wave function

$$\psi \simeq \frac{1}{2ikr} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos \theta) [(-1)^{\ell+1} e^{-ikr} + \eta_\ell e^{2i\delta_\ell} e^{ikr}] , \quad (3.54)$$

where the inelasticity parameter η_ℓ varies within the limits

$$0 \leq \eta_\ell \leq 1. \quad (3.55)$$

The equality $\eta = 0$ corresponds to a full absorption.

The partial amplitudes are equal to

$$T_\ell = \frac{\eta_\ell e^{2i\delta_\ell} - 1}{2i}. \quad (3.56)$$

A complete absorption is related to $T_\ell = i/2$. The value of T_ℓ is imaginary and maximal in the case of the Breit–Wigner resonance at $k^2 = k_0^2$, when $T_\ell = i$.

The k^2 -dependence (or the energy dependence) of T_ℓ can be displayed on the Argand diagram (see Fig. 3.2): the points on the Argand diagram correspond to T_ℓ at different energies.

The unitarity condition for the scattering amplitude (3.56) reads:

$$\text{Im } T_\ell = T_\ell T_\ell^* + \frac{1}{4} (1 - \eta_\ell^2). \quad (3.57)$$

In a graphical form, this unitarity condition is shown in Fig. 3.3: the term $(1 - \eta_\ell^2)/4$ in Eq. (3.57) corresponds to the contribution of inelastic processes to the imaginary part of the scattering amplitude. The first term in the r.h.s. of Eq. (3.57) describes elastic rescattering.

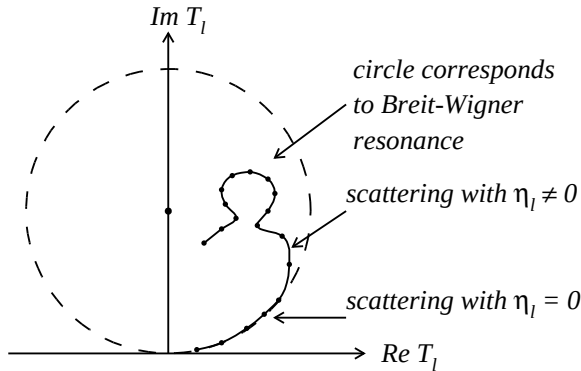


Fig. 3.2 Argand diagram for T_ℓ : the sequence of points gives values of T_ℓ at different and growing energies (or k^2).

$$Im \text{ (circle) } = \text{ (two circles with crosses) } + \Sigma \text{ (two circles with crosses and asterisk) }$$

Fig. 3.3 Unitarity condition for the scattering amplitude. The crosses denote particles in the intermediate states, over the phase volumes of which the integrations are carried out. The asterisk stands for the complex conjugated amplitude.

3.2 Analytical Properties of the Amplitudes

This section is devoted to the discussion of analytical properties of the amplitudes. The extraction of leading singularities of the amplitudes is a standard way of searching for new hadrons (resonances). The study of analytical properties is performed using the language of Green functions and Feynman diagrams.

3.2.1 Propagator function in quantum mechanics: the coordinate representation

To analyse the scattering amplitude, it is convenient to introduce the propagator function or the Green function. The propagator function determines the time evolution of the wave function:

$$\Psi(\mathbf{r}, t) = \int d^3r' K(\mathbf{r}, t; \mathbf{r}', t_0) \Psi(\mathbf{r}', t_0). \quad (3.58)$$

Here $\bar{\Psi}(\mathbf{r}', t_0)$ is the wave function determined at time t_0 ; $K(\mathbf{r}, t; \mathbf{r}', t_0)$ with $t \geq t_0$ is the propagator function. The propagator function has to satisfy

the boundary condition:

$$K(\mathbf{r}, t; \mathbf{r}', t_0)|_{t=t_0} = \delta(\mathbf{r} - \mathbf{r}'). \quad (3.59)$$

The propagator function allows us to find the wave function at any time t if the initial wave function at time t_0 is known. ($t_0 < t$). This means that the function K determines the scattering amplitude $f(\theta)$.

The function K can be constructed if a full set of wave functions Ψ_n , which satisfy the Schrödinger equation (3.6), is known. Then,

$$K(\mathbf{r}, t; \mathbf{r}', t_0) = \sum_n \Psi_n(\mathbf{r}, t) \Psi_n^*(\mathbf{r}', t_0), \quad (3.60)$$

where

$$\Psi_n(\mathbf{r}, t) = \psi_n(\mathbf{r}) e^{-iE_n t}, \quad (3.61)$$

and summation is performed over all eigenstates. The boundary condition (3.59) is equivalent to the completeness condition of the set of wave functions used:

$$\sum_n \psi_n(\mathbf{r}) \psi_n^*(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (3.62)$$

In the scattering process, we deal with a continuous spectrum of states; therefore, the summation over n should be replaced by the integration over states of the continuous spectrum. The interval d^3k contains $d^3k/(2\pi)^3$ quantum states, so we have to replace in (3.62):

$$\sum_n \longrightarrow \int \frac{d^3k}{(2\pi)^3}. \quad (3.63)$$

Let us consider in detail the propagation function of a free particle described by the plane wave:

$$\Psi_k(\mathbf{r}, t) = e^{i\mathbf{k}\mathbf{r} - i(\mathbf{k}^2/2m)t}. \quad (3.64)$$

Then

$$K_0(\mathbf{r}, t; 0, 0) = \int \frac{d^3k}{(2\pi)^3} \exp \left[i\mathbf{k}\mathbf{r} - i\frac{\mathbf{k}^2}{2m}t \right] = \left(\frac{2m}{i\pi t} \right)^{3/2} \exp \left[i\frac{m\mathbf{r}^2}{2t} \right]. \quad (3.65)$$

It is taken into account here that the free particle propagation function $K_0(\mathbf{r}, t; \mathbf{r}', t_0)$ depends on $\mathbf{r} - \mathbf{r}'$ and $t - t_0$ only, so we can put $\mathbf{r}' = t_0 = 0$.

The propagation function K describes the time evolution of a quantum state at $t > t_0$; it is convenient to use a propagator which is equal to zero at $t < t_0$. This is the Green function

$$G(\mathbf{r}, t; \mathbf{r}', t') = \theta(t - t') K(\mathbf{r}, t; \mathbf{r}', t'). \quad (3.66)$$

Here $\theta(t)$ is the step function: $\theta(t) = 1$ at $t \geq 0$ and $\theta(t) = 0$ at $t < 0$. Since the K -function satisfies the equation

$$\left(i \frac{\partial}{\partial t} - \hat{H}(r)\right) K(\mathbf{r}, t; \mathbf{r}', t') = 0$$

$$\hat{H}(r) = -\frac{\Delta}{2m} + V(r), \quad (3.67)$$

the Green function obeys the following equation

$$\left(i \frac{\partial}{\partial t} - \hat{H}(r)\right) G(\mathbf{r}, t; \mathbf{r}', t') = i\delta(t - t')\delta(\mathbf{r} - \mathbf{r}'). \quad (3.68)$$

Here the boundary condition (3.59) is used:

$$K(\mathbf{r}, t; \mathbf{r}', t') \frac{\partial}{\partial t} \theta(t - t') = K(\mathbf{r}, t; \mathbf{r}', t') \delta(t - t') = \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'). \quad (3.69)$$

Likewise, the Green function of a free particle is determined by the function K_0 :

$$G_0(\mathbf{r}, t) = K_0(\mathbf{r}, t; 0, 0)\theta(t). \quad (3.70)$$

If so, equation (3.65) gives us

$$G_0(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^3} \theta(t) \exp \left[i\mathbf{k}\mathbf{r} - i\frac{k^2}{2m}t \right]. \quad (3.71)$$

This expression can be rewritten as an integral over the four-vector $(E, \mathbf{k})$:

$$G_0(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dE}{2\pi i} \frac{1}{-E + (k^2/2m) - i0} e^{i\mathbf{k}\mathbf{r} - iEt}. \quad (3.72)$$

Here the symbol $i0$ is an infinitely small and positive imaginary quantity. For $t > 0$, the contour of integration over E is enclosed by the large circle in the lower half-plane (see Fig. 3.4a): the factor $\exp[-iEt]$ guarantees an infinitesimally small contribution to the integral from this circle. The integral is equal to the residue at the pole $E = k^2/2m$, therefore we can replace at $t > 0$:

$$\left[-E + \frac{k^2}{2m} - i0\right]^{-1} \rightarrow 2\pi i \delta\left(E - \frac{k^2}{2m}\right). \quad (3.73)$$

For $t < 0$, the factor $\exp[-iEt]$ is infinitesimally small on the large circle in the upper half-plane (see Fig. 3.4b): inside the enclosed contour there is no singularity, so the integral (3.72) at $t < 0$ is equal to zero. We see that Eq. (3.71) exactly reproduces the definition of $G_0(\mathbf{r}, t)$ given by Eq. (3.71).

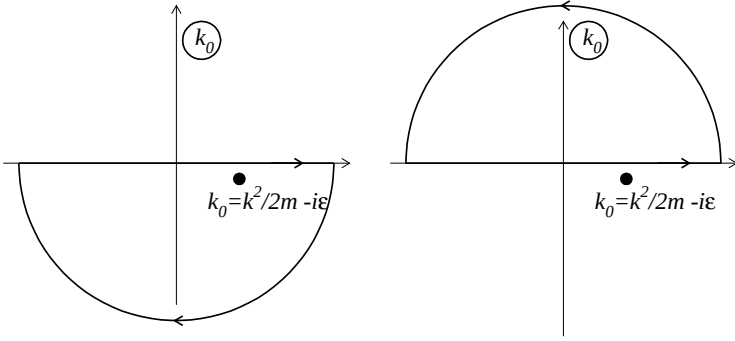


Fig. 3.4 Contours of integration over E in Eq. (3.72) for $t > 0$ and $t < 0$.

It should be pointed out that the factor $[-E + (k^2/2m) - i0]^{-1}$ in Eq. (3.72) is the operator $[-i\partial/\partial t + \hat{H}_0(r)]^{-1}$ in the momentum representation. It is important that the shift of the pole in the complex E plane is determined by the value of $-i0$. This shift suggests the evolution of the quark system in the positive time direction.

The Green function $G(\mathbf{r}, t; \mathbf{r}', t')$ satisfies the following integral equation:

$$G_0(\mathbf{r} - \mathbf{r}', t - t') + \int d^3r'' dt'' G_0(\mathbf{r} - \mathbf{r}'', t - t'') (-i)V(r'') G(\mathbf{r}'', t''; \mathbf{r}', t') = G(\mathbf{r}, t; \mathbf{r}', t'). \quad (3.74)$$

To justify Eq. (3.74), let us apply $\{i(\partial/\partial t) - \hat{H}_0(r)\}$ to Eq. (3.74) where $\hat{H}_0 = -\Delta^2/2m$. As a result, we have

$$\left(i \frac{\partial}{\partial t} - \hat{H}_0\right) G(\mathbf{r}, t; \mathbf{r}', t') = i\delta(\mathbf{r} - \mathbf{r}')\delta(t - t') + \int d^3r'' dt'' \delta(\mathbf{r} - \mathbf{r}'')\delta(t - t'') V(r'') G(\mathbf{r}'', t''; \mathbf{r}', t'). \quad (3.75)$$

After integrating over $d^3r'' dt''$, we arrive at Eq. (3.68).

The equation (3.74) can be written in a graphical form shown in Fig. 3.5: the thin lines correspond to free Green functions, G_0 , while the thick ones correspond to full Green functions, G .

The iteration of Eq. (3.74) demonstrates that the full Green function is an infinite set of diagrams of the type shown in Fig. 3.6. These diagrams describe the scattering of the effective particle on the field $V(r)$.

$$\overline{r, t} \rightarrow \overline{r_0, t_0} + \overline{r, t} \xrightarrow{V(r')} \overline{r_0, t_0} = \overline{r, t} \rightarrow \overline{r_0, t_0}$$

$\downarrow$
 $\times$
 $\vec{r}', t'$

Fig. 3.5 Graphical form of Eq. (3.74) for Green function.

$$\overline{r, t} \rightarrow \overline{r_0, t_0} = \overline{r, t} \rightarrow \overline{r_0, t_0} + \overline{r, t} \xrightarrow{V(r')} \overline{r_0, t_0} +$$

$$+ \overline{r, t} \xrightarrow{V(r') V(r'')} \overline{r_0, t_0} + \overline{r, t} \xrightarrow{V(r') V(r'') V(r''')} \overline{r_0, t_0} + \dots$$

Fig. 3.6 Full Green function represented as an infinite set of the scattering diagrams.

3.2.2 Propagator function in quantum mechanics: the momentum representation

Let us consider the Green functions in the momentum representation. The free Green function is determined as

$$G_0(k) = i \int d^3 r dt G_0(\mathbf{r}, t) e^{-i\mathbf{k}\mathbf{r} + iEt} = \frac{1}{-E + (\mathbf{k}^2/2m) - i0}. \quad (3.76)$$

The full Green function depends on two four-momenta:

$$G(k, p) = i \int d^3 r dt \int d^3 r' dt' G(\mathbf{r}, t; \mathbf{r}', t') \exp[-i\mathbf{k}\mathbf{r} + iEt] \exp[i\mathbf{p}\mathbf{r}' - iE_p t']. \quad (3.77)$$

Equation (3.74) for the Green function is rewritten in the momentum representation as follows:

$$G(k, p) = (2\pi)^4 \delta^{(4)}(k - p) G_0(k) - G_0(k) \int \frac{d^4 k'}{(2\pi)^4} V(k - k') G(k', p). \quad (3.78)$$

Here the potential V in the momentum representation is defined as

$$V(q) = \int d^3 r dt V(\mathbf{r}, t) e^{-i\mathbf{q}\mathbf{r} + iq_0 t}. \quad (3.79)$$

If $V(r)$ does not depend on t , then

$$V(q) = 2\pi\delta(q_0) \int d^3r V(\mathbf{r})e^{-i\mathbf{q}\mathbf{r}} = 2\pi\delta(q_0) V(\mathbf{q}) . \quad (3.80)$$

The iteration of Eq. (3.78) leads to the representation of $G(k, p)$ in a series over V :

$$\begin{aligned} G(k, p) &= (2\pi)^4 \delta^{(4)}(k - p) G_0(k) - G_0(k) V(k - p) G_0(p) \\ &\quad + G_0(k) \int \frac{d^4 k'}{(2\pi)^4} V(k - k') G_0(k') V(k' - p) G_0(p) \\ &\quad - G_0(k) \int \frac{d^4 k'}{(2\pi)^4} \frac{d^4 k''}{(2\pi)^4} V(k - k') G_0(k') V(k' - k'') G_0(k'') V(k'' - p) G_0(p) + \dots \end{aligned} \quad (3.81)$$

The formula (3.81) corresponds to the set of diagrams shown in Fig. 3.7. These are Feynman diagrams for the scattering of non-relativistic particle in the field V .

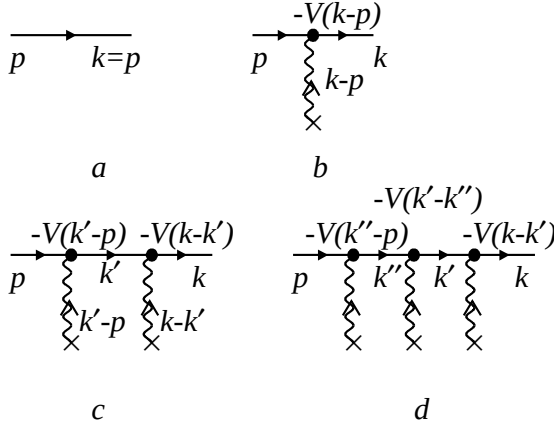


Fig. 3.7 Scattering diagrams for the full Green function in the momentum representation.

The scattering amplitude $f(\theta)$ introduced in Eq. (3.20) is determined by the Green function via the relation

$$G(k, p) = (2\pi)^4 \delta^{(4)}(k - p) G_0(k) + G_0(k) \frac{(2\pi)^2}{m} \delta(E - E_p) f(k, p) G_0(p). \quad (3.82)$$

Here we redenote $f(\theta)$ as $f(k, p)$, namely, $f(\theta) \equiv f(k, p)$.

3.2.3 Equation for the scattering amplitude $f(k, p)$

One can write an equation directly for the amplitude $f(k, p)$ keeping in mind that we consider here a time-independent interaction. The equation for $f(k, p)$ may be easily derived substituting (3.82) into (3.78) (taking into account Eq. (3.80) as well). Then

$$f(k, p) = -\frac{m}{2\pi}V(\mathbf{k} - \mathbf{p}) - \frac{m}{2\pi} \int \frac{d^3k'}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}') G_0(E, \mathbf{k}') f(k', p). \quad (3.83)$$

Here $E' = E = E_p$ and the propagator of the free particle is rewritten in the form which underlines energy conservation in the intermediate states:

$$G_0(k')|_{E'=E} \equiv G_0(E, \mathbf{k}') = \frac{1}{-E + (k'^2/2m) - i0}. \quad (3.84)$$

The amplitude $f(k, p)$ may be represented as a series over V :

$$\begin{aligned} f(k, p) = & -\frac{m}{2\pi}V(\mathbf{k} - \mathbf{p}) + \left(\frac{m}{2\pi}\right)^2 \int \frac{d^3k'}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}') G_0(E, \mathbf{k}') V(\mathbf{k}' - \mathbf{p}) \\ & - \left(\frac{m}{2\pi}\right)^3 \int \frac{d^3k'}{(2\pi)^3} \frac{d^3k''}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}') G_0(E, \mathbf{k}') V(\mathbf{k}' - \mathbf{k}'') G_0(E, \mathbf{k}'') V(\mathbf{k}'' - \mathbf{p}) \\ & + \dots \end{aligned} \quad (3.85)$$

If the propagator is small, we may restrict ourselves to a few terms on the r.h.s. of Eq. (3.85). If only the first term is taken into account, we obtain

$$f(k, p) \simeq -\frac{m}{2\pi}V(\mathbf{k} - \mathbf{p}). \quad (3.86)$$

This is the Born approximation for the scattering amplitude.

3.2.4 Propagators in the description of the two-particle scattering amplitude

Up to now our guideline was as follows: we considered the Schrödinger equation for two interacting particles, then we reduced it, in the c.m.s., to the equation for one particle scattered from the external field V . To this aim, we determined the scattering amplitude and the propagator of the non-relativistic particle.

In a number of cases, however, it is more convenient to work with two particles directly. The technique which uses propagators allows us to calculate the scattering amplitude, without reducing the Schrödinger equation beforehand to the one-particle case. We can start with Eq. (3.85) and transform it into a form which manifests a propagation of two particles.

Let us consider the scattering of particles 1 and 2, which in the initial state have the four-momenta

$$k_1 = \left(\frac{k^2}{2m_1}, \mathbf{k} \right) \equiv (E_1, \mathbf{k}), \quad k_2 = \left(\frac{k^2}{2m_2}, -\mathbf{k} \right) \equiv (E_2, -\mathbf{k}). \quad (3.87)$$

The centre-of-mass system is used here, as it has been done before. The four-momenta of the final state are

$$p_1 = \left(\frac{k^2}{2m_1}, \mathbf{p} \right), \quad p_2 = \left(\frac{k^2}{2m_2}, -\mathbf{p} \right). \quad (3.88)$$

The energy conservation is taken into account here, for the potential is time-independent (see Eq. (3.80)). The total energy of particles 1 and 2 is

$$E = \frac{k^2}{2m_1} + \frac{k^2}{2m_2} = \frac{k^2}{2m}, \quad (3.89)$$

where m is the reduced mass.

The equation (3.83) can be rewritten with an explicit form for G_0 :

$$f(k, p) = -\frac{m}{2\pi} V(\mathbf{k} - \mathbf{p}) - \frac{m}{2\pi} \int \frac{d^3 k'}{(2\pi)^3} \frac{V(\mathbf{k} - \mathbf{k}') f(k', p)}{-E + k'^2/2m - i0}. \quad (3.90)$$

The propagator $[-E + k'^2/2m - i0]^{-1}$ stands for the free motion of the two-particle system; it may be represented as a product of free propagators of the particles 1 and 2:

$$\frac{1}{-E + k'^2/2m - i0} = \int_{-\infty}^{\infty} \frac{dE'_1/2\pi i}{[-E'_1 + k'^2/2m_1 - i0] [-(E - E'_1) + k'^2/2m_2 - i0]}. \quad (3.91)$$

The integration in the r.h.s. is performed according to the Cauchy theorem: the integration contour may be closed in the lower half-plane E'_1 as was shown in Fig. 3.4a. If so, $(-E'_1 + k'^2/2m_1 - i0)^{-1} \rightarrow 2\pi i \delta(-E'_1 + k'^2/2m_1)$.

Let us write Eq. (3.90), according to Eq. (3.91), as follows:

$$f(k, p) = -\frac{m}{2\pi} V(\mathbf{k} - \mathbf{p}) - \frac{m}{2\pi} \int \frac{d^3 k'}{(2\pi)^3} \frac{dE'_1}{2\pi i} \frac{V(\mathbf{k} - \mathbf{k}') f(k', p)}{(-E'_1 + k'^2/2m_1 - i0)(-E'_2 + k'^2/2m_2 - i0)}, \quad (3.92)$$

where $E'_2 = E - E'_1$. The product of two propagators in the r.h.s. of Eq. (3.92) clearly manifests the propagation of two particles in the intermediate state.

The equation (3.92) is written in the c.m.s. of the scattering particles 1 and 2, but it is easy to present it in an arbitrary system: the frame-independent consideration of the two-particle interaction amplitude is given in the next subsection, where the relativistic generalisation of Eq. (3.92), the Bethe–Salpeter equation, is discussed.

3.2.5 Relativistic propagator for a free particle

The wave functions of a non-relativistic particle are eigenstates of the Schrödinger operator $[i\partial/\partial t - \hat{H}_0]$, therefore the Green function is defined by the same operator:

$$\left[i\frac{\partial}{\partial t} - \frac{1}{2m} \left(\frac{\partial}{\partial \mathbf{r}} \right)^2 \right] G_0(\mathbf{r}, t) = i\delta(\mathbf{r})\delta(t). \quad (3.93)$$

The wave function of a free scalar particle obeys the Klein–Gordon equation

$$\left[\left(\frac{\partial}{\partial t} \right)^2 - \left(\frac{\partial}{\partial \mathbf{r}} \right)^2 + m^2 \right] \varphi(x) = 0. \quad (3.94)$$

Likewise, the relativistic Green function is defined by the Klein–Gordon operator:

$$\left[\left(\frac{\partial}{\partial t} \right)^2 - \left(\frac{\partial}{\partial \mathbf{r}} \right)^2 + m^2 \right] D(x) = i\delta^{(4)}(x). \quad (3.95)$$

So the propagator of a free relativistic particle in the momentum representation is equal to

$$D(k) = \frac{1}{m^2 - k^2 - i0}, \quad (3.96)$$

where k is the four-momentum of a particle with $k = (k_0, \mathbf{k})$ and $k^2 = k_0^2 - \mathbf{k}^2$. In the non-relativistic approximation, $D(k)$ turns into the quantum mechanical propagator discussed in the previous sections. To see this, let us introduce $E = k_0 - m$ and consider the case $E \ll m$. Then,

$$\begin{aligned} \frac{1}{m^2 - k^2 - i0} &= \frac{1}{m^2 - (m + E)^2 + \mathbf{k}^2 - i0} \\ &\simeq \frac{1}{-2mE + \mathbf{k}^2 - i0} = \frac{1}{2m} \frac{1}{-E + (\mathbf{k}^2/2m) - i0}. \end{aligned} \quad (3.97)$$

The r.h.s. of Eq. (3.97), up to the factor $(2m)^{-1}$, coincides with Eq. (3.76) for the non-relativistic propagator in quantum mechanics. The relativistic Feynman propagator of Eq. (3.96) describes the propagation of a particle and its antiparticle:

$$\begin{aligned} \frac{1}{m^2 - k^2 - i0} &= \frac{1}{2\sqrt{m^2 + \mathbf{k}^2}} \\ &\times \left[\frac{1}{-k_0 + \sqrt{m^2 + \mathbf{k}^2} - i0} + \frac{1}{k_0 + \sqrt{m^2 + \mathbf{k}^2} - i0} \right]. \end{aligned} \quad (3.98)$$

The first term in the square brackets corresponds to the propagation of the relativistic particle with energy $\sqrt{m^2 + \mathbf{k}^2}$, while the second one describes the propagation of the particle with negative energy $-\sqrt{m^2 + \mathbf{k}^2}$.

3.2.6 Mandelstam plane

Feynman diagrams provide information about analytical properties of amplitudes (see [4] and references therein for more detail). The analytical properties of the scattering amplitude $1 + 2 \rightarrow 1' + 2'$ (see Fig. 3.8a) can be considered conveniently if we use the Mandelstam plane. For the sake of simplicity, let us take the masses of scattered particles in the process of Fig. 3.8a to be

$$p_1^2 = p_2^2 = p_1'^2 = p_2'^2 = m^2. \quad (3.99)$$

The scattering amplitude of spinless particles depends on two independent

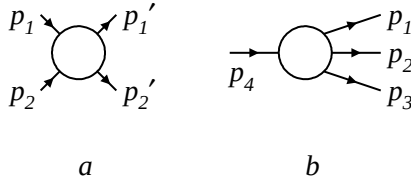


Fig. 3.8 Four-point amplitudes: a) scattering process $1 + 2 \rightarrow 1' + 2'$; b) decay $4 \rightarrow 1 + 2 + 3$.

variables. However, there are three variables for the description of the scattering amplitude on the Mandelstam plane:

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_1' + p_2')^2, \\ t &= (p_1 - p_1')^2 = (p_2 - p_2')^2, \\ u &= (p_1 - p_2')^2 = (p_2 - p_1')^2. \end{aligned} \quad (3.100)$$

These variables obey the condition

$$s + t + u = 4m^2. \quad (3.101)$$

The Mandelstam plane of the variables s , t and u is shown in Fig. 3.9. The physical region of the s -channel corresponds to the case shown in Fig. 3.8a: the incoming particles are 1 and 2, while particles $1'$ and $2'$ are outgoing; s is the energy squared, t and u are the momentum transfers squared. The physical region of the t -channel corresponds to the case when particles 1 and $1'$ collide, while the u -channel describes the collision of particles 1 and $2'$.

The Feynman diagram technique is a good guide for finding analytical properties of scattering amplitudes. Below, we consider typical singularities as examples.

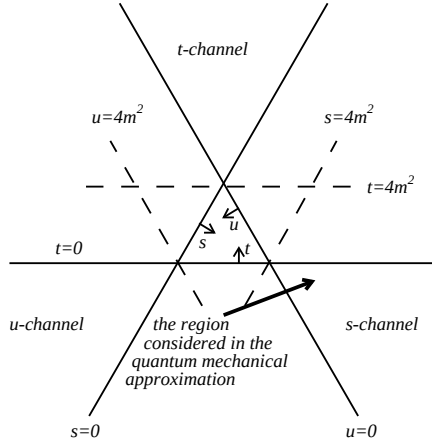
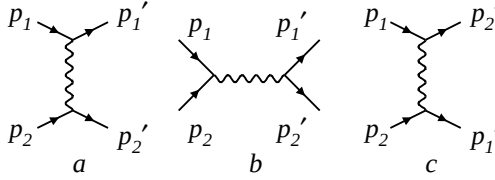


Fig. 3.9 The Mandelstam plane.

Fig. 3.10 One-particle exchange diagrams, with pole singularities in: a) t -channel, b) s -channel, and c) u -channel.

(i) One-particle exchange diagrams are shown in Fig. 3.10a,b,c: they provide pole singularities of the scattering amplitude, which are written as

$$\frac{g^2}{\mu^2 - t}, \quad \frac{g^2}{\mu^2 - s}, \quad \frac{g^2}{\mu^2 - u}, \quad (3.102)$$

where μ is the mass of a particle in the intermediate state, while g is its coupling constant with external particles.

(ii) The two-particle exchange diagram is shown in Fig. 3.11a. It has square-root singularities in the s -channel (the corresponding cut is shown in Fig. 3.11b) and in the t -channel (the cutting marked by crosses is shown in Fig. 3.11c). The s -channel cutting corresponds to the replacement of the Feynman propagators in the following way:

$$(k^2 - m^2)^{-1} \rightarrow \delta(k^2 - m^2), \quad (3.103)$$

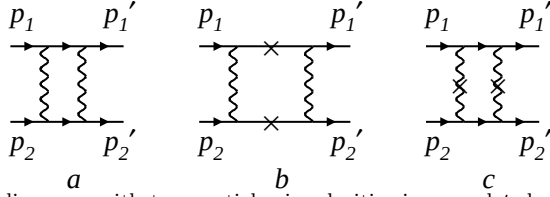


Fig. 3.11 Box diagrams with two-particle singularities in s - and t -channels. Cuttings of diagrams which indicate singularities are marked by crosses.

thus providing us with the imaginary part of the diagram Fig. 3.11a in the s -channel. The s -channel two-particle singularity is located at $s = (m_1 + m_2)^2 = 4m^2$; the singularity is of the type

$$\sqrt{s - (m_1 + m_2)^2} = \sqrt{s - 4m^2}; \quad (3.104)$$

it is the threshold singularity for the s -channel scattering amplitude. The t -channel singularity is at $t = 4\mu^2$, see Fig. 3.11c. It is of the type

$$\sqrt{t - 4\mu^2}. \quad (3.105)$$

(iii) An example of the three-particle singularity in the s -channel is represented by the diagram of Fig. 3.12a. The singularity is located at

$$s = (m_1 + m_2 + \mu)^2 = (2m + \mu)^2. \quad (3.106)$$

The type of singularity is as follows:

$$\begin{aligned} [s - (m_1 + m_2 + \mu)^2]^2 \ln [s - (m_1 + m_2 + \mu)^2] = \\ = [s - (2m + \mu)^2]^2 \ln [s - (2m + \mu)^2]. \end{aligned} \quad (3.107)$$

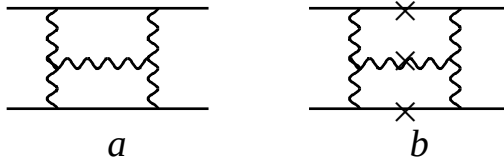


Fig. 3.12 Examples of the diagram with three-particle intermediate state in the s -channel; the crosses mark the appearance of threshold singularity.

3.2.7 Dalitz plot

The four-point amplitude has an additional physical region when particle masses (m_1, m_2, m_3) and m_4 are different and one of them is larger than the sum of all others:

$$m_4 > m_1 + m_2 + m_3. \quad (3.108)$$

It leads to a possibility of the decay process, see Fig. 3.8b (as before, we put $m_1 = m_2 = m_3 = m$):

$$4 \rightarrow 1 + 2 + 3. \quad (3.109)$$

The Mandelstam plane is shown for this case in Fig. 3.13. The physical region of the decay process is located in the centre of the plane. This region of the decay process is shown separately in Fig. 3.14: it is called the Dalitz-plot.

The energies squared of the outgoing particles, $s_{ij} = (p_i + p_j)^2$, obey the constraint

$$\begin{aligned} s_{12} + s_{13} + s_{23} - (m_1^2 + m_2^2 + m_3^2) &= \\ &= s_{12} + s_{13} + s_{23} - 3m^2 = m_4^2. \end{aligned} \quad (3.110)$$

The threshold singularities at

$$s_{ij} = (m_i + m_j)^2 = 4m^2 \quad (3.111)$$

are touching the physical region of the decay.

3.3 Dispersion Relation N/D -Method and Bethe–Salpeter Equation

In this section the basic features of the dispersion integration method are considered for the scattering amplitude $1 + 2 \rightarrow 1' + 2'$ (see [4, 8]). We show how the dispersion technique is related to other methods: the Feynman diagram technique and the light cone variable approach. We consider here the Bethe–Salpeter equation [9] as well as other approaches to the analysis of the partial amplitudes like the method of propagator matrices and the K -matrix method.

3.3.1 N/D -method for the one-channel scattering amplitude of spinless particles

Consider the analytical properties of scattering amplitudes for two spinless particles (with mass m) which interact via the exchange of another spinless particle (with mass μ). This amplitude, $A(s, t)$, has s - and t -channel

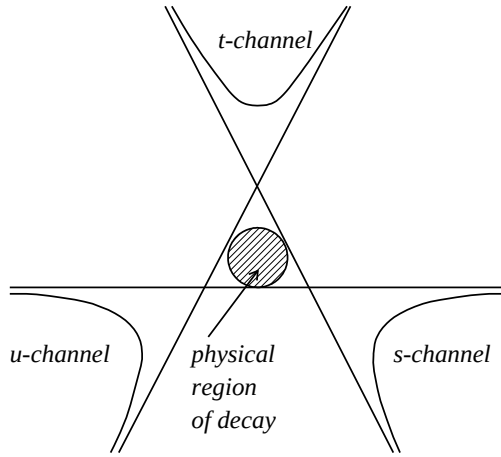


Fig. 3.13 Mandelstam plane and physical region of the decay $4 \rightarrow 1 + 2 + 3$.

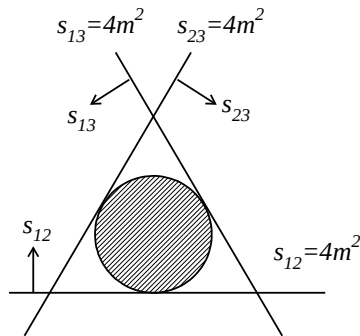


Fig. 3.14 Dalitz plot of the decay $4 \rightarrow 1 + 2 + 3$ for the case $m_1 = m_2 = m_3 = m$.

singularities. In the t -plane there are singularities at $t = \mu^2, 4\mu^2, 9\mu^2$, etc., which correspond to one- or many-particle exchanges. In the s -plane the amplitude has a singularity at $s = 4m^2$ (elastic rescattering) and singularities at $s = (2m + n\mu)^2$, with $n = 1, 2, \dots$, corresponding to the production of n particles with mass μ in the s -channel intermediate state. If a bound state with mass M exists, the pole singularity is at $s = M^2$. If the mass of this bound state $M > 2m$, this is a resonance and the corresponding pole is located on the second sheet of the complex s -plane.

In the N/D -method we deal with partial wave amplitudes. Partial amplitudes in the s -channel depend on s only. They have all the

s -channel right-hand side singularities of $A(s, t)$ at $s = M^2$, $s = 4m^2$, $s = (2m + \mu)^2, \dots$ shown in Fig. 3.15.

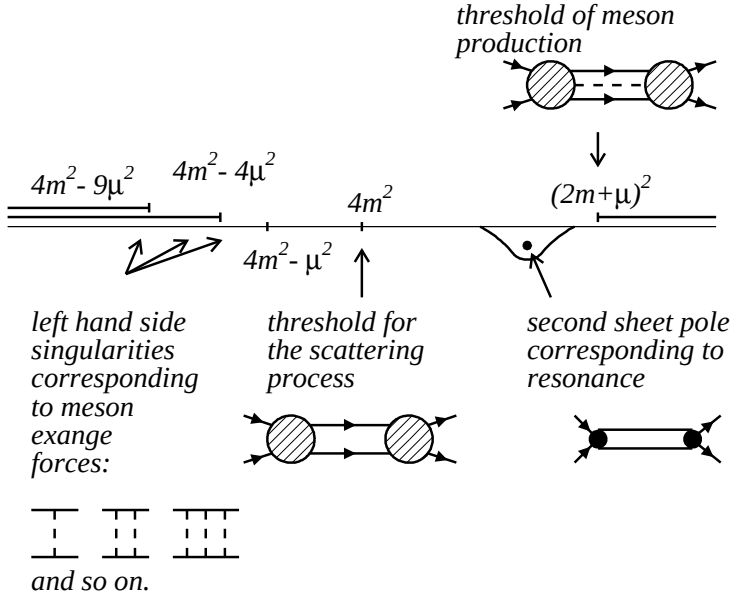


Fig. 3.15 Singularities of partial wave amplitudes in the s -plane.

Left-hand side singularities of the partial amplitudes are connected with the t -channel exchanges contributing to $A(s, t)$. The S -wave partial amplitude is equal to

$$A(s) = \int_{-1}^1 \frac{dz}{2} A(s, t(z)), \quad (3.112)$$

where $t(z) = -2(s/4 - m^2)(1 - z)$ and $z = \cos \theta$. Left-hand side singularities correspond to

$$t(z = -1) = (n\mu)^2, \quad (3.113)$$

they are located at $s = 4m^2 - \mu^2$, $s = 4m^2 - 4\mu^2$, and so on.

The dispersion relation N/D -method [8] provides us the possibility to reconstruct the relativistic two-particle partial amplitude in the region of low and intermediate energies.

Let us restrict ourselves to the consideration of the region in the vicinity of $s = 4m^2$. The unitarity condition for the partial wave amplitude (we consider the S -wave amplitude as an example) reads:

$$\text{Im } A(s) = \rho(s) |A(s)|^2. \quad (3.114)$$

Here $\rho(s)$ is the two-particle phase space integrated at fixed s :

$$\begin{aligned} \rho(s) &= \int d\Phi_2(P; k_1, k_2) = \frac{1}{16\pi} \sqrt{\frac{s - 4m^2}{s}}, \\ d\Phi_2(P; k_1, k_2) &= \frac{1}{2} (2\pi)^4 \delta^4(P - k_1 - k_2) \frac{d^3 k_1}{(2\pi)^3 2k_{10}} \frac{d^3 k_2}{(2\pi)^3 2k_{20}}, \end{aligned} \quad (3.115)$$

where P is the total momentum, $P^2 = s$; k_1 and k_2 are momenta of particles in the intermediate state. In the N/D -method the amplitude $A(s)$ is represented as

$$A(s) = \frac{N(s)}{D(s)}. \quad (3.116)$$

Here $N(s)$ has only left-hand side singularities, whereas $D(s)$ has only right-hand side ones. So, the N -function is real in the physical region $s > 4m^2$. The unitarity condition can be rewritten as:

$$\text{Im } D(s) = -\rho(s)N(s). \quad (3.117)$$

The solution of this equation is

$$D(s) = 1 - \int_{4m^2}^{\infty} \frac{d\tilde{s}}{\pi} \frac{\rho(\tilde{s})N(\tilde{s})}{\tilde{s} - s} \equiv 1 - B(s). \quad (3.118)$$

In Eq. (3.118) we neglect the so-called CDD-poles [10] and normalise $N(s)$ by the condition $D(s) \rightarrow 1$ as $s \rightarrow \infty$.

Let us introduce the vertex function

$$G(s) = \sqrt{N(s)}. \quad (3.119)$$

We assume here that $N(s)$ is positive (the cases with negative $N(s)$ or if $N(s)$ changes sign need a special and more cumbersome treatment). Then the partial wave amplitude $A(s)$ can be expanded in a series

$$A(s) = G(s)[1 + B(s) + B^2(s) + B^3(s) + \dots]G(s), \quad (3.120)$$

where $B(s)$ is a loop-diagram

$$B(s) = \text{---} \circ \text{---} \quad (3.121)$$

The graphical interpretation of Eq. (3.120) is as follows:

$$\text{Diagram: a circle with four external lines} = \text{Diagram: two vertices connected by a line} + \text{Diagram: two vertices connected by two lines} + \text{Diagram: two vertices connected by three lines} + \dots \quad (3.122)$$

so the amplitude $A(s)$ is a set of terms with different numbers of rescatterings.

3.3.2 N/D -amplitude and K -matrix

As was shown above, in the N/D method the amplitude A is written as

$$A(s) = \frac{N(s)}{1 - \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{N(s')\rho(s')}{s'-s}} = \frac{N(s)}{1 - \text{P} \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{N(s')\rho(s')}{s'-s} - iN(s)\rho(s)} \quad (3.123)$$

where P means the principal value of the integral. P is real and does not contain the threshold singularity, so we have for the K -matrix representation

$$T(s) = \rho(s)A(s) = \frac{K(s)}{1 - iK(s)} \quad (3.124)$$

with

$$K(s) = \frac{\rho(s)N(s)}{1 - \text{P} \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{N(s')\rho(s')}{s'-s}}. \quad (3.125)$$

It is the K -matrix representation of the scattering amplitude for the one-channel case (see Eq. (3.45)).

An important point is that in the considered case the principal-valued integral does not contain a threshold singular term: this is a property of the two-particle threshold singularity. A singular term related to the two-particle threshold exists in the semi-residue term only.

3.3.3 Dispersion relation representation and light-cone variables

The loop diagram $B(s)$ plays the main role for the whole dispersion amplitude; below, we compare the dispersion and Feynman expressions for $B(s)$ in detail.

The Feynman expression for $B_F(s)$, with a special choice of separable interaction $G(4k_\perp^2 + 4m^2)$, may be proved to be equal to the dispersion integral representation, where the four-vector $k_\perp$ is defined as

$$2k_\perp = k_1 - k_2 - \frac{k_1^2 - k_2^2}{P^2}P, \quad k_1 + k_2 = P, \quad k_1 \equiv k. \quad (3.126)$$

The Feynman expression for the loop diagram reads:

$$B_F(P^2) = \int \frac{d^4k}{(2\pi)^4 i} \frac{G^2(4(Pk)^2/P^2 - 4k^2 + 4m^2)}{(m^2 - k^2 - i0)(m^2 - (P - k)^2 - i0)}. \quad (3.127)$$

Let us introduce the light-cone coordinates:

$$\begin{aligned} k_- &= \frac{1}{\sqrt{2}}(k_0 - k_z), \quad k_+ = \frac{1}{\sqrt{2}}(k_0 + k_z), \\ k^2 &= 2k_+k_- - m_\perp^2, \quad m_\perp^2 = m^2 + k_\perp^2. \end{aligned} \quad (3.128)$$

The four-vector P is written as $P = (P_0, \mathbf{P}_\perp, P_z)$. Let us choose a reference frame in which $\mathbf{P}_\perp = 0$. Then,

$$Pk = P_+k_- + P_-k_+, \quad (3.129)$$

and Eq. (3.127) takes the form for $G = 1$:

$$\begin{aligned} B_F(P^2) &= \frac{1}{(2\pi)^4 i} \\ &\times \int \frac{dk_+ dk_- d^2k_\perp}{(2k_+k_- - m_\perp^2 + i0)(P^2 - 2(P_+k_- + P_-k_+) + 2k_+k_- - m_\perp^2 + i0)}. \end{aligned} \quad (3.130)$$

If $G \equiv 1$, one can integrate over k_- right now closing the integration contour around the pole

$$k_- = \frac{m_\perp^2 - i0}{2k_+} \quad (3.131)$$

and obtaining the standard dispersion representation for the Feynman loop graph ($x = k_+/P_+$):

$$\begin{aligned} &\int \frac{d^2k_\perp}{(2\pi)^4 i} \int_0^1 \frac{dx}{2} \frac{(-2\pi i)}{P^2 x(1-x) - m_\perp^2 + i0} \\ &= \int \frac{ds}{\pi(s - P^2 - i0)} \int \frac{dx dk_\perp^2}{16\pi x(1-x)} \delta\left(s - \frac{m_\perp^2}{x(1-x)}\right) = \int_{4m^2}^{\infty} \frac{ds \rho(s)}{\pi(s - P^2 - i0)}. \end{aligned} \quad (3.132)$$

The variable x changes from 0 to 1, because for $x < 0$ and $x > 1$ both poles in k_- are located on the same side of the integration contour and the integral equals zero. The dispersion integral (3.132) is divergent at $s \rightarrow \infty$

because $G = 1$, and it is just G which provides the convergence of B_F in Eq. (3.127). The convergence of the integral (3.132) can be restored by a subtraction (or cutting) procedure.

For $G \neq 1$, some additional steps are required to obtain the dispersion representation; we introduce new variables ξ_+ and ξ_-

$$P_+ k_- + P_- k_+ = \sqrt{P^2} \xi_+ , \quad P_+ k_- - P_- k_+ = \sqrt{P^2} \xi_- . \quad (3.133)$$

Using these variables, Eq. (3.127) takes the following form:

$$\begin{aligned} B_F(P^2) &= \frac{1}{(2\pi)^4 i} \\ &\times \int \frac{G^2 (4(\xi_-^2 + m_\perp^2)) d\xi_+ d\xi_- d^2 k_\perp}{(\xi_+^2 - \xi_-^2 - m_\perp^2 + i0)(P^2 - 2\sqrt{P^2} \xi_+ + \xi_+^2 - \xi_-^2 - m_\perp^2 + i0)} \\ &= \int_0^\infty 2\pi d\xi_- dk_\perp^2 G^2 (4(\xi_-^2 + m_\perp^2)) \\ &\times \int_{-\infty}^\infty \frac{d\xi_+}{(\xi_+^2 - (\xi_-^2 + m_\perp^2) + i0)[(\xi_+ - \sqrt{P^2})^2 - (\xi_-^2 + m_\perp^2) + i0]} . \end{aligned} \quad (3.134)$$

The integration over ξ_+ is performed by closing the integration contour in the upper half-plane, and two poles, $\xi_+ = -\sqrt{\xi_-^2 + m_\perp^2} + i0$ and $\xi_+ = \sqrt{P^2} - \sqrt{\xi_-^2 + m_\perp^2} + i0$, contribute. The result of the integration over ξ_+ is

$$\frac{2\pi i}{\sqrt{\xi_-^2 - m_\perp^2} (4(\xi_-^2 + m_\perp^2) - P^2)} . \quad (3.135)$$

The introduction of a new variable $s = 4(\xi_-^2 + m_\perp^2)$ yields

$$B_F(P^2) = \int_{4m^2}^\infty \frac{ds G^2(s)}{\pi(s - P^2)} \frac{1}{16\pi} \sqrt{1 - \frac{4m^2}{s}} , \quad (3.136)$$

that is just the dispersion representation (3.118).

Note that in rewriting the Feynman loop integral in the form of (3.136), the choice of the vertex in its separable form (Eq. (3.126)) was crucial.

3.3.4 *Bethe–Salpeter equations in the momentum representation*

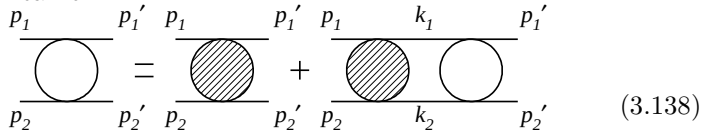
We discuss here the Bethe–Salpeter (BS) equation [9], which is widely used for scattering processes and bound systems, and compare it with a

treatment of the same amplitudes based on dispersion relations. The BS-equation is a straightforward generalisation of the non-relativistic Eq. (3.92) for the scattering amplitude.

The non-homogeneous BS-equation in the momentum representation reads:

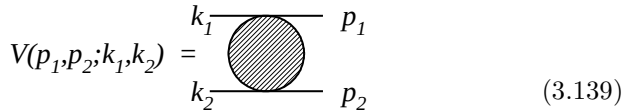
$$A(p'_1, p'_2; p_1, p_2) = V(p'_1, p'_2; p_1, p_2) + \int \frac{d^4 k_1 d^4 k_2}{i(2\pi)^4} A(p'_1, p'_2; k_1 k_2) \times \frac{\delta^4(k_1 + k_2 - P)}{(m^2 - k_1^2 - i0)(m^2 - k_2^2 - i0)} V(k_1, k_2; p_1, p_2) \quad (3.137)$$

or in the graphical form:



$$\text{Diagram (3.138)} \quad (3.138)$$

Here the momenta of the constituents obey the momentum conservation law $p_1 + p_2 = p'_1 + p'_2 = P$ and $V(p_1, p_2; k_1, k_2)$ is a two-constituent irreducible kernel:

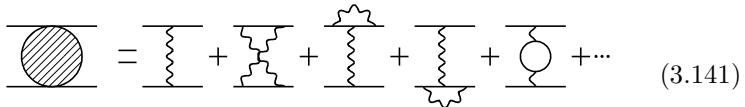


$$V(p_1, p_2; k_1, k_2) = \text{Diagram (3.139)} \quad (3.139)$$

For example, it can be a kernel induced by the meson-exchange interaction

$$\frac{g^2}{\mu^2 - (k_1 - p_1)^2} \quad (3.140)$$

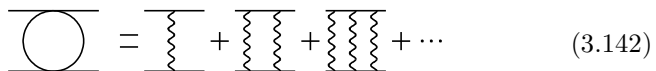
Generally, $V(p_1, p_2; k_1, k_2)$ is an infinite sum of irreducible two-particle graphs



$$\text{Diagram (3.141)} \quad (3.141)$$

We would like to emphasise that the amplitude A determined by the BS-equation is a mass-off-shell amplitude. Even if we put $p_1^2 = p_1'^2 = p_2^2 = p_2'^2 = m^2$ in the left-hand side of (3.137), the right-hand side contains the amplitude $A(k'_1, k_2; p'_1, p'_2)$ for $k_1^2 \neq m^2$, $k_2^2 \neq m^2$.

Let us restrict ourselves to one-meson exchange in the irreducible kernel V . By iterating Eq. (3.137), we come to infinite series of ladder diagrams:



$$\text{Diagram (3.142)} \quad (3.142)$$

Let us investigate the intermediate states in these ladder diagrams. Note that these diagrams have two-particle intermediate states which can appear as real states at c.m. energies $\sqrt{s} > 2m$. This corresponds to the cutting of the ladder diagrams across constituent lines:

$$(3.143)$$

Such a two-particle state manifests itself as a singularity of the scattering amplitude at $s = 4m^2$. However, the amplitude A being a function of s has not only this singularity but also an infinite set of singularities which correspond to the ladder diagram cuts across meson lines of the type:

$$(3.144)$$

The diagrams, which appear after this cutting procedure, are meson production diagrams, *e.g.*, one-meson production diagrams:

$$(3.145)$$

Hence, the amplitude $A(p'_1, p'_2; p_1, p_2)$ has the following cut singularity in the complex- s plane:

$$s = 4m^2, \quad (3.146)$$

which is related to the rescattering process. Other singularities are related to the meson production processes with the cuts starting at

$$s = (2m + n\mu)^2; \quad n = 1, 2, 3, \dots \quad (3.147)$$

The four-point amplitude, which is the subject of the BS-equation, depends on six variables:

$$p_1^2, p_2^2, p_1'^2, p_2'^2, \quad (3.148)$$

$$s = (p_1 + p_2)^2 = (p_1' + p_2')^2, \quad t = (p_1 - p_1')^2 = (p_2 - p_2')^2,$$

while the seventh variable, $u = (p_1 - p_2')^2 = (p_1' - p_2)^2$, is not independent because of the relation

$$s + t + u = p_1^2 + p_2^2 + p_1'^2 + p_2'^2. \quad (3.149)$$

It is possible to decrease the number of variables in Eq. (3.137) if we consider an amplitude with definite angular momentum. The standard way is to consider Eq. (3.137) in the c.m.s. of particles 1 and 2 and expand the

amplitude $A(p'_1, p'_2; p_1, p_2)$ as well as the interaction term over the angular momentum states

$$\begin{aligned} \langle L' M' | A(p'_1, p'_2; p_1, p_2) | L M \rangle &= A_L(s; p_1^2, p_2^2, p_1'^2, p_2'^2) \delta_{LL'} \delta_{MM'} \\ \langle L' M' | V(p'_1, p'_2; p_1, p_2) | L M \rangle &= V_L(s; p_1^2, p_2^2, p_1'^2, p_2'^2) \delta_{LL'} \delta_{MM'}. \end{aligned} \quad (3.150)$$

For spinless particles the states $|LM\rangle$ are spherical harmonics $Y_{LM}(\theta, \varphi)$. An alternative procedure related to the covariant angular momentum expansion is discussed in Chapter 4. Using (3.150), we get for the amplitude A_L the following equation:

$$\begin{aligned} A_L(s; p_1'^2, p_2'^2, p_1^2, p_2^2) &= V_L(s; p_1'^2, p_2'^2, p_1^2, p_2^2) + \int \frac{d^4 k}{i(2\pi)^4} V_L(s; p_1'^2, p_2'^2, k_1^2, k_2^2) \\ &\times \frac{|Y_{LM}(\mathbf{k}/|\mathbf{k}|)|^2}{(m^2 - k_1^2 - i0)(m^2 - k_2^2 - i0)} A_L(s, k_1^2, k_2^2, p_1^2, p_2^2), \end{aligned} \quad (3.151)$$

where $k = k_1$, $k_2 = P - k$ and $P = p_1 + p_2 = (\sqrt{s}, 0, 0, 0)$.

If a bound state of the constituents exist, the scattering partial amplitude has a pole at $s = (p_1 + p_2)^2 = M^2$, where M is the mass of the bound state. This pole appears both in the on- and off-shell scattering amplitudes. This means that the infinite sum of diagrams of Fig. 3.16a type may be rewritten as a pole term of Fig. 3.16b plus some regular terms at $s = M^2$.

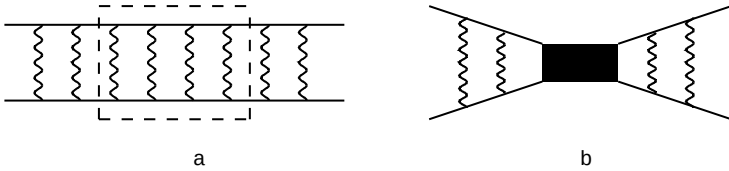


Fig. 3.16 a) Ladder diagram of the mass-on-shell scattering amplitude and the internal block which is the subject of consideration in Eq. (3.138); b) Pole diagram which corresponds to a composite particle and vertices of the transition “composite particle $\rightarrow$ constituents”.

The left and right blocks in Fig. 3.16b, $\chi(p_1, p_2; P)$ and $\chi(p'_1, p'_2; P)$, satisfy the homogeneous BS-equation

$$\begin{aligned} \chi(p_1, p_2; P) &= \int \frac{d^4 k_1 d^4 k_2}{i(2\pi)^4} V(p_1, p_2; k_1, k_2) \\ &\times \frac{\delta^4(k_1 + k_2 - P)}{(m^2 - k_1^2 - i0)(m^2 - k_2^2 - i0)} \chi(k_1, k_2; P), \end{aligned} \quad (3.152)$$

whose graphical form is

$$\text{Diagram (3.153)} \quad (3.153)$$

The n iterations of (3.153) give

$$\text{Diagram (3.154)} \quad (3.154)$$

The same cutting procedure of the interaction block in the right-hand side of (3.154) shows us that the amplitude $\chi(p_1, p_2; P)$ contains all the singularities of the amplitude A given by Eqs. (3.143), (3.144).

The three-point amplitude $\chi(p_1, p_2; P)$ depends on three variables

$$P^2 \text{ (or } s), p_1^2, p_2^2, \quad (3.155)$$

and again, as in the case of the scattering amplitude A , the BS-equation contains the mass-off-shell amplitude $\chi(k_1, k_2; P)$; χ is a solution of the homogeneous equation, hence the normalisation condition should be imposed independently.

For the normalisation, one can use the connection between χ and A at $P^2 \rightarrow M^2$:

$$A(p_1, p_2; p'_1, p'_2) = \frac{\chi(p_1, p_2; P^2 = M^2)\chi(P^2 = M^2; p'_1, p'_2)}{P^2 - M^2} + \text{regular terms.} \quad (3.156)$$

In the formulation of scattering theory, we start from a set of asymptotic states, containing constituent particles (with mass m) and mesons (with mass μ) only. We do not include in such a formulation of the scattering theory the composite particles as asymptotic states; we simply cannot know beforehand whether such bound states exist or not. But if we consider the production or decay of particles which are bound states, they should be included into the set of asymptotic states.

3.3.5 Spectral integral equation with separable kernel in the dispersion relation technique

As was demonstrated in Section 3.3.3, the Feynman diagram calculus of scattering amplitudes with separable interactions gives us the same result as the N/D dispersion relation method when the vertices in the c.m. system depend only on the space components of momenta. Here the BS-equation

with a separable kernel is expressed in terms of the dispersion relation integrals. So, the scattering amplitude A is defined as an infinite sum of dispersion relation loop diagrams:

$$A(s) = s \rangle \langle s + s \rangle \text{loop} \langle s + s \rangle \text{loop} \text{loop} \langle s + \dots \quad (3.157)$$

The energy-off-shell amplitude emerges when the cutting procedure of the series (3.157) is performed:

$$s \rangle \langle \tilde{s} | \tilde{s} \rangle \langle s + s \rangle \langle \tilde{s} | \tilde{s} \rangle \langle \tilde{s}' \rangle \langle s + \dots \quad (3.158)$$

This amplitude is also represented as an infinite sum of loop diagrams, where, however, the initial and final values $\tilde{s}$ and s are different:

$$A(\tilde{s}, s) = \tilde{s} \rangle \langle s + \tilde{s} \rangle \text{loop}(\tilde{s}) \langle s + \dots \quad (3.159)$$

It is the energy-off-shell amplitude which has to be considered in the general case. This amplitude satisfies the equation

$$A(\tilde{s}, s) = G(\tilde{s})G(s) + G(\tilde{s}) \int_{4m^2}^{\infty} \frac{d\tilde{s}'}{\pi} \frac{G(\tilde{s}')\rho(\tilde{s}')A(\tilde{s}', s)}{\tilde{s}' - s}. \quad (3.160)$$

Let us emphasise that in the dispersion approach we deal with the mass-on-shell amplitudes, *i.e.* amplitudes for real constituents, whereas in the BS-equation (3.137) the amplitudes are mass-off-shell. The appearance of the energy-off-shell amplitude in the dispersion method, Eq. (3.160), is the price we have to pay for keeping all the constituents on the mass shell.

The solution of Eq. (3.160) reads:

$$A(\tilde{s}, s) = G(\tilde{s}) \frac{G(s)}{1 - B(s)}. \quad (3.161)$$

For the physical processes $\tilde{s} = s$, so the partial wave amplitude $A(s)$ is $A(s) = A(s, s)$.

Consider the partial amplitude near the pole corresponding to the bound state. The pole appears when

$$B(M^2) = 1, \quad (3.162)$$

and in the vicinity of this pole we have:

$$A(s) = G(s) \frac{1}{1 - B(s)} G(s) \simeq \frac{G(s)}{\sqrt{B'(M^2)}} \cdot \frac{1}{M^2 - s} \cdot \frac{G(s)}{\sqrt{B'(M^2)}} + \dots \quad (3.163)$$

Here we take into account that $1 - B(\tilde{s}) \simeq 1 - B(M^2) - B'(M^2)(s - M^2)$.

The homogeneous equation for the bound state vertex $G_{vertex}(s, M^2)$ reads:

$$G_{vertex}(s, M^2) = G(s) \int_{4m^2}^{\infty} \frac{d\tilde{s}}{\pi} G(\tilde{s}) \frac{\rho(\tilde{s})}{\tilde{s} - M^2} G_{vertex}(\tilde{s}, M^2), \quad (3.164)$$

where $G_{vertex}(s, M^2)$ is the analogue of $\chi(p_1, p_2; P^2 = M^2)$.

The only s -dependent term in the right-hand side of Eq.(3.164) is the factor $G(s)$, so

$$G_{vertex}(s, M^2) \sim G(s). \quad (3.165)$$

As was mentioned above, the normalisation condition for $G_{vertex}(s, M^2)$ is the relation between $G_{vertex}(s, M^2)$ and $A(s, \tilde{s})$ in the vicinity of the pole. The equation (3.163) tells us:

$$G_{vertex}(s, M^2) = \frac{G(s)}{\sqrt{B'(M^2)}}. \quad (3.166)$$

The vertex function $G_{vertex}(s, M^2)$ enters all processes containing the bound state interaction. For example, this vertex determines the form factor of a bound state.

3.3.6 Composite system wave function, its normalisation condition and additive model for form factors

The vertex function represented by (3.166) gives way to a subsequent description of composite systems in terms of dispersion relations with separable interactions. To see this, one should consider not only the two-particle interaction (what we have dealt with before) but to go off the frame of this problem: we have to study the interaction of the two-particle composite system with the electromagnetic field. In principle, this is not a difficult task when interactions are separable.

Consider the dispersion representation of the triangle diagram shown in Fig. 3.17a. It can be written in a way similar to the one-fold representation for the loop diagram with a certain necessary complication (as before, we consider a simple case of equal masses $m_1 = m_2 = m$).

First, a double dispersion integral should be written in terms of the masses of the incoming and outgoing particles:

$$\int_{4m^2}^{\infty} \frac{ds}{\pi} \frac{1}{s - p^2 - i0} \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{1}{s' - p'^2 - i0} \times \dots \quad (3.167)$$

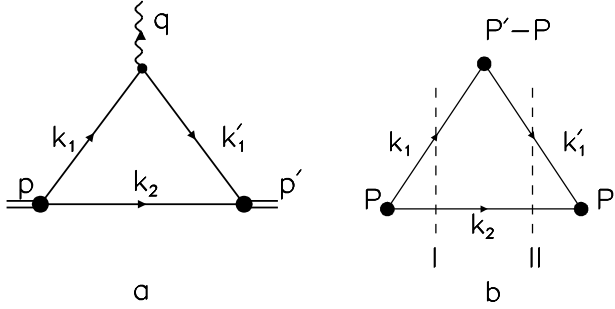


Fig. 3.17 a) Additive quark model diagram for composite system: one of constituents interacts with electromagnetic field; b) cut triangle diagram in the double spectral representation: $P^2 = s$, $P'^2 = s'$ and $(P' - P)^2 = q^2$.

The double spectral representation is inevitable when the interaction of the photon, though with one constituent only, divides the loop diagram into two pieces. Dots in (3.167) stand for the double discontinuity of the triangle diagram, with cutting lines I and II (see Fig. 3.17b); let us denote it as $disc_s disc_{s'} F(s, s', q^2)$. This double discontinuity is written analogously to the discontinuity of the loop diagram. Namely,

$$\begin{aligned}
 disc_s disc_{s'} F(s, s', q^2) &\sim G_{vertex}(s, M^2) d\Phi_{tr}(P, P'; k_1, k'_1, k_2) \\
 &\quad \times G_{vertex}(s', M^2), \\
 d\Phi_{tr}(P, P'; k_1, k'_1, k_2) &= d\Phi_2(P'; k_1, k_2) d\Phi_2(P'; k'_1, k'_2) \\
 &\quad \times (2\pi)^3 2k'_{20} \delta^3(\mathbf{k}_2 - \mathbf{k}'_2)
 \end{aligned} \tag{3.168}$$

Here the vertex G_{vertex} is defined according to (3.166), the two-particle phase volume is written following (3.115) and the factor $2(2\pi)^3 k'_{20} \delta^3(\mathbf{k}_2 - \mathbf{k}'_2)$ reflects the fact that the constituent spectator line was cut twice (that is, of course, impossible and requires to eliminate in (3.168) the extra phase space integration). Let us stress that in (3.168) the constituents are on the mass shell: $k_1^2 = k_2^2 = k'^2_1 = m^2$, the momentum transfer squared is fixed $(k'_1 - k_1)^2 = (P' - P)^2 = q^2$ but $P' - P \neq q$.

We did not write in (3.168) an equality sign, since there is one more factor in Fig. 3.17b.

In the diagram of Fig. 3.17b, the gauge invariant vertex for the interaction of a scalar (or pseudoscalar) constituent with a photon is written as $(k_{1\mu} + k'_{1\mu})$, from which one should separate a factor orthogonal to the

momentum transfer $P'_\mu - P_\mu$. This is not difficult using the kinematics of real particles:

$$\begin{aligned} k_{1\mu} + k'_{1\mu} &= \alpha(s, s', q^2) \left[P_\mu + P'_\mu - \frac{s' - s}{q^2} (P'_\mu - P_\mu) \right] + k_{\perp\mu}, \\ \alpha(s, s', q^2) &= -\frac{q^2(s + s' - q^2)}{\lambda(s, s', q^2)}, \\ \lambda(s, s', q^2) &= -2q^2(s + s') + q^4 + (s' - s)^2, \end{aligned} \quad (3.169)$$

where $k_{\perp\mu}$ is orthogonal to both $(P_\mu + P'_\mu)$ and $(P_\mu - P'_\mu)$. Hence,

$$\begin{aligned} &disc_s disc_{s'} F(s, s', q^2) \\ &= G_{vertex}(s, M^2) G_{vertex}(s', M^2) d\Phi_{tr}(P, P'; k_1, k'_1, k_2) \alpha(s, s', q^2), \end{aligned} \quad (3.170)$$

and the form factor of the composite system reads:

$$F(q^2) = \int_{4m^2}^{\infty} \frac{ds}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{disc_s disc_{s'} F(s, s', q^2)}{(s - M^2 - i0)(s' - M^2 - i0)}, \quad (3.171)$$

where we took into account that $p^2 = p'^2 = M^2$ and the term $k_{\perp\mu}$ equals zero after integrating over the phase space.

Let us underline that the full amplitude of the interaction of the photon with a composite system, when the charge of the composite system equals unity, is:

$$A_\mu(q^2) = (p_\mu + p'_\mu) F(q^2), \quad (3.172)$$

that is, the form factor of the composite system is an invariant coefficient in front of the transverse part of the amplitude A_μ :

$$(p + p') \perp q. \quad (3.173)$$

Likewise, the invariant coefficient $\alpha(s, s' q^2)$ defines the transverse part of the diagram shown in Fig. 3.17b:

$$\left[P + P' - \frac{s' - s}{q^2} (P' - P) \right] \perp (P' - P). \quad (3.174)$$

Formula (3.171) has a remarkable property: for the vertex $G_{vertex}(s)$ (3.166) it gives a correct normalisation of the charge form factor,

$$F(0) = 1. \quad (3.175)$$

It is easy to carry out the derivation of this normalisation condition, we shall do that below. For $F(q^2)$, after integrating in (3.171) over the momenta k_1 , k'_1 and k_2 at fixed s and s' , we obtain the following expression:

$$F(q^2) = \int_{4m^2}^{\infty} \frac{ds ds'}{\pi^2} \frac{G_{vertex}(s, M^2)}{s - M^2} \frac{G_{vertex}(s', M^2)}{s' - M^2} \times \frac{\Theta(-ss'q^2 - m^2\lambda(s, s', q^2))}{16\sqrt{\lambda(s, s', q^2)}} \alpha(s, s', q^2). \quad (3.176)$$

Here the Θ -function is defined as follows: $\Theta(X) = 1$ at $X \geq 0$ and $\Theta(X) = 0$ at $X < 0$.

To calculate (3.176) in the limit $q^2 \rightarrow 0$, let us introduce new variables:

$$\sigma = \frac{1}{2}(\tilde{s} + \tilde{s}'); \quad \Delta = \tilde{s} - \tilde{s}', \quad Q^2 = -q^2, \quad (3.177)$$

and then consider the case of interest, $Q^2 \rightarrow 0$. The form factor formula reads:

$$F(-Q^2 \rightarrow 0) = \int_{4m^2}^{\infty} \frac{d\sigma}{\pi} \frac{G_{vertex}^2(\sigma, M^2)}{(\sigma - M^2)(\sigma - M^2)} \int_{-b}^b d\Delta \frac{\alpha(\sigma, \Delta, Q^2)}{16\pi\sqrt{\Delta^2 + 4\sigma Q^2}}, \quad (3.178)$$

where

$$b = \frac{Q}{m} \sqrt{\sigma(\sigma - 4m^2)}, \quad \alpha(\sigma, \Delta, Q^2) = \frac{2\sigma Q^2}{\Delta^2 + 4\sigma Q^2}. \quad (3.179)$$

As a result we have:

$$F(0) = 1 = \int_{4m^2}^{\infty} \frac{ds}{\pi} \Psi^2(s) \rho(s),$$

$$\rho(s) = \frac{1}{16\pi} \sqrt{1 - 4m^2/s}, \quad \Psi(s) = \frac{G_{vertex}(s, M^2)}{s - M^2}. \quad (3.180)$$

We see that the condition $F(0) = 1$ means actually the normalisation condition for the wave function of the composite system $\Psi(s)$.

3.3.6.1 Separable interaction in the N/D method and the prospects of its application to the calculation of radiative decays

Formulae (3.176) and (3.180) are indeed remarkable. They show that we have a unified triad:

(i) the method of spectral integration for composite systems,

- (ii) the hypothesis of separable interaction for composite systems,
- (iii) the calculation technique for radiative transitions in composite systems with radiative transitions defined by the diagrams of the additive quark model.

This triad opens future prospects for the calculation of both wave functions (or vertices) of the composite systems and radiative processes with this composite systems.

Of course, the use of separable interactions imposes a model restriction on the treatment of physical processes (for example, within the above triad we do not account for the interaction of photons with exchange currents). But for composite systems the most important are additive processes, and the discussed model opens a possibility to carry out subsequent calculations of interaction processes with the electromagnetic field taking into account the gauge invariance.

The procedure of construction of gauge invariant amplitudes within the framework of the spectral integration method has been realised for the deuteron in [11, 12], and, correspondingly, for the elastic scattering and photodisintegration process. A generalisation of the method for the composite quark systems has been performed in [13, 14, 15].

3.4 The Matrix of Propagators

The D -matrix technique based on the dispersion N/D -method allows us to reconstruct the amplitude being analytical on the whole complex- s plane. We discuss effects owing to the overlap and the mixing of resonances: mass shifts and the accumulation of widths of the neighbouring resonances by one of the resonances.

We consider here the S -wave state. The method can be easily generalised for other waves.

3.4.1 The mixing of two unstable states

In case of two resonances, the distribution function of state 1 is determined by the diagrams shown in Fig. 3.18a.

Taking into account all the presented processes, the propagator of state 1 can be written as

$$D_{11}(s) = \left(m_1^2 - s - B_{11}(s) - \frac{B_{12}(s)B_{21}(s)}{m_2^2 - s - B_{22}(s)} \right)^{-1}. \quad (3.181)$$

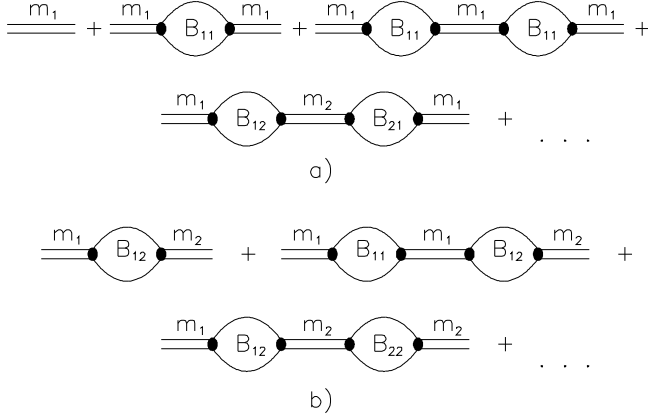


Fig. 3.18 Diagrams that determine the mixing of two unstable particles.

Here m_1 and m_2 are masses of the states 1 and 2, and the loop diagrams $B_{ij}(s)$ are determined by the expressions (3.134)–(3.136), with the substitution $G^2(s) \rightarrow g_i(s)g_j(s)$. It is useful to introduce the propagator matrix D_{ij} , where the non-diagonal terms $D_{12} = D_{21}$ describe the $1 \rightarrow 2$ and $2 \rightarrow 1$ transitions (see Fig. 3.18b). The matrix is

$$\hat{D} = \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} \quad (3.182)$$

$$= \frac{1}{(M_1^2 - s)(M_2^2 - s) - B_{12}B_{21}} \begin{vmatrix} M_2^2 - s, & B_{12} \\ B_{21}, & M_1^2 - s \end{vmatrix}.$$

We use here the following notation:

$$M_i^2 = m_i^2 - B_{ii}(s) \quad i = 1, 2. \quad (3.183)$$

The zeros of the denominator in the propagator matrix (3.182) determine the complex masses of the mixed resonances, M_A^2 and M_B^2 :

$$\Pi(s) = (M_1^2 - s)(M_2^2 - s) - B_{12}B_{21} = 0. \quad (3.184)$$

We denote the complex masses of the mixed states as M_A and M_B .

Consider now a simple model. Let us assume that the s -dependence of the functions $B_{ij}(s)$ in the regions $s \sim M_A^2$ and $s \sim M_B^2$ can be neglected. Taking M_i^2 and B_{12} as constants, we have

$$M_{A,B}^2 = \frac{1}{2}(M_1^2 + M_2^2) \pm \sqrt{\frac{1}{4}(M_1^2 - M_2^2)^2 + B_{12}B_{21}}. \quad (3.185)$$

When the widths of the initial resonances 1 and 2 are small (and, hence, the imaginary part of the transition diagram B_{12} is also small), Eq. (3.185) is nothing but the standard quantum-mechanical expression for the splitting of the mixed levels which, as a result of the mixing, are repelled. Then

$$\hat{D} = \begin{vmatrix} \frac{\cos^2 \theta}{M_A^2 - s} + \frac{\sin^2 \theta}{M_B^2 - s} & \frac{-\cos \theta \sin \theta}{M_A^2 - s} + \frac{\sin \theta \cos \theta}{M_B^2 - s} \\ \frac{-\cos \theta \sin \theta}{M_A^2 - s} + \frac{\sin \theta \cos \theta}{M_B^2 - s} & \frac{\sin^2 \theta}{M_A^2 - s} + \frac{\cos^2 \theta}{M_B^2 - s} \end{vmatrix}, \quad (3.186)$$

$$\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \frac{\frac{1}{2}(M_1^2 - M_2^2)}{\sqrt{\frac{1}{4}(M_1^2 - M_2^2)^2 + B_{12}B_{21}}}.$$

The states $|A\rangle$ and $|B\rangle$ are superpositions of the initial states $|1\rangle$ and $|2\rangle$:

$$|A\rangle = \cos \theta |1\rangle - \sin \theta |2\rangle, \quad |B\rangle = \sin \theta |1\rangle + \cos \theta |2\rangle. \quad (3.187)$$

The procedure of representing the states $|A\rangle$ and $|B\rangle$ as superpositions of the initial states remains valid in the general case, when the s -dependence of the functions $B_{ij}(s)$ cannot be neglected and the imaginary parts are not small. Let us consider the propagator matrix near $s = M_A^2$:

$$\hat{D} = \frac{1}{\Pi(s)} \begin{vmatrix} M_2^2(s) - s & B_{12}(s) \\ B_{21}(s) & M_1^2(s) - s \end{vmatrix} \quad (3.188)$$

$$\simeq \frac{-1}{\Pi'(M_A^2)(M_A^2 - s)} \begin{vmatrix} M_2^2(M_A^2) - M_A^2 & B_{12}(M_A^2) \\ B_{21}(M_A^2) & M_1^2(M_A^2) - M_A^2 \end{vmatrix}.$$

In the right-hand side of (3.188), we keep singular (pole) terms only. The determinant of the matrix in the right-hand side of (3.188) equals zero:

$$[M_2^2(M_A^2) - M_A^2][M_1^2(M_A^2) - M_A^2] - B_{12}(M_A^2)B_{21}(M_A^2) = 0, \quad (3.189)$$

this is the consequence of Eq. (3.184) stating that $\Pi(M_A^2) = 0$. The equality (3.189) allows us to introduce a complex-valued mixing angle:

$$|A\rangle = \cos \theta_A |1\rangle - \sin \theta_A |2\rangle. \quad (3.190)$$

In this case the right-hand side of (3.188) assumes the form

$$[\hat{D}]_{s \sim M_A^2} = \frac{N_A}{M_A^2 - s} \begin{vmatrix} \cos^2 \theta_A & -\cos \theta_A \sin \theta_A \\ -\sin \theta_A \cos \theta_A & \sin^2 \theta_A \end{vmatrix}, \quad (3.191)$$

where

$$N_A = \frac{1}{\Pi'(M_A^2)} [2M_A^2 - M_1^2 - M_2^2], \quad (3.192)$$

$$\cos^2 \theta_A = \frac{M_A^2 - M_2^2}{2M_A^2 - M_1^2 - M_2^2}, \quad \sin^2 \theta_A = \frac{M_A^2 - M_1^2}{2M_A^2 - M_1^2 - M_2^2}.$$

Let us remind that in (3.192) the functions $M_1^2(s)$, $M_2^2(s)$ and $B_{12}(s)$ are fixed in the point $s = M_A^2$. As the angle θ_A is complex, the probabilities to find the states $|1\rangle$ and $|2\rangle$ in $|A\rangle$ are $|\cos \theta_A|^2$ and $|\sin \theta_A|^2$ rather than the usual $\cos^2 \theta_A$ and $\sin^2 \theta_A$.

To analyse the contents of the $|B\rangle$ state, a similar expansion of the matrix propagator has to be carried out near $s = M_B^2$. Introducing

$$|B\rangle = \sin \theta_B |1\rangle + \cos \theta_B |2\rangle, \quad (3.193)$$

we obtain the following expression for $\hat{D}$ in the neighbourhood of the second pole $s = M_B^2$:

$$\left[\hat{D} \right]_{s \sim M_B^2} = \frac{N_B}{M_B^2 - s} \begin{vmatrix} \sin^2 \theta_B & \cos \theta_B \sin \theta_B \\ \sin \theta_B \cos \theta_B & \cos^2 \theta_B \end{vmatrix}, \quad (3.194)$$

where

$$N_B = \frac{1}{\Pi'(M_B^2)} [2M_B^2 - M_1^2 - M_2^2], \quad (3.195)$$

$$\cos^2 \theta_B = \frac{M_B^2 - M_1^2}{2M_B^2 - M_1^2 - M_2^2}, \quad \sin^2 \theta_B = \frac{M_B^2 - M_2^2}{2M_B^2 - M_1^2 - M_2^2}.$$

In (3.195) the functions $M_1^2(s)$, $M_2^2(s)$ and $B_{12}(s)$ are fixed in the point $s = M_B^2$.

If there is only a weak s -dependence of B_{12} so that it can be neglected, the angles θ_A and θ_B coincide; in general, however, they are different, and the expressions for the propagator matrices differ from those in the standard quantum-mechanical description.

Another difference is related to the behaviour of levels in the mixing: in quantum mechanics the levels “repel” from the mean value $(E_1 + E_2)/2$ (see also Eq. (3.185)). Generally speaking, (3.184) may lead to either a “repulsion” or an “attraction” of the masses squared with respect to the mean value $(M_1^2 + M_2^2)/2$: this takes place because the levels are shifted in the complex plane (we discuss it in detail in the next subsection).

Up to now we have considered the case when both resonances transfer into the same state (single-channel case). The scattering amplitude for such a state is determined by the expression

$$A(s) = g_i(s) D_{ij}(s) g_j(s). \quad (3.196)$$

The existence of many decay channels leads to the redefinition of the block of loop diagrams. In the multichannel case $B_{ij}(s)$ is the sum of loop diagrams:

$$B_{ij}(s) = \sum_n B_{ij}^{(n)}(s), \quad (3.197)$$

where $B_{ij}^{(n)}$ is the loop diagram in the n channel with vertex functions $g_i^{(n)}$, $g_j^{(n)}$ and the phase space ρ_n . The partial scattering amplitude in the n channel is written as

$$A_n(s) = g_i^{(n)}(s) D_{ij}(s) g_j^{(n)}(s). \quad (3.198)$$

3.4.2 The case of many overlapping resonances: construction of propagator matrices

The above considerations can be easily expanded to the case of an arbitrary number N of resonance states. The propagator matrix $\hat{D}$, which describes the transitions of states, should satisfy the set of linear equations

$$\hat{D} = \hat{D}\hat{B}\hat{d} + \hat{d}, \quad (3.199)$$

where $\hat{B}$ is the matrix of one-loop diagrams similar to those in Fig. 3.18 and $\hat{d}$ is the diagonal propagator matrix for the initial states

$$\hat{d} = \text{diag} \left((m_1^2 - s)^{-1}, (m_2^2 - s)^{-1}, (m_3^2 - s)^{-1} \dots \right). \quad (3.200)$$

The poles in the matrix elements $D_{ij}(s)$ of the propagator matrix correspond to physical resonances appearing as a result of mixing. Let us denote the complex masses of these resonances as

$$s = M_A^2, \quad M_B^2, \quad M_C^2, \dots \quad (3.201)$$

Near the pole (*e.g.* $s = M_A^2$) only the leading pole term can be left in the propagator matrix. In this case, the matrix elements $D_{ij}(s \sim M_A^2)$ do not depend on the initial index i , and the solution assumes the factorised form

$$\left[\hat{D}^{(N)} \right]_{s \sim M_A^2} = \frac{N_A}{M_A^2 - s} \cdot \begin{vmatrix} \alpha_1^2 & \alpha_1 \alpha_2 & \alpha_1 \alpha_3 & \dots \\ \alpha_2 \alpha_1 & \alpha_2^2 & \alpha_2 \alpha_3 & \dots \\ \alpha_3 \alpha_1 & \alpha_3 \alpha_2 & \alpha_3^2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}, \quad (3.202)$$

where N_A is the normalisation factor, and the complex coupling constants are normalised by the condition

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \dots + \alpha_N^2 = 1. \quad (3.203)$$

The constants α_i are normalised transition amplitudes *resonance* $A \rightarrow$ *state* i . The probability to find the state i in a physical resonance A is

$$w_i = |\alpha_i|^2. \quad (3.204)$$

Analogous expansions of the propagator matrix can be carried out also near other poles:

$$D_{ij}^{(N)}(s \sim M_B^2) = N_B \frac{\beta_i \beta_j}{M_B^2 - s}, \quad D_{ij}^{(N)}(s \sim M_C^2) = N_C \frac{\gamma_i \gamma_j}{M_C^2 - s}, \dots \quad (3.205)$$

The coupling constants satisfy normalisation conditions similar to (3.203):

$$\beta_1^2 + \beta_2^2 + \dots + \beta_N^2 = 1, \quad \gamma_1^2 + \gamma_2^2 + \dots + \gamma_N^2 = 1, \dots \quad (3.206)$$

In the general case, however, the condition of completeness is not fulfilled for the inverse expansion, *i.e.*

$$\alpha_i^2 + \beta_i^2 + \gamma_i^2 + \dots \neq 1. \quad (3.207)$$

For two resonances, this means that $\cos^2 \Theta_A + \sin^2 \Theta_B \neq 1$. The reason for this incompleteness is the s -dependence of the loop diagrams B_{ij} . Could we neglect this dependence, as we did it in the expressions (3.185)–(3.187), the left-hand side of (3.207) would be equal to unity, that is, the inverse expansion would be also complete.

3.4.3 *A complete overlap of resonances: the effect of accumulation of widths by a resonance*

We consider here two examples which describe idealised cases of the complete overlap of two and three resonances. In these examples we observe the unperturbed effect of width accumulation by one of the neighbouring resonances.

a) A complete overlap of two resonances

For the sake of simplicity, let us discuss the case when B_{ij} depends weakly on s : we use (3.185). Suppose

$$M_1^2 = M_R^2 - iM_R\Gamma_1, \quad M_2^2 = M_R^2 - iM_R\Gamma_2, \quad (3.208)$$

and

$$\text{Re } B_{12}(M_R^2) = P \int_{(\mu_1 + \mu_2)^2}^{\infty} \frac{ds' g_1(s') g_2(s') \rho(s')}{\pi (s' - M_R^2)} \rightarrow 0. \quad (3.209)$$

For positive g_1 and g_2 , $\text{Re } B_{12}(M_R^2)$ can turn into zero, if the contribution of the integration over the region $s' < M_R^2$ is compensated by the contribution coming from the region $s' > M_R^2$. In this case,

$$B_{12}(M_R^2) \rightarrow i g_1(M_R^2) g_2(M_R^2) \rho(M_R^2) = i M_R \sqrt{\Gamma_1 \Gamma_2}. \quad (3.210)$$

Substituting (3.208)–(3.210) in the expression (3.185), we obtain:

$$M_A^2 \rightarrow M_R^2 - iM_R(\Gamma_1 + \Gamma_2) \quad M_B^2 \rightarrow M_R^2. \quad (3.211)$$

Hence, after the mixing one of the states accumulates the widths of the initial resonances, $\Gamma_A \rightarrow \Gamma_1 + \Gamma_2$, while the other state becomes a virtually stable particle, $\Gamma_B \rightarrow 0$.

b) A complete overlap of three resonances

The poles of the $N \times N$ matrix $\hat{D}$ are determined by the zeros of its determinant $\Pi^{(N)}(s)$. Consider the equation

$$\Pi^{(3)}(s) = 0 \quad (3.212)$$

in the same approximation as in the previous example. Thus, we assume

$$\text{Re } B_{ab}(M_R^2) \rightarrow 0, (a \neq b); \quad M_i^2 = M_R^2 - s - iM_R\Gamma_i = x - i\gamma_i. \quad (3.213)$$

We introduced here a new variable $x = M_R^2 - s$, and denoted $M_R\Gamma_i = \gamma_i$. Taking into account $B_{ij}B_{ji} = -\gamma_i\gamma_j$ and $B_{12}B_{23}B_{31} = -i\gamma_1\gamma_2\gamma_3$, Eq. (3.212) can be rewritten as

$$x^3 + x^2(i\gamma_1 + i\gamma_2 + i\gamma_3) = 0. \quad (3.214)$$

Hence, if the resonances overlap completely,

$$M_A^2 \rightarrow M_R^2 - iM_R(\Gamma_1 + \Gamma_2 + \Gamma_3), \quad M_B^2 \rightarrow M_R^2, \quad M_C^2 \rightarrow M_R^2. \quad (3.215)$$

The resonance A accumulates the widths of all three initial resonances, and the states B and C turn out to be virtually stable and degenerate.

3.5 K-Matrix Approach

In the experimental investigation of multichannel amplitudes the use of the K -matrix representation [7] turns out to be rather productive.

3.5.1 One-channel amplitude

First, let us remind the case of one resonance in a single channel scattering, when the amplitude is determined as

$$A(s) = \frac{g^2(s)}{m_0^2 - s - B(s)}, \quad (3.216)$$

and $B(s)$ is the loop diagram.

The K -matrix representation of the amplitude $A(s)$ is related to the separation of the imaginary part of the loop diagram:

$$\begin{aligned} A(s) &= \frac{g^2(s)}{m_0^2 - s - \text{Re } B(s) - i\rho(s)g^2(s)} = \frac{K(s)}{1 - i\rho(s)K(s)}, \\ K(s) &= \frac{g^2(s)}{m_0^2 - s - \text{Re } B(s)}. \end{aligned} \quad (3.217)$$

The function $\text{Re } B(s)$ in the two-particle loop diagram is analytical at $s = 4m^2$. We redefined the K -matrix term here, extracting the phase space, $K(s) = \rho(s)K(s)$ (to compare, see Eq. (3.124)). This means that the only possible singularities of $K(s)$ at $s > 0$ are the poles. In the left half-plane s , however, the function $K(s)$ contains singularities owing to the t -channel exchange.

The pole of the amplitude $A(s)$, determined by the equality

$$m_0^2 - s - B(s) = 0, \quad (3.218)$$

corresponds to the existence of a particle with quantum numbers of the considered partial wave.

If the K -matrix pole is above the threshold $s = 4m^2$, the corresponding state is a resonance: in what follows we consider just such a case. Let the condition (3.218) be satisfied at the point

$$s = M^2 \equiv \mu^2 - i\Gamma\mu. \quad (3.219)$$

We expand the real part of the denominator (3.216) in a series near $s = \mu^2$:

$$m_0^2 - s - \text{Re } B(s) \simeq (1 + \text{Re } B'(\mu^2))(\mu^2 - s) - i g^2(s)\rho(s). \quad (3.220)$$

The standard Breit–Wigner approximation appears if $\text{Im } B(s)$ is fixed in the point $s = \mu^2$. If the pole is close to the threshold singularity $s = 4m^2$, the s -dependence of the phase volume should be preserved. In this case we use a modified Breit–Wigner formula:

$$A(s) = \frac{\gamma}{\mu^2 - s - i\gamma\rho(s)}, \quad \gamma = \frac{g^2(\mu^2)}{1 + \text{Re } B'(\mu^2)}. \quad (3.221)$$

A similar resonance approximation can be carried out also when we use the K -matrix description of the amplitude. This corresponds to expanding in a series the function $K(s)$ represented in the form (3.217) near the point $s = \mu^2$:

$$K(s) = \frac{g^2(K)}{\mu^2 - s} + f, \quad (3.222)$$

where

$$g^2(K) = \frac{g^2(\mu^2)}{1 + \text{Re } B'(\mu^2)}, \quad f = \frac{g^2(\mu^2)}{2(1 + \text{Re } B'(\mu^2))} - \frac{2g(\mu^2)g'(\mu^2)}{1 + \text{Re } B'(\mu^2)}. \quad (3.223)$$

3.5.2 Multichannel amplitude

The resonance amplitude (3.216) can be generalised to a multichannel case. We consider here both the one-resonance amplitude and the multichannel one, with an arbitrary number of resonances.

(i) One-resonance amplitude: the Flatté formula

The multichannel one-resonance transition amplitude $b \rightarrow a$ reads:

$$A_{ab}(s) = \frac{g_a(s)g_b(s)}{m_0^2 - s - B(s)}, \quad B(s) = \sum_{c=1}^n B_{cc}(s), \quad (3.224)$$

where B_{cc} is the loop diagram with c -channel particles. Expanding (3.224) near the pole in s , as it was done in the previous section, we obtain the K -matrix form:

$$A_{ab}(s) = \frac{\gamma_a \gamma_b}{\mu^2 - s - i \sum_{c=1}^n \gamma_c^2 \rho(s)}. \quad (3.225)$$

This is the Flatté formula [16]. In the case of the two-channel amplitude ($\pi\pi, K\bar{K}$) it is widely used for the description of $f_0(980)$ (for example, see [17]). Actually, the Flatté formula is not quite precise for this purpose. A more adequate description of the data can be achieved either by using the two channel K -matrix or by modifying the resonance formula, introducing the transition length $\pi\pi \rightarrow K\bar{K}$, see Appendix 3.A.

(ii) Two-channel amplitude

The two-channel K -matrix amplitude can be easily obtained starting from the one-channel amplitude (3.124) by inserting the second-channel interactions into the block K :

$$K \rightarrow K_{11} + K_{12} \frac{1}{1 - iK_{22}} K_{21}. \quad (3.226)$$

The first term, K_{11} , gives us a direct transition *channel 1* $\rightarrow$ *channel 1*, while the second one describes the transition into channel 2 (block K_{12}), rescatterings in this channel (factor $(1 - iK_{22})^{-1}$) and the return into channel 1 (block K_{21}). We have as a result:

$$A_{11}(s) = \frac{K_{11} + i[K_{12}K_{21} - K_{11}K_{22}]}{1 - iK_{11} - iK_{22} + [K_{12}K_{21} - K_{11}K_{22}]} . \quad (3.227)$$

The transition amplitude reads:

$$A_{11}(s) = \frac{K_{12}}{1 - iK_{11} - iK_{22} + [K_{12}K_{21} - K_{11}K_{22}]} . \quad (3.228)$$

The two-channel amplitude satisfies the unitarity condition:

$$\text{Im } A_{11}(s) = \frac{1}{2i} (A_{11}(s) - A_{11}^*(s)) = \sum_a A_{1a}^*(s) A_{a1}(s), \quad (3.229)$$

and the amplitude can be presented in the matrix form

$$\hat{A} = \frac{\hat{K}}{1 - i\hat{K}}, \quad (3.230)$$

where $\hat{A}$ and $\hat{K}$ are 2×2 matrices:

$$\hat{A} = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}, \quad \hat{K} = \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix}. \quad (3.231)$$

For example, for the cases $1 = \pi\pi$ and $2 = K\bar{K}$ the amplitude A_{11} refers to the scattering amplitude $\pi\pi \rightarrow \pi\pi$, and A_{12} is the transition amplitude $\pi\pi \rightarrow K\bar{K}$; just these two channels give the main contribution into the wave $I = 0, J^{PC} = 0^{++}$ in the region ~ 1000 MeV.

The matrix elements K_{ab} contain threshold singularities. To extract these singularities, one has to redefine the K -matrix elements:

$$K_{ab} = \sqrt{\rho_a} K_{ab} \sqrt{\rho_b}, \quad (3.232)$$

where $\sqrt{\rho_a}$ and $\sqrt{\rho_b}$ are space factors of the states a and b . In strong interactions, $K_{21} = K_{12}$. Matrix elements K_{ab} are real and do not contain threshold singularities; they, however, may have pole singularities.

(iii) Multichannel amplitude with an arbitrary number of resonances

Describing meson-meson spectra, it is convenient to work with the elements K_{ab} , where threshold singularities are extracted, see (3.232). If so, the n -channel amplitude has the form

$$\hat{A} = \hat{K} \frac{I}{I - i\hat{\rho}\hat{K}}, \quad (3.233)$$

where $\hat{K}$ is the $n \times n$ matrix, with $K_{ab}(s) = K_{ba}(s)$; I is a unit $n \times n$ matrix, $I = \text{diag}(1, 1, \dots, 1)$, and $\hat{\rho}$ is the diagonal matrix of phase volumes

$$\hat{\rho} = \text{diag}(\rho_1(s), \rho_2(s), \dots, \rho_n(s)). \quad (3.234)$$

The elements of the K -matrix are constructed as sums of pole terms and the smooth, non-singular in the physical region, term $f_{ab}(s)$:

$$K_{ab}(s) = \sum_{\alpha} \frac{g_a^{(\alpha)} g_b^{(\alpha)}}{\mu_{\alpha}^2 - s} + f_{ab}(s). \quad (3.235)$$

In Appendix 3.B we present K -matrix analyses of the partial wave amplitudes ($IJ^{PC} = 00^{++}$) for the reactions $\pi\pi \rightarrow \pi\pi, K\bar{K}, \eta\eta, \eta\eta', \pi\pi\pi\pi$ in the mass region 450–1950 MeV. The K -matrix analysis of reactions $\pi K \rightarrow \pi K, \eta' K$ and $K\pi\pi\pi$ is given in Appendix 3.C.

3.5.3 The problem of short and large distances

Classifying quark–antiquark and gluonium states, we face the closely related problems of the quark–hadron duality and the role of short and large distances to the meson spectrum formation.

Let us discuss these problems using the language of the potential quark model, when the levels of the $q\bar{q}$ states are determined by a potential increasing infinitely at large r : $V(r) \sim \alpha r$ (see Fig. 3.19a). The infinitely growing potential produces an infinite set of stable $q\bar{q}$ levels. This is, obviously, a simplified picture, since only the lowest $q\bar{q}$ levels are stable with respect to hadronic decays. Higher states decay into hadrons: an excited $(q\bar{q})_a$ state produces a new $q\bar{q}$ pair, after which the quarks $(q\bar{q})_a + (q\bar{q})$ recombine into mesons, which leave the confinement trap for the continuous spectrum, see Fig. 3.20.

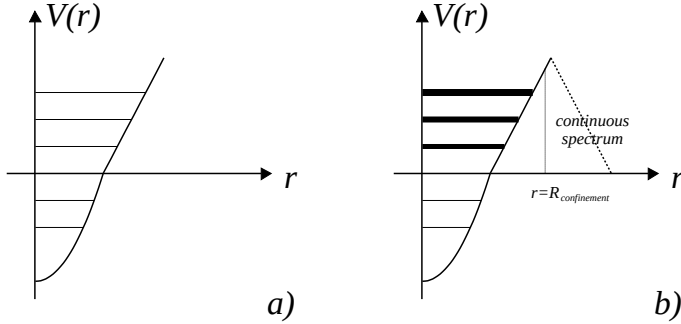


Fig. 3.19 (a) Potential of the standard quark model with stable $q\bar{q}$ levels; (b) potential with unstable upper levels, which imitates the actual situation for the highly excited $q\bar{q}$ states.

Figure 3.19b displays the schematic structure of the meson level spectrum, when decay processes are included into consideration.

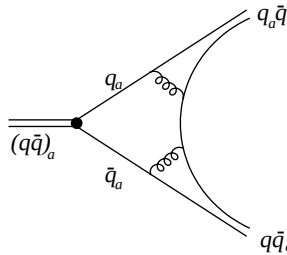


Fig. 3.20 Decay of the $(q\bar{q})_a$ level due to the production of a new $q\bar{q}$ pair state.

The interaction related to confinement is represented here by a potential barrier: the interaction at $r < R_{\text{confinement}}$ forms the discrete spectrum of $q\bar{q}$ levels, while the transitions into the $r > R_{\text{confinement}}$ region provide the continuous meson spectrum. It is just this meson spectrum which is observed experimentally, and the task of reconstructing $q\bar{q}$ levels formed at $r < R_{\text{confinement}}$ is directly connected to the problem of understanding the impact of mesonic decay spectra on the level shift. Carrying out a $q\bar{q}$ classification of the levels requires the elimination of the effect of meson decays.

This problem can be roughly solved in the framework of the K -matrix description of meson spectra, when the contribution of transitions into real meson states is killed in the K -matrix amplitude. Formally, this is equivalent to the transition to the limit $\rho_a \rightarrow 0$ in (3.233). If only leading pole singularities are taken into account, the transition amplitude $b \rightarrow a$ can be written in the form

$$A_{ab}^{\text{bare}}(s) = K_{ab}(s) = \frac{g_a(K)g_b(K)}{\mu^2 - s} + f_{ab}. \quad (3.236)$$

Hence, the pole of the K -matrix corresponds to a state where the “coat” of the real mesons is eliminated. This is the reason for calling the corresponding states “bare mesons” [18, 19, 20]. Let us remind that this definition is different from the definition of bare particles in field theory, where the “coat” includes virtual states off the mass shell.

In the case when the $q\bar{q}$ spectrum includes several states with identical quantum numbers, the amplitude $A_{ab}^{\text{bare}}(s)$ is determined by the sum of the corresponding poles:

$$A_{ab}^{\text{bare}}(s) = \sum_{\alpha} \frac{g_a^{(\alpha)}(K)g_b^{(\alpha)}(K)}{\mu_{\alpha}^2 - s} + f_{ab}. \quad (3.237)$$

The approximation of the amplitude in terms of a series of poles at $r < R_{\text{confinement}}$ is not new: it is widely used in dual models and when considering the leading contributions in the $1/N_c$ -expansion. From the point of view of such models, the term f_{ab} independent of s is just the sum of pole contributions which are far from the considered region.

The coupling constants of the bare states, $g_a^{(\alpha)}(K)$, serve us as a source of information on the quark–gluon content of this state.

3.5.4 *Overlapping resonances: broad locking states and their role in the formation of the confinement barrier*

Resonance decay processes may play another important role in the physics of mesons. Indeed, in the case of overlapping resonances broad states can be formed via the accumulation of widths of the neighbouring states, thus playing the role of “locking states” for their neighbours (we have seen this when we investigated the D -matrix in Section 3.4). This fact leads to the idea that the existence of a broad state is instrumental in forming the confinement barrier.

Resonances with the same quantum numbers can easily overlap when a state of different nature, formed by different forces (*e.g.* a gluonium gg) appears among the $q\bar{q}$ - levels. If the direct transition $q\bar{q} \rightarrow gg$ is, by some reasons, suppressed at small distances, then the transition $q\bar{q} \rightarrow \text{mesons} \rightarrow gg$ begins to take place. As a consequence, the state of “different nature” (gg in our consideration) accumulates the widths of the closest $q\bar{q}$ states. Hence, the formation of broad states may be a general phenomenon for exotic states.

3.5.4.1 *Accumulation of widths in the K -matrix approach*

To examine the mixing of non-stable states in a pure form, consider as an example three resonances decaying into the same channel. In the K -matrix approach, the amplitude we consider reads:

$$A = K(1 - i\rho K)^{-1}, \quad K = g^2 \sum_{a=1,2,3} \frac{1}{(M_a^2 - s)}. \quad (3.238)$$

Here, to be illustrative, we take g^2 to be the same for all three resonances, and make the approximations that:

- (i) the phase space factor ρ is constant, and
- (ii) $M_1^2 = m^2 - \delta$, $M_2^2 = m^2$, $M_3^2 = m^2 + \delta$. Figure 3.21 shows the location of poles in the complex- M plane ($M = \sqrt{s}$) as the coupling g increases. At large g , which corresponds to a strong overlapping of the resonances, one resonance accumulates the widths of the others while two counterparts of the broad state become nearly stable.

The idea according to which the exotic states, when appearing among the usual $q\bar{q}$ -mesons, transform into broad resonances and play the role of locking states for the neighbouring $q\bar{q}$ levels, was formulated in [21].

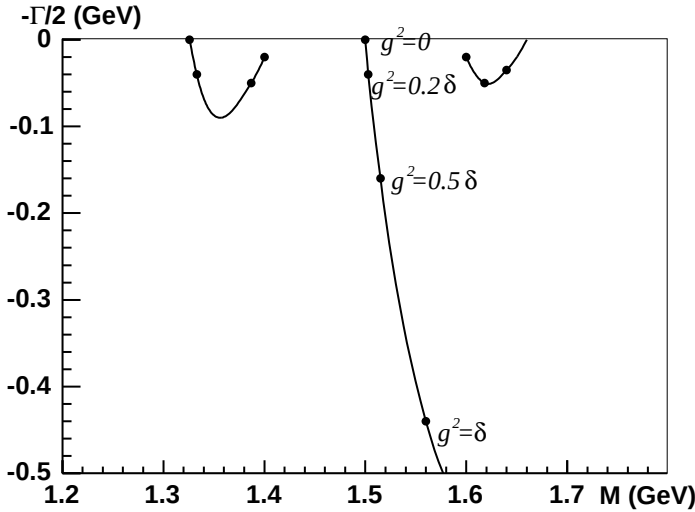


Fig. 3.21 Position of the poles of the amplitude of Eq. (3.238) in the complex- $\sqrt{s}$ plane ($\sqrt{s} = M - i\Gamma/2$) with the increase of g^2 . In this example $m = 1.5$ GeV, $\delta = 0.5$ GeV² and the phase space factor is fixed, $\rho = 1$.

3.6 Elastic and Quasi-Elastic Meson–Meson Reactions

Meson–meson amplitudes are not a subject of direct experimental study, they are extracted from the study of meson–nucleon (or meson–nucleus) collisions with the production of mesons.

3.6.1 Pion exchange reactions

The most popular way to get information about meson–meson amplitudes is to consider a meson–nucleon reaction, with meson production at small momentum transfer squared to nucleon (t); examples are shown in Fig. 3.22.

At small t the pion exchange, as a rule, dominates. This simplifies the extraction of meson–meson amplitudes. For example, for the amplitude of Fig. 3.22a one can suggest at $t \sim 0$:

$$A_{\pi N \rightarrow \pi \pi N} = A_{\pi \pi \rightarrow \pi \pi} \frac{G_N}{\mu_\pi^2 - t} + \text{smooth term}, \quad (3.239)$$

where $A_{\pi \pi \rightarrow \pi \pi}$ is the pion–pion scattering amplitude and G_N pion–nucleon vertex. Such a representation can be justified at rather small t only. Hence,

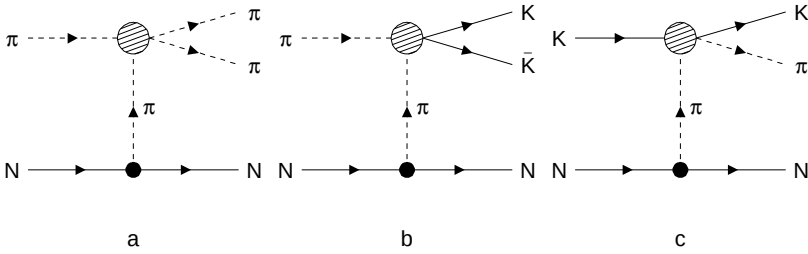


Fig. 3.22 Reactions $\pi N \rightarrow \pi\pi N$ (a), $\pi N \rightarrow \pi\pi N$ (b) and $KN \rightarrow K\pi N$ (c) determined by the t -channel pion exchange.

to study the pion–pion amplitude in a broad interval of the pion–pion mass ($M_{\pi\pi} \sim 500 - 2000$ MeV), one should work at large total energies ($s_{\pi N} \gg M_{\pi\pi}^2$).

At $|t| \gtrsim 0.1$ GeV², the contributions of other exchanges may be essential. To take into account other meson exchanges, it is convenient to use the Regge pole technique.

3.6.2 Regge pole propagators

Here we present Regge pole propagators using as an example the two-body reactions.

If we have a look at the Mandelstam plane (Fig. 3.9), we find there an interesting and important region: the region of high energies (s , for example) and small momentum transfers (let it be t). In this region the Regge phenomenology, which can be considered as a generalisation of pole phenomenology, is rather successful.

Let us turn our attention to the pole diagrams, presented in Fig. 3.22. In the region of high $s = (p_1 + p_2)^2$ and small $t = (p_1 - p'_1)^2$ the nearest strong singularity is given by the pole diagram Fig. 3.10a: $g^2/(\mu^2 - t)$. In the framework of Regge phenomenology we can, making use of the Regge pole propagators, take into account the exchange of a whole series of poles lying on the Regge trajectories (see Chapter 2, where linear trajectories in the (J, M^2) plane are presented, and Fig. 3.23).

The Regge pole theory, which was developed in the framework of the quantum mechanical problem of particle scattering [22], was later generalised for the relativistic two-particle scattering processes [23, 24].

Let us consider a one-reggeon exchange amplitude (Fig. 3.23). Such an

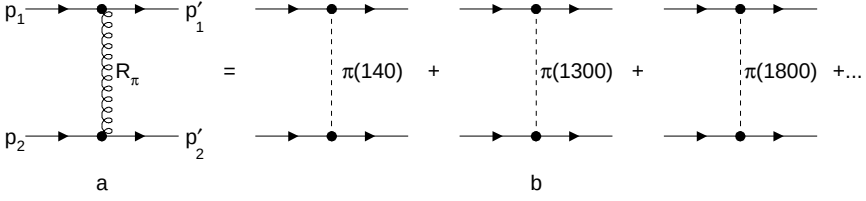


Fig. 3.23 Pion reggeon exchange (a) as an account for strong t -channel pole singularities: $\pi(140)$, $\pi(1300)$, $\pi(1800)$, and so on.

amplitude has the structure

$$\hat{G}_1(t)R(\nu, t)\hat{G}_2(t), \quad (3.240)$$

where $\hat{G}_1$ and $\hat{G}_2$ are vertices (the upper and lower blocks in Fig. 3.23). They depend on t and the masses of the blocks (*e.g.* in Fig. 3.23, $\hat{G}_1(t)$ depends on m_1^2 and $m_1'^2$). The reggeon amplitude of the process $1 + 2 \rightarrow 1' + 2'$ is supposed to describe also the crossing process $1 + \bar{2}' \rightarrow 1' + \bar{2}$ in which the high energy is u at small t , see Fig. 3.24. Thus, to write the reggeon propagator correctly, we have to use the variable

$$\nu = \frac{s - u}{2}. \quad (3.241)$$

However, $s + t + u = \sum_{i=1,2,3,4} m_i^2$. Consequently, the use of the variables s and u is equivalent in the region where the reggeon propagator is considered, *i.e.* at large s and $|u|$ and relatively small $|t|$ and m_i^2 , since $\nu \simeq s \simeq |u|$.

In Fig. 3.24 the physical regions of the reactions $1 + 2 \rightarrow 1' + 2'$ and $1 + \bar{2}' \rightarrow 1' + \bar{2}$ are presented at small $|t|$ values. Presuming a power dependence of the Regge amplitude at large s (or $|u|$) values and making use of its analytical properties, we can write the amplitudes $1 + 2 \rightarrow 1' + 2'$ and $1 + \bar{2}' \rightarrow 1' + \bar{2}$ in the form

$$A_R^{1+2 \rightarrow 1'+2'}(s, t) = \hat{G}_{1 \rightarrow 1'}(t) \frac{s^{\alpha_R(t)} \pm (-s)^{\alpha_R(t)}}{\sin[\pi\alpha_R(t)]} \hat{G}_{2 \rightarrow 2'}(t). \quad (3.242)$$

The factor

$$R(s, t) = \frac{s^{\alpha_R(t)} + \xi_R(-s)^{\alpha_R(t)}}{\sin[\pi\alpha_R(t)]}, \quad \xi_R = \pm 1 \quad (3.243)$$

is the reggeon propagator. The reggeon propagator satisfies the analytical properties reflected in Fig. 3.24.

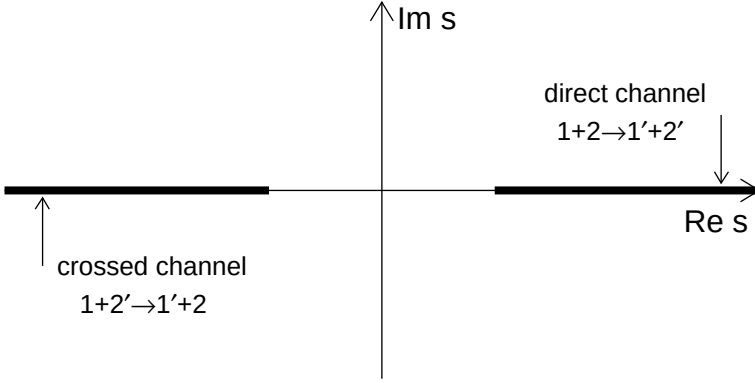


Fig. 3.24 Physical regions of the direct $1 + 2 \rightarrow 1' + 2'$ and crossed $1 + \bar{2}' \rightarrow 1' + \bar{2}$ channels of the reaction.

Indeed, at $s \gg m^2$ the phase is determined as

$$(-s)^{\alpha_R(t)} = \exp[-i\pi\alpha_R(t)]s^{\alpha_R(t)}. \quad (3.244)$$

So in the region $Re s \simeq 0$ the propagator (and, hence, the scattering amplitude) is real, as it is required (see the Mandelstam plane in Fig. 3.9).

Depending on the signature ξ_R , the Regge amplitude of the transition of the direct channel (with s the total energy squared) to the crossing channel (where u is the total energy squared) is either an even ($\xi_R = +1$) function, or an odd ($\xi_R = -1$) one.

Let us make another remark to Eqs. (3.242)–(3.244). Usually, in numerical calculations, the parameter s_0 is introduced to replace $s \rightarrow s/s_0$; here s_0 is of the order of the hadron mass squared. Using (3.244), the propagators for $\xi_R = +1$ and $\xi_R = -1$ can be rewritten:

$$\begin{aligned} \xi_R = +1 : & \quad \frac{\exp\left[-i\frac{\pi}{2}\alpha_R(t)\right]}{\sin\left[\frac{\pi}{2}\alpha_R(t)\right]} \left(\frac{s}{s_0}\right)^{\alpha_R(t)}, \\ \xi_R = -1 : & \quad \frac{i \exp\left[-i\frac{\pi}{2}\alpha_R(t)\right]}{\cos\left[\frac{\pi}{2}\alpha_R(t)\right]} \left(\frac{s}{s_0}\right)^{\alpha_R(t)}. \end{aligned} \quad (3.245)$$

This is the standard form of the reggeon propagators, see, *e.g.*, [25, 26].

We see that the factor $1/\sin\left[\frac{\pi}{2}\alpha_R(t)\right]$ has poles when $\alpha_R(t)$ is integer and even. It reproduces the poles corresponding to the meson states $J = 0, 2, 4, 6, \dots$. The factor $1/\cos\left[\frac{\pi}{2}\alpha_R(t)\right]$ provides us with poles of states with odd J values, namely, $J = 1, 3, 5, \dots$

The trajectories $\alpha_R(t)$, denoted in Chapter 2 as $\alpha_R(M^2)$, are presented in Fig. 2.4 for different states. We saw that they are linear at $t = M^2 > 0$.

Moving now with these linear trajectories into the region of negative t values, we arrive at an obviously incorrect result: poles at $M^2 < 0$ appear. But this is quite understandable: we just wrote a simplified denominator in the propagator $R(s, t)$ (Eqs. (3.243) and (3.245)) which has to be corrected. To carry out the required modification, we can start with (3.245): it is reasonable to introduce Γ functions so that their poles compensate the zeros in the denominators of the propagators. (Recall that $\Gamma(z)$ turns into ∞ at $z = 0, -1, -2, -3, \dots$). Thus, we write

$$\begin{aligned} \xi_R = +1: \quad R(s, t) &= \frac{\exp \left[-i \frac{\pi}{2} \alpha_R(t) \right]}{\sin \left[\frac{\pi}{2} \alpha_R(t) \right] \cdot \Gamma \left(\frac{1}{2} \alpha_R(t) + 1 \right)} \left(\frac{s}{s_0} \right)^{\alpha_R(t)}, \\ \xi_R = -1: \quad R(s, t) &= i \frac{\exp \left[-i \frac{\pi}{2} \alpha_R(t) \right]}{\cos \left[\frac{\pi}{2} \alpha_R(t) \right] \cdot \Gamma \left(\frac{1}{2} \alpha_R(t) + \frac{1}{2} \right)} \left(\frac{s}{s_0} \right)^{\alpha_R(t)}. \end{aligned} \quad (3.246)$$

Now there are no false poles in the region $t < 0$ any more.

Since the Γ -functions obey the relations

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad \Gamma\left(\frac{1}{2} + z\right)\Gamma\left(\frac{1}{2} - z\right) = \frac{\pi}{\sin \pi z}, \quad \Gamma(z+1) = z\Gamma(z), \quad (3.247)$$

equations (3.246) can be written in different forms.

Let us turn our attention to the fact that there are certain trajectories in the (J, M^2) plane where some states are lacking (they are absent for $q\bar{q}$ systems in the quark model). The trajectories in question are those for a_J mesons or f_J mesons with $J = 2, 4, \dots$. For these trajectories the Γ -function in the denominator has to be modified: $\Gamma(\frac{1}{2}\alpha_R(t) + 1) \rightarrow \Gamma(\frac{1}{2}\alpha_R(t))$.

The f_2 trajectory (it is also called the $\mathbf{P}'$ trajectory) may serve us as an example. In this case the propagator is written in the form

$$R_{f_2(\text{leading})}(s, t) = \frac{\exp \left[-i \frac{\pi}{2} \alpha_{f_2}(t) \right]}{\sin \left(\frac{\pi}{2} \alpha_{f_2}(t) \right) \Gamma \left(\frac{1}{2} \alpha_{f_2}(t) \right)} \left(\frac{s}{s_0} \right)^{\alpha_{f_2}(t)}. \quad (3.248)$$

The first meson state placed on this trajectory is the tensor meson $f_2(1275)$, and there are no scalar mesons on this trajectory (see Chapter 2).

3.7 Appendix 3.A: The $f_0(980)$ in Two-Particle and Production Processes

Concerning $f_0(980)$, there exists a strong $K\bar{K}$ threshold near the pole, so the resonance in the amplitude is described not as a generalised Breit–Wigner formula (the Flatté pole term [16]) but in a more complicated way.

(i) Two-channel amplitudes $\pi\pi \rightarrow \pi\pi$, $\pi\pi \rightarrow K\bar{K}$ and $K\bar{K} \rightarrow K\bar{K}$

For the $\pi\pi \rightarrow \pi\pi$, $\pi\pi \rightarrow K\bar{K}$ and $K\bar{K} \rightarrow K\bar{K}$ transitions near $f_0(980)$, a reasonably good description of data can be given by the following resonance amplitudes [27]:

$$\begin{aligned} R_{f_0(980)}^{(\pi\pi \rightarrow \pi\pi)} &= \frac{G_{\pi\pi}^2 + i\rho_{K\bar{K}}F(s)}{D(s)}, & R_{f_0(980)}^{(K\bar{K} \rightarrow K\bar{K})} &= \frac{G_{K\bar{K}}^2 + i\rho_{\pi\pi}F(s)}{D(s)}, \\ R_{f_0(980)}^{(\pi\pi \rightarrow K\bar{K})} &= \frac{G_{\pi\pi}G_{K\bar{K}} + if_{\pi\pi \rightarrow K\bar{K}}(\rho_{K\bar{K}}G_{K\bar{K}}^2 + \rho_{\pi\pi}G_{\pi\pi}^2)}{D(s)}, \end{aligned} \quad (3.249)$$

where

$$\begin{aligned} \rho_{K\bar{K}} &= \frac{1}{m_0} \sqrt{s - 4m_K^2}, & \rho_{\pi\pi} &= \frac{1}{m_0} \sqrt{s - 4m_\pi^2}, \\ F(s) &= 2G_{\pi\pi}G_{K\bar{K}}f_{\pi\pi \rightarrow K\bar{K}} + f_{\pi\pi \rightarrow K\bar{K}}^2(m_0^2 - s), \\ D(s) &= m_0^2 - s - i\rho_{\pi\pi}G_{\pi\pi}^2 - i\rho_{K\bar{K}}G_{K\bar{K}}^2 + \rho_{\pi\pi}\rho_{K\bar{K}}F. \end{aligned} \quad (3.250)$$

Here m_0 is the input mass of $f_0(980)$, $G_{\pi\pi}$ and $G_{K\bar{K}}$ are coupling constants to pion and kaon channels. The dimensionless constant $f_{\pi\pi \rightarrow K\bar{K}}$ stands for the prompt transition $\pi\pi \rightarrow K\bar{K}$: the value f/m_0 is the “transition length” which is analogous to the scattering length of the low-energy hadronic interaction. The constants m_0 , $G_{\pi\pi}$, $G_{K\bar{K}}$, $f_{\pi\pi \rightarrow K\bar{K}}$ are parameters which are to be chosen to reproduce the $f_0(980)$ characteristics.

The $\pi\pi$ scattering amplitude in the region 900–1100 MeV has two components: a smooth background and a contribution of the $f_0(980)$. It reads:

$$A_{\pi\pi \rightarrow \pi\pi} = e^{i\theta} R_{f_0(980)}^{(\pi\pi, \pi\pi)} + e^{i\frac{\theta}{2}} \sin \frac{\theta}{2}. \quad (3.251)$$

The background term in (3.251) is fixed by the requirement that the $\pi\pi$ scattering amplitude below the $K\bar{K}$ threshold has the form $\exp(i\delta) \sin \delta$.

Let us graphically illustrate different terms in (3.249). For that purpose we neglect the self-energy part in the $f_0(980)$ propagator: $1/D(s) \simeq 1/(m_0^2 - s)$. Then $R_{f_0(980)}^{(\pi\pi \rightarrow \pi\pi)}$ is given by four diagrams of Fig. 3.25a (we denote 1 = $\pi\pi$ and 2 = $K\bar{K}$), $R_{f_0(980)}^{(K\bar{K} \rightarrow K\bar{K})}$ by diagrams of Fig. 3.25b and $R_{f_0(980)}^{(\pi\pi \rightarrow K\bar{K})}$ by diagrams of Fig. 3.25c.

(ii) Production of $f_0(980)$ in multiparticle processes

The production of $f_0(980)$ in multiparticle process with the subsequent decay $f_0(980) \rightarrow \pi\pi$ is given in the approximation $1/D(s) \simeq 1/(m_0^2 - s)$ by diagrams of Fig. 3.26. Correspondingly, we write:

$$\begin{aligned} A(\text{initial state} \rightarrow [f_0(980) \rightarrow \pi\pi] + \text{outgoing particles}) &= \\ &= \Lambda_{f_0(980)} \frac{G_{\pi\pi} + if_{\pi\pi \rightarrow K\bar{K}}\rho_{K\bar{K}}G_{K\bar{K}}}{D(s)}, \end{aligned} \quad (3.252)$$

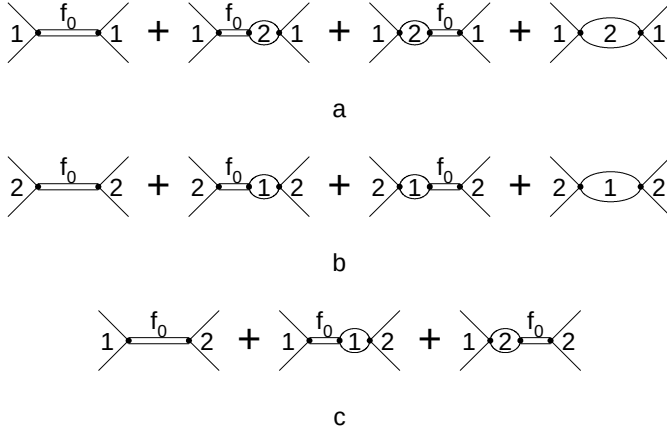


Fig. 3.25 Diagrams describing processes $\pi\pi \rightarrow \pi\pi$ (a), $K\bar{K} \rightarrow K\bar{K}$ (b), and $\pi\pi \rightarrow K\bar{K}$ (c) in the region of the resonance $f_0(980)$.

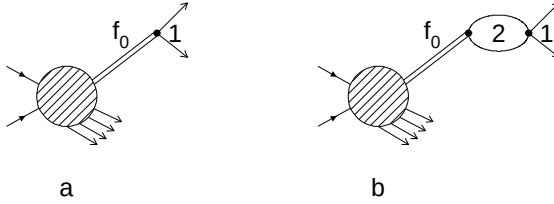


Fig. 3.26 Production of $f_0(980)$ and its subsequent decay $f_0(980) \rightarrow \pi\pi$ in multiparticle reactions.

where $\Lambda_{f_0(980)}$ is the multiparticle production block. Considering the decay $f_0(980) \rightarrow K\bar{K}$, one should replace in (3.252):

$$G_{\pi\pi} + if_{\pi\pi \rightarrow K\bar{K}} \rho_{K\bar{K}} G_{K\bar{K}} \rightarrow G_{K\bar{K}} + if_{\pi\pi \rightarrow K\bar{K}} \rho_{\pi\pi} G_{\pi\pi} . \quad (3.253)$$

(iii) Parameters

Two sets of parameters exist with sufficiently correct values of the $f_0(980)$ pole position and couplings. They are (in GeV units):

$$\text{Solution A: } m_0 = 1.000, f = 0.516, G = 0.386, G_{K\bar{K}} = 0.447,$$

$$\text{Solution B: } m_0 = 0.952, f = -0.478, G = 0.257, G_{K\bar{K}} = 0.388. \quad (3.254)$$

The above parameters provide us with a reasonable description of the $\pi\pi$ scattering amplitude. The phase shift δ_0^0 and the inelasticity parameter η_0^0

are shown in Fig. 3.27; the angle θ for the background term in Solutions A and B, determined as

$$\theta = \theta_1 + \left(\frac{\sqrt{s}}{m_0} - 1\right)\theta_2, \quad (3.255)$$

is numerically

$$\begin{aligned} \text{Solution A: } & \theta_1 = 189^\circ, \quad \theta_2 = 146^\circ, \\ \text{Solution B: } & \theta_1 = 147^\circ, \quad \theta_2 = 57^\circ. \end{aligned} \quad (3.256)$$

Solutions A and B give significantly different predictions for η_0^0 ; however, the existing data do not allow us to discriminate between them.

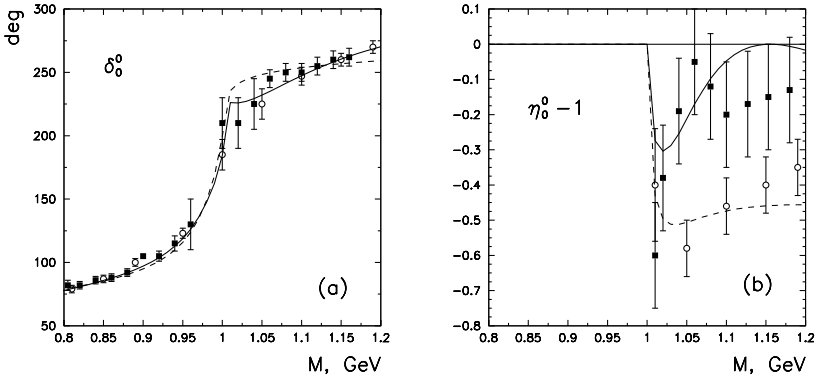


Fig. 3.27 Reaction $\pi\pi \rightarrow \pi\pi$: description of δ_0^0 and η_0^0 in the region of $f_0(980)$. Solid and dashed curves correspond to the parameter sets A and B. Data are taken from [19] (full squares) and [33] (open circles).

3.8 Appendix 3.B: K -Matrix Analyses of the $(IJ^{PC} = 00^{++})$ -Wave Partial Amplitude for Reactions $\pi\pi \rightarrow \pi\pi, K\bar{K}, \eta\eta, \eta\eta', \pi\pi\pi\pi$

To be illustrative, we give here, following [28], a detailed description of the technique of the K -matrix analysis of the partial wave $IJ^{PC} = 00^{++}$ in the reactions $\pi\pi \rightarrow \pi\pi, K\bar{K}, \eta\eta, \eta\eta', \pi\pi\pi\pi$. We demonstrate that, in the framework of the K -matrix approach, the analytical amplitude can be reconstructed on the basis of the available data [29, 30, 31, 32, 33] in the mass region $450 \text{ MeV} < \sqrt{s} < 1950 \text{ MeV}$. The following

scalar–isoscalar states are seen: comparatively narrow resonances $f_0(980)$, $f_0(1300)$, $f_0(1500)$, $f_0(1750)$ and the broad state $f_0(1200–1600)$. The positions of the amplitude poles (masses and total widths of the resonances) are determined as well as the pole residues (partial widths to meson channels $\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$, $\pi\pi\pi\pi$). The fitted amplitude gives us the positions of the K -matrix poles (bare states) and the values of the bare state couplings to meson channels thus allowing the quark-antiquark nonet classification of bare states.

A detailed story presented below on the fitting procedure and on obtaining several different solutions aims to emphasise that, when working with as many as possible samples of experimental data, there still exist the uncertainties in the 00^{++} amplitude. It is indeed astonishing that some groups have worked with a limited set of data (these papers are quoted in [34]) and obtained a unique solution with a rather high accuracy. We learned from our investigations [28, 33] that one should be rather careful with the recognition of results of such incomplete studies of the 00^{++} channel.

3.8.0.1 Scattering amplitude

For the S -wave scattering amplitude in the scalar–isoscalar sector we use a parametrisation similar to that of [28, 33]:

$$K_{ab}^{00}(s) = \left(\sum_{\alpha} \frac{g_a^{(\alpha)} g_b^{(\alpha)}}{M_{\alpha}^2 - s} + f_{ab} \frac{1 \text{ GeV}^2 + s_0}{s + s_0} \right) \frac{s - s_A}{s + s_{A0}}, \quad (3.257)$$

where K_{ab}^{IJ} is a 5×5 matrix ($a, b = 1, 2, 3, 4, 5$), with the following notations for the meson channels: $1 = \pi\pi$, $2 = K\bar{K}$, $3 = \eta\eta$, $4 = \eta\eta'$ and $5 =$ multimeson states (four-pion states were measured at $\sqrt{s} < 1.6 \text{ GeV}$). The $g_a^{(\alpha)}$ is the coupling constant of the bare state α to the meson channel; the parameters f_{ab} and s_0 describe the smooth part of the K -matrix elements ($1 \leq s_0 \leq 5 \text{ GeV}^2$). The factor $(s - s_A)/(s + s_{A0})$ is used to suppress the false kinematical singularity at $s = 0$ in the physical region near the $\pi\pi$ threshold. The parameters s_A and s_{A0} are kept to be of the order of $s_A \sim (0.1 - 0.5)m_{\pi}^2$ and $s_{A0} \sim (0.1 - 0.5) \text{ GeV}^2$; for these intervals, the results practically do not depend on the precise values of s_A and s_{A0} .

For the two-particle states, $\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$, the phase space matrix elements are written as:

$$\rho_a(s) = \sqrt{\frac{s - (m_{1a} + m_{2a})^2}{s}}, \quad a = 1, 2, 3, 4, \quad (3.258)$$

where m_{1a} and m_{2a} are the masses of pseudoscalars. The multi-meson phase space factor is determined as follows:

$$\begin{aligned}\rho_5(s) &= \begin{cases} \rho_{51} & \text{at } s < 1 \text{ GeV}^2, \\ \rho_{52} & \text{at } s > 1 \text{ GeV}^2, \end{cases} \\ \rho_{51} &= \rho_0 \int \frac{ds_1}{\pi} \int \frac{ds_2}{\pi} \\ &\quad \times \frac{M^2 \Gamma(s_1) \Gamma(s_2) \sqrt{(s + s_1 - s_2)^2 - 4ss_1}}{s[(M^2 - s_1)^2 + M^2 \Gamma^2(s_1)][(M^2 - s_2)^2 + M^2 \Gamma^2(s_2)]}, \\ \rho_{52} &= \left(\frac{s - 16m_\pi^2}{s} \right)^n. \end{aligned} \quad (3.259)$$

Here s_1 and s_2 are the two-pion energies squared, M is the mass of the ρ -meson and $\Gamma(s)$ refers to its energy-dependent width, $\Gamma(s) = \gamma \rho_1^3(s)$. The factor ρ_0 provides the continuity of $\rho_5(s)$ at $s = 1 \text{ GeV}^2$. The power parameter n is taken to be 1, 3, 5 for different versions of the fitting; the results are weakly dependent on these values (in the analysis [33] the value $n = 5$ was used).

3.8.0.2 The fitting procedure

For the decay couplings of bare states, $g_a^{(\alpha)}$, quark combinatorial relations in the leading terms of $1/N$ -expansion are imposed, see Chapter 2.

The rules of quark combinatorics were first suggested for the high energy hadron production [35] and then extended to hadronic J/Ψ decays [36]. The quark combinatorial relations were used for the decay couplings of the scalar–isoscalar states in the analysis of the quark–gluonium content of resonances in [37] and later on in a set of papers, see [28, 33] and references therein.

Remind that the flavour wave functions of the f_0 -states were supposed to be a mixture of the quark–antiquark and gluonium components, $q\bar{q} \cos \gamma + g\bar{g} \sin \gamma$, where the $q\bar{q}$ -state is determined as $q\bar{q} = n\bar{n} \cos \varphi + s\bar{s} \sin \varphi$ and $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$.

Using formulae given in Chapter 2 for the vertices $q\bar{q} \rightarrow \pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$ together with analogous couplings for the transition $g\bar{g} \rightarrow \pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$, we obtain the following coupling constants squared for the decays $f_0 \rightarrow \pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$:

$$\begin{aligned}
g_{\pi\pi}^2 &= \frac{3}{2} \left(\frac{g}{\sqrt{2}} \cos \varphi + \frac{G}{\sqrt{2+\lambda}} \right)^2, \\
g_{K\bar{K}}^2 &= 2 \left(\frac{g}{2} (\sin \varphi + \sqrt{\frac{\lambda}{2}} \cos \varphi) + G \sqrt{\frac{\lambda}{2+\lambda}} \right)^2, \\
g_{\eta\eta'}^2 &= \frac{1}{2} \left(g \left(\frac{\cos^2 \Theta}{\sqrt{2}} \cos \varphi + \sqrt{\lambda} \sin \varphi \sin^2 \Theta \right) + \frac{G}{\sqrt{2+\lambda}} (\cos^2 \Theta + \lambda \sin^2 \Theta) \right)^2, \\
g_{\eta\eta'}^2 &= \sin^2 \Theta \cos^2 \Theta \left(g \left(\frac{1}{\sqrt{2}} \cos \varphi - \sqrt{\lambda} \sin \varphi \right) + G \frac{1-\lambda}{\sqrt{2+\lambda}} \right)^2. \quad (3.260)
\end{aligned}$$

Here $g = g_0 \cos \gamma$ and $G = G_0 \sin \gamma$, where g_0 is a universal constant for all nonet members and G_0 is a universal decay constant for the gluonium state. The value $g_{\pi\pi}^2$ is determined as a sum of couplings squared for the transitions to $\pi^+\pi^-$ and $\pi^0\pi^0$, when the identity factor for $\pi^0\pi^0$ is taken into account. Likewise, $g_{K\bar{K}}^2$ is the sum of coupling constants squared for the transitions to $K\bar{K}$ and $K^0\bar{K}^0$. The angle Θ stands for the mixing of $n\bar{n}$ and $s\bar{s}$ components in the η and η' mesons, we use $\Theta = 36.9^\circ$ [38].

Quark combinatorics make it possible to perform the nonet classification of bare states. In doing that in [28, 33], we refer to $f_0^{(bare)}$ as pure states, either $q\bar{q}$ or a glueball. For the $f_0^{(bare)}$ states this means:

- (1) The angle difference between isoscalar nonet partners should be 90° :
$$\varphi[f_0^{(bare)}(1)] - \varphi[f_0^{(bare)}(2)] = 90^\circ \pm 5^\circ. \quad (3.261)$$
- (2) Coupling constants g_0 should be roughly equal for all nonet partners:
$$g_0[f_0^{(bare)}(1)] \simeq g_0[f_0^{(bare)}(2)] \simeq g_0[a_0^{(bare)}] \simeq g_0[K_0^{(bare)}]. \quad (3.262)$$
- (3) Decay couplings for the bare gluonium should obey the relations for a glueball ($\varphi_{\text{gleball}} \simeq 27^\circ - 33^\circ$, see Chapter 2).

The conventional quark model requires an exact coincidence of the couplings g_0 . The energy dependence of the decay loop diagram, $B(s)$, may, however, violate the coupling-constant balance because of the mass splitting inside a nonet. The K -matrix coupling constant contains an additional s -dependent factor as compared to the coupling of the N/D-amplitude [39]: $g^2(K) = g^2(N/D)/[1 + B'(s)]$. The factor $[1 + B'(s)]^{-1}$ affects mostly the low- s region due to the threshold and left-hand side singularities of the partial amplitude. Therefore, the coupling constant equality is mostly violated for the lightest 00^{++} nonet, 1^3P_0 $q\bar{q}$. We allow for the members of this nonet $1 \leq g[f_0(1)]/g[f_0(2)] \leq 1.3$. For the 2^3P_0 $q\bar{q}$ nonet members, we put the two-meson couplings equal for isoscalar and isovector mesons.

3.8.0.3 Description of data and the results for the 00^{++} -wave

For the description of the 00^{++} wave in the mass region below 1900 MeV, five K -matrix poles are needed (a four-pole amplitude fails to describe the set of data under consideration). Accordingly, five bare states are introduced. We have found two solutions in which one bare state satisfies constraints inherent to the glueball; others can be considered as members of $q\bar{q}$ nonets with $n = 1, 2$, namely, 1^3P_0 and 2^3P_0 .

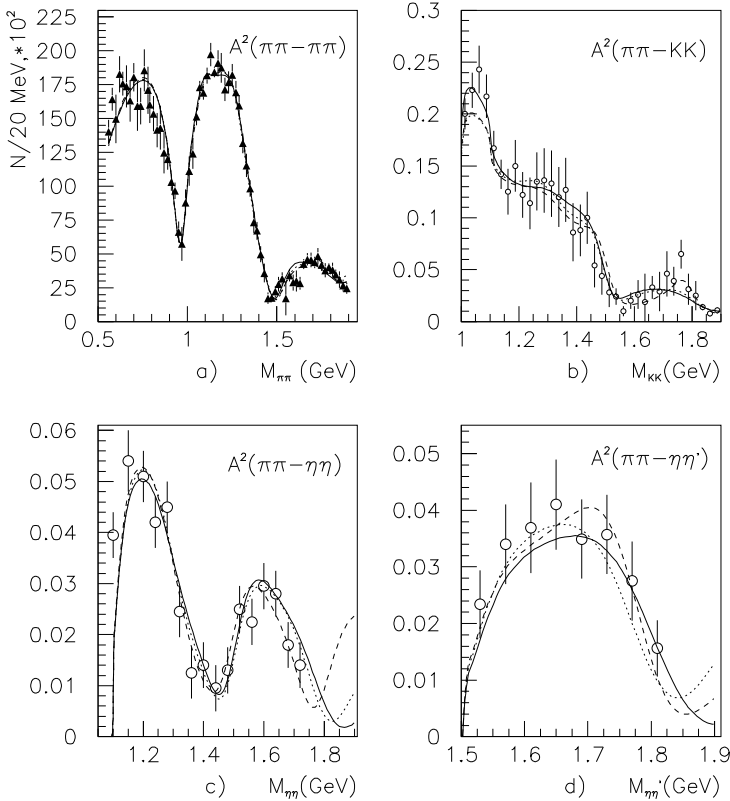


Fig. 3.28 S -wave amplitudes squared as functions of the $M_{\pi\pi} \equiv \sqrt{s}$ [29, 30, 31, 32] and their description in [28]: solid curve stands for Solution II.

In [28] we have found three solutions which are denoted as Solutions I, II-1 and II-2. They are similar to those found in [33].

Examples of description of the data are shown in Figs. 3.28 and 3.29.

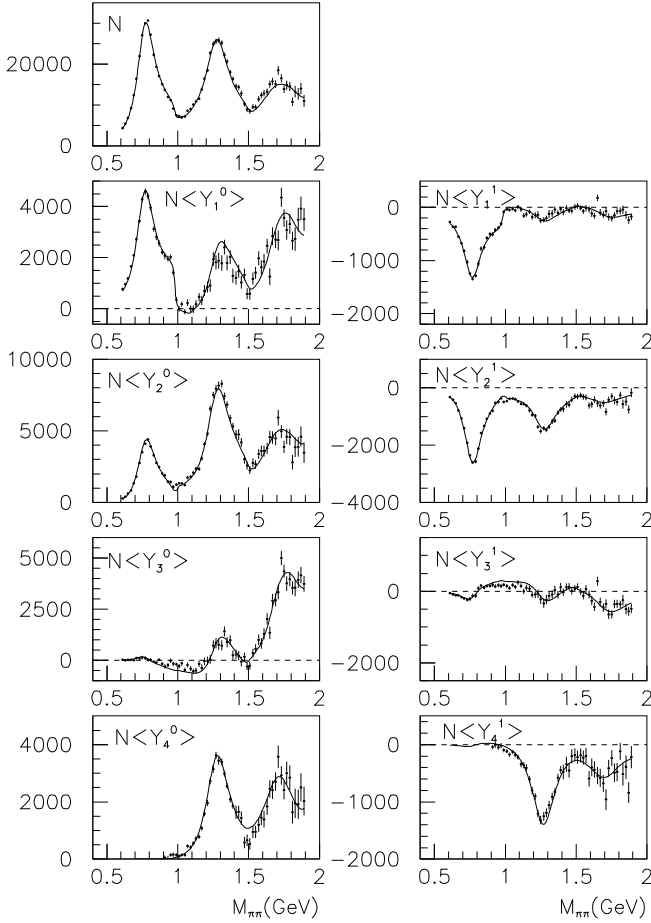


Fig. 3.29 Description of the angle moments for the $\pi^-\pi^+$ distributions (in cms of the $\pi^-\pi^+$) measured in the reaction $\pi^-p \rightarrow n\pi^-\pi^+$ [32], Solution **II** [28].

3.8.0.4 Bare f_0 -states and resonances

In the K -matrix analysis of the 00^{++} -wave five bare states have been found, see Tables 3.1, 3.2 and 3.3. The bare states can be classified as nonet partners of the $q\bar{q}$ multiplets 1^3P_0 and 2^3P_0 or a scalar glueball. The K -matrix solutions give us two versions for the glueball definition: either it is a bare state with a mass near 1250 MeV, or it is located near 1600 MeV.

After having imposed the constraints (3.261) and (3.262), we found the following versions for the nonet classification.

Solution I:

$f_0^{\text{bare}}(700 \pm 100)$ and $f_0^{\text{bare}}(1245 \pm 40)$ are 1^3P_0 nonet partners with $\varphi[f_0^{\text{bare}}(700)] = -70^\circ \pm 10^\circ$ and $\varphi[f_0^{\text{bare}}(1245)] = 20^\circ \pm 10^\circ$.

For members of the 2^3P_0 nonet, there are two versions:

- 1) either $f_0^{\text{bare}}(1220 \pm 30)$ and $f_0^{\text{bare}}(1750 \pm 40)$ are 2^3P_0 nonet partners, with $\varphi[f_0^{\text{bare}}(1220)] = 33^\circ \pm 8^\circ$ and $\varphi[f_0^{\text{bare}}(1750)] = -60^\circ \pm 10^\circ$, while $f_0^{\text{bare}}(1630 \pm 30)$ is the glueball, with $\varphi[f_0^{\text{bare}}(1630)] = 27^\circ \pm 10^\circ$; or
- 2) $f_0^{\text{bare}}(1630 \pm 30)$ and $f_0^{\text{bare}}(1750 \pm 40)$ are 2^3P_0 nonet partners, and $f_0^{\text{bare}}(1220 \pm 30)$ is the glueball.

Solution II-1:

$f_0^{\text{bare}}(670 \pm 100)$ and $f_0^{\text{bare}}(1215 \pm 40)$ are 1^3P_0 nonet partners with $\varphi[f_0^{\text{bare}}(670)] = -65^\circ \pm 10^\circ$ and $\varphi[f_0^{\text{bare}}(1215)] = 15^\circ \pm 10^\circ$;
 $f_0^{\text{bare}}(1560 \pm 40)$ and $f_0^{\text{bare}}(1820 \pm 40)$ are 2^3P_0 nonet partners with $\varphi[f_0^{\text{bare}}(1560)] = 15^\circ \pm 10^\circ$ and $\varphi[f_0^{\text{bare}}(1820)] = -80^\circ \pm 10^\circ$,
 $f_0^{\text{bare}}(1220 \pm 30)$ is the glueball, $\varphi[f_0^{\text{bare}}(1220)] = 40^\circ \pm 10^\circ$.

Solution II-2:

$f_0^{\text{bare}}(700 \pm 100)$ and $f_0^{\text{bare}}(1220 \pm 40)$ are 1^3P_0 nonet partners with $\varphi[f_0^{\text{bare}}(700)] = -70^\circ \pm 10^\circ$ and $\varphi[f_0^{\text{bare}}(1220)] = 15^\circ \pm 10^\circ$. In this solution there are two versions for the 2^3P_0 nonet:

- 1) either $f_0^{\text{bare}}(1230 \pm 30)$ and $f_0^{\text{bare}}(1830 \pm 40)$ are 2^3P_0 nonet partners with $\varphi[f_0^{\text{bare}}(1230)] = 45^\circ \pm 10^\circ$ and $\varphi[f_0^{\text{bare}}(1830)] = -55^\circ \pm 10^\circ$,
 $f_0^{\text{bare}}(1560 \pm 30)$ is the glueball, with $\varphi[f_0^{\text{bare}}(1560)] = 15^\circ \pm 10^\circ$, or
- 2) $f_0^{\text{bare}}(1560 \pm 30)$ and $f_0^{\text{bare}}(1830 \pm 40)$ are nonet partners and $f_0^{\text{bare}}(1230)$ is the glueball with $\varphi[f_0^{\text{bare}}(1230)] = 45^\circ \pm 10^\circ$.

Tables 3.1, 3.2 and 3.3 present parameters which correspond to these three solutions.

3.8.0.5 f_0 -resonances: masses, decay couplings and partial widths

The resonance masses and decay couplings cannot be determined directly from the fitting procedure. To calculate these quantities, one needs to carry out the analytical continuation of the K -matrix amplitude into the lower complex- s half-plane. One is allowed to do it, for the K -matrix amplitude takes into account correctly the threshold singularities related to the $\pi\pi$, $\pi\pi\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$ channels which are important in the 00^{++} -wave.

Masses of resonances

The complex masses of the resonances $f_0(980)$, $f_0(1300)$, $f_0(1500)$, $f_0(1200 - 1600)$ obtained in Solutions I, II-1 and II-2 do not differ seriously.

Table 3.1 Masses, coupling constants (in GeV) and mixing angles (in degree) for the f_0^{bare} -resonances for Solution **I**. The errors reflect the boundaries for a satisfactory description of the data. Sheet II is under the $\pi\pi$ and 4π cuts; sheet IV is under the $\pi\pi$, 4π , $K\bar{K}$ and $\eta\eta$ cuts; sheet V is under the $\pi\pi$, 4π , $K\bar{K}$, $\eta\eta$ and $\eta\eta'$ cuts.

	Solution I				
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
M	$0.650^{+.120}_{-.050}$	$1.245^{+.040}_{-.030}$	$1.220^{+.030}_{-.030}$	$1.630^{+.030}_{-.020}$	$1.750^{+.040}_{-.040}$
$g^{(\alpha)}$	$0.940^{+.80}_{-.100}$	$1.050^{+.080}_{-.080}$	$0.680^{+.060}_{-.060}$	$0.680^{+.060}_{-.060}$	$0.790^{+.080}_{-.080}$
$g_5^{(\alpha)}$	0	0	$0.960^{+.100}_{-.150}$	$0.900^{+.070}_{-.150}$	$0.280^{+.100}_{-.100}$
$\varphi_\alpha(deg)$	$-(72^{+5}_{-10})$	18.0^{+8}_{-8}	33^{+8}_{-8}	27^{+10}_{-10}	-59^{+10}_{-10}
	$a = \pi\pi$	$a = K\bar{K}$	$a = \eta\eta$	$a = \eta\eta'$	$a = 4\pi$
f_{1a}	$-0.050^{+.100}_{-.100}$	$0.250^{+.100}_{-.100}$ $f_{ba} = 0$	$0.440^{+.100}_{-.100}$ $b = 2, 3, 4, 5$	$0.320^{+.100}_{-.100}$	$-0.540^{+.100}_{-.100}$
sheet II	Position of pole				
	$1.031^{+.008}_{-.008}$ $-i(0.032^{+.008}_{-.008})$				
sheet IV		$1.306^{+.020}_{-.020}$ $-i(0.147^{+.015}_{-.025})$	$1.489^{+.008}_{-.004}$ $-i(0.051^{+.005}_{-.005})$	$1.480^{+.100}_{-.150}$ $-i(1.030^{+.080}_{-.170})$	
sheet V					$1.732^{+.015}_{-.015}$ $-i(0.072^{+.015}_{-.015})$

Table 3.2 Masses, coupling constants (in GeV) and mixing angles (in degree) for the f_0^{bare} -resonances for Solution **II-1**. The errors reflect the boundaries for a satisfactory description of the data. Sheet II is under the $\pi\pi$ and 4π cuts; sheet IV is under the $\pi\pi$, 4π , $K\bar{K}$ and $\eta\eta$ cuts; sheet V is under the $\pi\pi$, 4π , $K\bar{K}$, $\eta\eta$ and $\eta\eta'$ cuts.

	Solution II-1				
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
M	$0.670^{+.100}_{-.100}$	$1.215^{+.40}_{-.040}$	$1.220^{+.015}_{-.030}$	$1.560^{+.030}_{-.040}$	$1.830^{+.030}_{-.050}$
$g^{(\alpha)}$	$0.990^{+.080}_{-.120}$	$1.100^{+.080}_{-.100}$	$0.670^{+.100}_{-.120}$	$0.500^{+.060}_{-.060}$	$0.410^{+.060}_{-.060}$
$g_5^{(\alpha)}$	0	0	$0.870^{+.100}_{-.100}$	$0.600^{+.100}_{-.100}$	$-0.850^{+.080}_{-.080}$
$\varphi_\alpha(\text{deg})$	$-(66^{+8}_{-10})$	13^{+8}_{-5}	40^{+12}_{-12}	15^{+08}_{-15}	-80^{+10}_{-10}
	$a = \pi\pi$	$a = K\bar{K}$	$a = \eta\eta$	$a = \eta\eta'$	$a = 4\pi$
f_{1a}	$0.050^{+.100}_{-.100}$	$0.100^{+.080}_{-.080}$ $f_{ba} = 0$	$0.360^{+.100}_{-.100}$ $b = 2, 3, 4, 5$	$0.320^{+.100}_{-.100}$	$-0.350^{+.060}_{-.060}$
sheet II	Position of pole				
	$1.020^{+.008}_{-.008}$ $-i(0.035^{+.008}_{-.008})$				
sheet IV		$1.320^{+.020}_{-.020}$ $-i(0.130^{+.015}_{-.025})$	$1.485^{+.005}_{-.006}$ $-i(0.055^{+.008}_{-.008})$	$1.530^{+.150}_{-.100}$ $-i(0.900^{+.100}_{-.200})$	
sheet V					$1.785^{+.015}_{-.015}$ $-i(0.135^{+.025}_{-.010})$

Table 3.3 Masses, coupling constants (in GeV) and mixing angles (in degree) for the f_0^{bare} -resonances for Solution **II-2**. The errors reflect the boundaries for a satisfactory description of the data. Sheet II is under the $\pi\pi$ and 4π cuts; sheet IV is under the $\pi\pi$, 4π , $K\bar{K}$ and $\eta\eta$ cuts; sheet V is under the $\pi\pi$, 4π , $K\bar{K}$, $\eta\eta$ and $\eta\eta'$ cuts.

	Solution II-2				
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
M	$0.650^{+.120}_{-.050}$	$1.220^{+.040}_{-.030}$	$1.230^{+.030}_{-.030}$	$1.560^{+.030}_{-.020}$	$1.830^{+.040}_{-.040}$
$g^{(\alpha)}$	$1.050^{+.80}_{-.100}$	$0.980^{+.080}_{-.080}$	$0.470^{+.050}_{-.050}$	$0.420^{+.040}_{-.040}$	$0.420^{+.050}_{-.050}$
$g_5^{(\alpha)}$	0	0	$0.870^{+.100}_{-.100}$	$0.560^{+.070}_{-.070}$	$-0.780^{+.070}_{-.070}$
$\varphi_\alpha(\text{deg})$	$-(68^{+3}_{-15})$	14^{+8}_{-8}	43^{+8}_{-8}	15^{+10}_{-10}	-55^{+10}_{-10}
	$a = \pi\pi$	$a = K\bar{K}$	$a = \eta\eta$	$a = \eta\eta'$	$a = 4\pi$
f_{1a}	$0.260^{+.100}_{-.100}$	$0.100^{+.100}_{-.100}$ $f_{ba} = 0$	$0.260^{+.100}_{-.100}$ $b = 2, 3, 4, 5$	$0.260^{+.100}_{-.100}$	$-0.140^{+.060}_{-.060}$
sheet II	Position of pole				
	$1.020^{+.008}_{-.008}$ $-i(0.035^{+.008}_{-.008})$				
sheet IV		$1.325^{+.020}_{-.030}$ $-i(0.170^{+.020}_{-.040})$	$1.490^{+.010}_{-.010}$ $-i(0.060^{+.005}_{-.005})$	$1.450^{+.150}_{-.100}$ $-i(0.800^{+.100}_{-.150})$	
sheet V					$1.740^{+.020}_{-.020}$ $-i(0.160^{+.025}_{-.010})$

Solutions I and II differ essentially in the characteristics of the $f_0(1750)$.

For the positions of the poles the following values have been found (in MeV):

$$\begin{aligned}
 \text{Solution I :} \quad & f_0(980) \quad \rightarrow 1031 - i \, 32 \\
 & f_0(1300) \quad \rightarrow 1306 - i \, 147 \\
 & f_0(1500) \quad \rightarrow 1489 - i \, 51 \\
 & f_0(1750) \quad \rightarrow 1732 - i \, 72 \\
 & f_0(1200 - 1600) \rightarrow 1480 - i \, 1030 \, , \quad (3.263)
 \end{aligned}$$

$$\begin{aligned}
 \text{Solution II - 1 :} \quad & f_0(980) \quad \rightarrow 1020 - i \, 33 \\
 & f_0(1300) \quad \rightarrow 1320 - i \, 130 \\
 & f_0(1500) \quad \rightarrow 1485 - i \, 55 \\
 & f_0(1750) \quad \rightarrow 1785 - i \, 135 \\
 & f_0(1200 - 1600) \rightarrow 1530 - i \, 900 \, , \quad (3.264)
 \end{aligned}$$

$$\begin{aligned}
 \text{Solution II - 2 :} \quad & f_0(980) \quad \rightarrow 1020 - i \, 35 \\
 & f_0(1300) \quad \rightarrow 1325 - i \, 170 \\
 & f_0(1500) \quad \rightarrow 1490 - i \, 60 \\
 & f_0(1750) \quad \rightarrow 1740 - i \, 160 \\
 & f_0(1200 - 1600) \rightarrow 1450 - i \, 800 \, . \quad (3.265)
 \end{aligned}$$

We see that Solutions I and II give different values for the total width of the $f_0(1750)$.

3.9 Appendix 3.C: The K-Matrix Analyses of the $(IJ^P = \frac{1}{2}0^+)$ -Wave Partial Amplitude for Reaction $\pi K \rightarrow \pi K$

The partial wave analysis of the $K^-\pi^+$ system for the reaction $K^-p \rightarrow K^-\pi^+n$ at 11 GeV/c was carried out in [40], where two alternative solutions (A and B), which differ only in the region above 1800 MeV, were found for the S -wave. In [40], the T -matrix fit on the $K\pi$ S -wave was performed independently for the regions 850 – 1600 MeV and 1800 – 2100 MeV. In the lower mass region the resonance $K_0^*(1430)$ was found:

$$M_R = 1429 \pm 9 \text{ MeV}, \quad \Gamma = 287 \pm 31 \text{ MeV} \, , \quad (3.266)$$

while at higher masses Solutions A and B provided us with the following parameters for the description of the resonance $K_0^*(1950)$:

$$\begin{array}{ll} \text{Solution A} & M_R = 1934 \pm 28 \text{ MeV}, \quad \Gamma = 174 \pm 98 \text{ MeV}, \\ \text{Solution B} & M_R = 1955 \pm 18 \text{ MeV}, \quad \Gamma = 228 \pm 56 \text{ MeV}. \end{array} \quad (3.267)$$

The necessity to improve this analysis was obvious. First, the mass region 1600 – 1800 MeV, where the amplitude varies quickly, must be included into consideration. As was emphasised above, it is well known that, due to a strong interference, the resonance reveals itself not only as a bump in the spectrum but also as a dip or a shoulder (in this way the resonances appear in the 00^{++} wave, see Section 3.8). Second, the interference effects are a source of ambiguities. It is worth noting that ambiguities in scalar–isoscalar 00^{++} wave were successfully eliminated owing to a simultaneous fitting to different meson spectra only. The available data are not copious for the wave $\frac{1}{2}0^+$, hence one may suspect that the solution found in [40] is not unique.

The K -matrix reanalysis of the $K\pi$ S -wave has been carried out in [41] with the purpose

(i) to restore the masses and coupling constants of the bare states for the wave $\frac{1}{2}0^+$, in order to establish the $q\bar{q}$ -classification;

(ii) to find all possible K -matrix solutions for the $K\pi$ S -wave in the mass region up to 2000 MeV.

The S -wave $K\pi$ scattering amplitude extracted from the reaction $K^-p \rightarrow K^-\pi^+n$ at small momentum transfers is a sum of two components, with isotopic spins $\frac{1}{2}$ and $\frac{3}{2}$:

$$A_S = A_S^{1/2} + \frac{1}{2}A_S^{3/2} = |A_S| e^{i\phi_S}, \quad (3.268)$$

where $|A_S|$ and ϕ_S are measurable quantities entering the S -wave amplitude [40]. The part of the S -wave amplitude with the isotopic spin $I = 3/2$ is of non-resonance behaviour at the considered energies, so it can be parametrised as follows:

$$A_S^{3/2}(s) = \frac{\rho_{K\pi}(s)a_{3/2}(s)}{1 - i\rho_{K\pi}(s)a_{3/2}(s)}, \quad (3.269)$$

where $a_{3/2}(s)$ is a smooth function and $\rho_{K\pi}(s)$ is the $K\pi$ phase space factor.

For the description of the $A_S^{1/2}$ amplitude, in [41] the 3×3 K -matrix was used, with the following channel notations:

$$1 = K\pi, \quad 2 = K\eta', \quad 3 = K\pi\pi\pi + \text{multimeson states}.$$

Table 3.4 Coupling constants for the transitions $K_0^0 \rightarrow \text{two mesons}$ and $a_0^- \rightarrow \text{two mesons}$ in the leading and next-to-leading terms of the $1/N$ expansion.

Channel	Couplings for leading terms	Couplings for next-to-leading terms
$K^+\pi^-$	$g^L/2$	0
$K^0\pi^0$	$-g^L/\sqrt{8}$	0
$K^0\eta$	$(\cos\Theta/\sqrt{2} - \sqrt{\lambda}\sin\Theta)g^L/2$	$(\sqrt{2}\cos\Theta - \sqrt{\lambda}\sin\Theta)g^{NL}/2$
$K^0\eta'$	$(\sin\Theta/\sqrt{2} + \sqrt{\lambda}\cos\Theta)g^L/2$	$(\sqrt{2}\sin\Theta - \sqrt{\lambda}\cos\Theta)g^{NL}/2$
K^-K^0	$g^L\sqrt{\lambda}/2$	0
$\pi^-\eta$	$g^L\cos\Theta/\sqrt{2}$	$(\sqrt{2}\cos\Theta - \sqrt{\lambda}\sin\Theta)g^{NL}/2$
$\pi^-\eta'$	$g^L\sin\Theta/\sqrt{2}$	$(\sqrt{2}\sin\Theta - \sqrt{\lambda}\cos\Theta)g^{NL}/2$

The account for the channel $K\eta$ does not influence the data description, since the transition $K\pi \rightarrow K\eta$ is suppressed [40]. The latter is in agreement with the results of quark combinatorics, see Table 3.4.

In [41] the fitting to the wave $\frac{1}{2}0^+$ was performed in the following way. The analysed data on the reaction $K^-p \rightarrow K^-\pi^+n$ were extracted with small momentum transfers ($|t| < 0.2 \text{ GeV}^2$), and, at the first stage, the data were fitted to the unitary amplitude. At the next stage, the t -dependence was introduced into the K -matrix amplitude. The amplitude $K\pi(t) \rightarrow K\pi$, where $\pi(t)$ stands for a virtual pion, is equal to:

$$A_S^{1/2} = \sum_{a=1,2,3} K_{1a}(t) \left[\frac{\hat{I}}{\hat{I} - i\hat{p}\hat{K}(m_\pi^2)} \right]_{a1}, \quad (3.270)$$

with the parametrisation of the matrix $K_{1a}(t)$ written in the form:

$$K_{1a}(t) = \left(\Sigma_\alpha \frac{g_1^{(\alpha)}(t)g_a^{(\alpha)}}{M_\alpha^2 - s} + f_{1a}(t) \frac{1 \text{ GeV}^2 + s_0}{s + s_0} \right); \quad (3.271)$$

here $g_1^{(\alpha)}(t = m_\pi^2) = g_1^{(\alpha)}$ and $f_{1a}(t = m_\pi^2) = f_{1a}$. Coupling constants are determined by the rules of quark combinatorics, they are presented in Table

3.4. In [41], only the leading terms in the $1/N$ expansion were taken into consideration: in this case all coupling constants are defined by the same parameter g^L (g^L is a common quantity for all the nonet members).

As follows from the K -matrix fit on the ($IJ^P = \frac{1}{2}0^+$) wave [41], for a good description of the $K\pi$ -spectrum in the region 800–2000 MeV at least two K_0 -states are necessary. Correspondingly, the $\frac{1}{2}0^+$ -amplitude of this minimal solution has poles near the physical region on the 2nd sheet (under the $K\pi$ -cut) and on the 3rd sheet (under the $K\pi$ - and $K\eta'$ -cuts) at the following complex masses:

$$(1415 \pm 30) - i(165 \pm 25) \text{ MeV}, \quad (1820 \pm 40) - i(125 \pm 35) \text{ MeV}. \quad (3.272)$$

In the fits A and B (see Fig. 3.30) the poles appeared to be close to one another, that resulted in small error bars in (3.272). The $K\eta'$ threshold, being in the vicinity of the resonance (at 1458 MeV), strongly influences the $\frac{1}{2}0^+$ amplitude, so the lowest K_0 -state has a second pole which is located above the $K\eta'$ -cut, at $M = (1525 \pm 125) - i(420 \pm 80) \text{ MeV}$: the situation is analogous to that observed for the $f_0(980)$ -meson, which also has a two-pole structure of the amplitude due to the $K\bar{K}$ -threshold. As was said above, the $K\eta$ channel influences weakly the $\frac{1}{2}0^+$ $K\pi$ amplitude. Experimental data [40] prove it as well as the rules of quark combinatorics do.

The minimal solution contains two K_0^{bare} states:

$$K_0^{\text{bare}}(1200_{-110}^{+60}), \quad K_0^{\text{bare}}(1820_{-75}^{+40}). \quad (3.273)$$

The errors in (3.273) take into account the existence of two solutions, A and B, see Fig. 3.30. In the minimal solution, the lightest bare scalar kaon appears to be 200 MeV lower than the amplitude pole, and this latter circumstance makes it easier to build the basic scalar nonet, with masses in the range 900–1200 MeV.

The $K\pi$ spectra allow also solutions with three poles and with a much better χ^2 ; still, for these solutions the lightest kaon state, K_0^{bare} , does not leave the range 900–1200 MeV. In the three-pole Solution B-3 (see Fig. 3.31) we have the bare states

$$K_0^{\text{bare}}(1090 \pm 40), \quad K_0^{\text{bare}}(1375_{-40}^{+125}), \quad K_0^{\text{bare}}(1950_{-20}^{+70}), \quad (3.274)$$

while the $K\pi$ -amplitude has the following poles:

$$\begin{aligned} \text{II sheet } M &= 998 \pm 15 - i(80 \pm 15) \text{ MeV} \\ \text{II sheet } M &= 1426 \pm 15 - i(182 \pm 15) \text{ MeV} \\ \text{III sheet } M &= 1468 \pm 30 - i(309 \pm 15) \text{ MeV} \\ \text{III sheet } M &= 1815 \pm 25 - i(130 \pm 25) \text{ MeV}. \end{aligned} \quad (3.275)$$

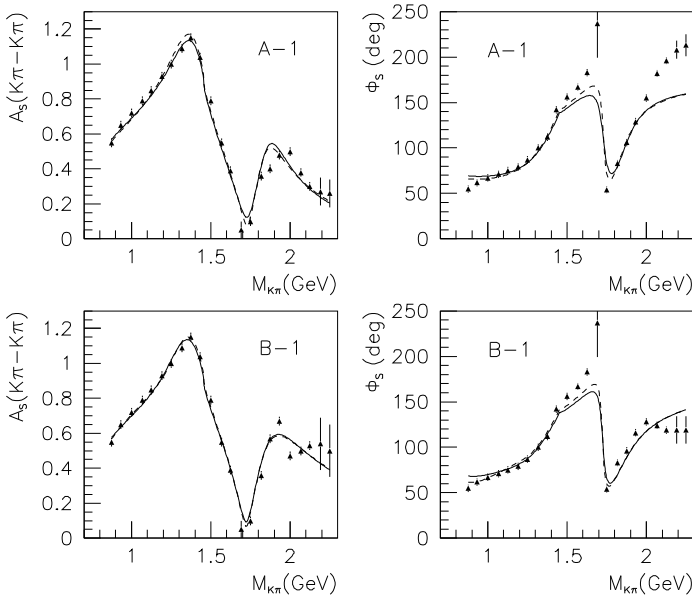


Fig. 3.30 Description of data in [40] in the two-pole K -matrix fit: Solutions (A-1) and (B-1). Solid curves correspond to the solution found for the unitary amplitude, dashed line stands for the fit with the t -dependent K -matrix.

One can see that the bare state $K_0^{bare}(1375_{-40}^{+125})$, being near the $K\eta'$ threshold, leads to a doubling of the amplitude poles around 1400 MeV.

It should be underlined that masses of the lightest bare kaon states obtained by the two- and three-pole solutions coincide within the errors.

3.10 Appendix 3.D: The Low-Mass σ -Meson

In the framework of the dispersion relation N/D -method, we restore the low-energy $\pi\pi$ ($IJ^{PC} = 00^{++}$)-wave amplitude sewing it with the previously obtained K -matrix solution for the region 450–1900 MeV. The restored N/D -amplitude has a pole on the second sheet of the complex- s plane near the $\pi\pi$ threshold.

An important result obtained in [28, 33, 42] is that the K -matrix 00^{++} -amplitude has no pole singularities in the region 500–800 MeV. The $\pi\pi$ -scattering phase δ_0^0 increases smoothly in this energy region reaching 90° at 800–900 MeV. A straightforward explanation of such a behaviour of δ_0^0 might be the presence of a broad resonance, with a mass about 600–900

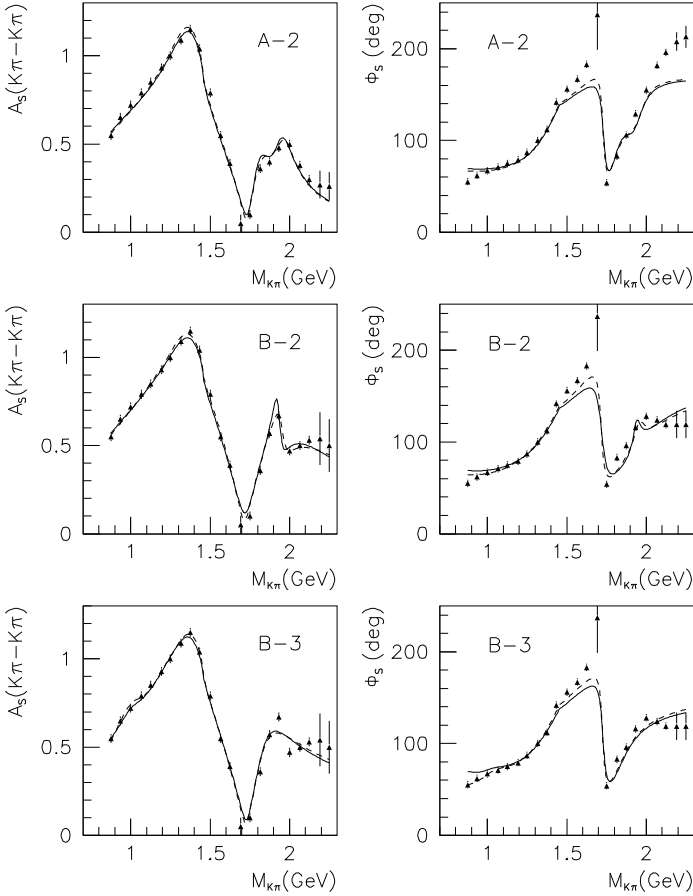


Fig. 3.31 Description of data in [40] in the three-pole K -matrix fit: Solutions (A-2) and (B-2) have two poles in the region of large masses, Solution (B-3) has two poles at low masses.

MeV and width $\Gamma \sim 500$ MeV [43, 44, 45, 46]. However, according to the K -matrix solution [28, 33, 42], the 00^{++} -amplitude does not contain pole singularities on the second sheet of the complex- $M_{\pi\pi}$ plane inside the interval $450 \leq \text{Re } M_{\pi\pi} \leq 900$ MeV: the K -matrix amplitude has only a low-mass pole, which is located on the second sheet either near the $\pi\pi$ threshold or even below it. In [28, 33, 42], the presence of this pole was not emphasised, for the left-hand cut, which is important for the reconstruction of analytical structure of the low-energy partial amplitude, was taken into

account only in an indirect way. A proper way for the description of the low-mass amplitude must be the use of the dispersion relation representation.

Here, following [47], the dispersion relation $\pi\pi$ scattering amplitude is reconstructed in the region of small $M_{\pi\pi}$ being attached to the K -matrix solution of [33, 42] which was found for $M_{\pi\pi} \sim 450 - 1950$ MeV. On the basis of data for δ_0^0 , we construct the N/D amplitude below 900 MeV sewing it with the K -matrix amplitude; our aim is a continuation to the region $s = M_{\pi\pi}^2 \sim 0$. By this sewing, we strictly follow the results obtained for the K -matrix amplitude in the region 450-900 MeV, that is, the region where we can be confident in the results of the K -matrix representation. Let us remind that the K -matrix representation allows one to restore correctly the analytical structure of the amplitude in the region $s > 0$ (threshold and pole singularities) but not for the left-hand singularities at $s \leq 0$ (singularities related to forces). Hence, we cannot be quite sure in the K -matrix results below the $\pi\pi$ threshold.

Using the approximation method of the left-hand cut suggested in [48], we can find the dispersion relation amplitude. The constructed N/D -amplitude provides a good description of δ_0^0 from threshold to 900 MeV, thus including the region $\delta_0^0 \sim 90^\circ$. This amplitude has no pole in the region 500–900 MeV; instead, the pole is located near the $\pi\pi$ threshold.

We suppose that the low-mass pole in the scalar–isoscalar wave is related to a fundamental phenomenon at large distances (in hadronic scale). In Chapter 2 we argued that the low-mass pole is related to singularities of the amplitude owing to confinement forces.

3.10.1 *Dispersion relation solution for the $\pi\pi$ -scattering amplitude below 900 MeV*

The partial pion–pion scattering amplitude being a function of the invariant energy squared, $s = M_{\pi\pi}^2$, can be represented as a ratio $N(s)/D(s)$. Here $N(s)$ has a left-hand cut due to the “forces” (the interactions due to t - and u -channel exchanges), while the function $D(s)$ is determined by the rescatterings in the s -channel. $D(s)$ is given by the dispersion integral along the right-hand cut in the complex- s plane:

$$A(s) = \frac{N(s)}{D(s)}, \quad D(s) = 1 - \int_{4\mu_\pi^2}^{\infty} \frac{ds'}{\pi} \frac{\rho(s')N(s')}{s' - s + i0}. \quad (3.276)$$

Here $\rho(s)$ is the invariant $\pi\pi$ phase space, $\rho(s) = (16\pi)^{-1} \sqrt{(s - 4\mu_\pi^2)/s}$. It supposed in (3.276) that $D(s) \rightarrow 1$ with $s \rightarrow \infty$ and CDD-poles are absent (a detailed presentation of the N/D -method can be found in [4]).

The N -function can be written as an integral along the left-hand cut as follows:

$$N(s) = \int_{-\infty}^{s_L} \frac{ds'}{\pi} \frac{L(s')}{s' - s}, \quad (3.277)$$

where the value s_L marks the beginning of the left-hand cut. For example, for the one-meson exchange diagram $g^2/(m^2 - t)$, the left-hand cut starts at $s_L = 4\mu_\pi^2 - m^2$, and the N -function in this point has a logarithmic singularity; for the two-pion exchange, $s_L = 0$.

Below, we work with the amplitude $a(s)$, which is defined as:

$$a(s) = \frac{N(s)}{8\pi\sqrt{s}} \left(1 - \text{P} \int_{4\mu_\pi^2}^{\infty} \frac{ds'}{\pi} \frac{\rho(s')N(s')}{s' - s} \right)^{-1}. \quad (3.278)$$

The amplitude $a(s)$ is related to the scattering phase shift: $a(s)\sqrt{s/4 - \mu_\pi^2} = \tan \delta_0^0$. In equation (3.278) the threshold singularity is singled out explicitly, so the function $a(s)$ contains only a left-hand cut and poles corresponding to zeros of the denominator of the right-hand side (3):

$1 = \text{P} \int_{4\mu_\pi^2}^{\infty} (ds'/\pi) \cdot \rho(s')N(s')/(s' - s)$. The pole of $a(s)$ at $s > 4\mu_\pi^2$ corresponds to the phase shift value $\delta_0^0 = 90^\circ$. The phase of the $\pi\pi$ scattering reaches the value $\delta_0^0 = 90^\circ$ at $\sqrt{s} = M_{90} \simeq 850$ MeV. Because of that, the amplitude $a(s)$ may be represented in the form

$$a(s) = \int_{-\infty}^{s_L} \frac{ds'}{\pi} \frac{\alpha(s')}{s' - s} + \frac{C}{s - M_{90}^2} + D. \quad (3.279)$$

For the reconstruction of the low-mass amplitude, the parameters D, C, M_{90} and $\alpha(s)$ have been determined by fitting to the experimental data. In the fit we have used a method approved in the analysis of the low-energy nucleon–nucleon amplitudes [48]. Namely, the integral in the right-hand side of (3.279) has been replaced by the sum

$$\int_{-\infty}^{s_L} \frac{ds'}{\pi} \frac{\alpha(s')}{s' - s} \rightarrow \sum_n \frac{\alpha_n}{s_n - s} \quad (3.280)$$

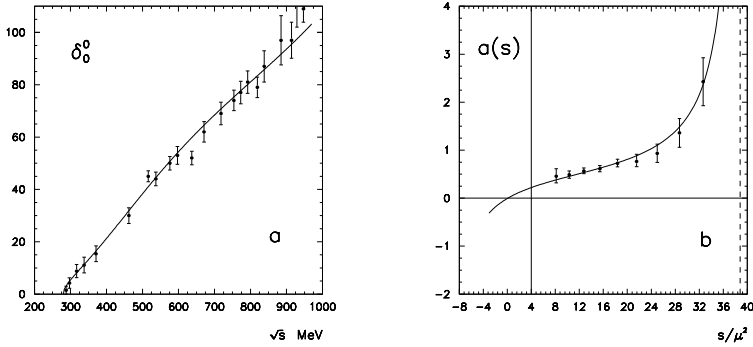


Fig. 3.32 a) Fit to the data on δ_0^0 by using the N/D -amplitude. b) Amplitude $a(s)$ in the N/D -solution (solid curve) and the K -matrix approach [28, 47] (points with error bars).

with $-\infty < s_n \leq s_L$.

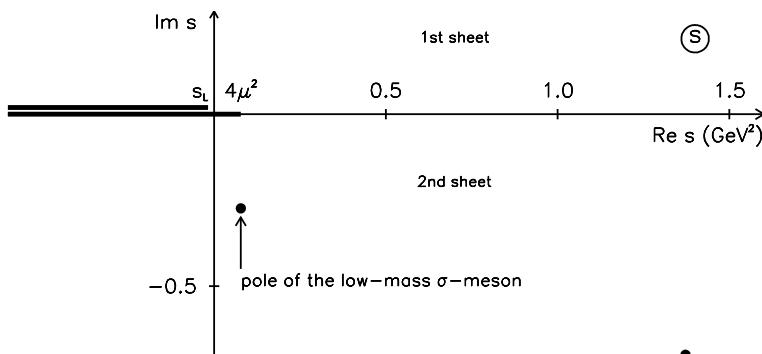
The description of data within the N/D -solution, which uses six terms in the sum (3.280), is demonstrated in Fig. 3.32a. The parameters of the solution are as follows:

$s_n \mu_\pi^{-2}$	-9.56	-10.16	-10.76	-32	-36	-40	(3.281)
$\alpha_n \mu_\pi^{-1}$	2.21	2.21	2.21	0.246	0.246	0.246	
$M_{90} = 6.228 \mu_\pi, \quad C = -13.64 \mu_\pi, \quad D = 0.316 \mu_\pi^{-1}$							

The scattering length in this solution is equal to $a_0^0 = 0.22 \mu_\pi^{-1}$, the Adler zero is at $s = 0.12 \mu_\pi^2$. The N/D -amplitude is attached to the K -matrix amplitude of [33, 42], and figure 3.32b demonstrates the level of the coincidence of the amplitudes $a(s)$ for both solutions.

The dispersion relation solution has a correct analytical structure in the region $|s| < 1 \text{ GeV}^2$. The amplitude has no poles on the first sheet of the complex- s plane. After the replacement given by (3.280), the left-hand cut of the N -function is transformed into a set of poles on the negative part of the real- s axis: six poles of the amplitude (at $s/\mu_\pi^2 = -5.2, -9.6, -10.4, -31.6, -36.0, -40.0$) represent the left-hand singularity of $N(s)$.

On the second sheet (under the $\pi\pi$ -cut) the amplitude has two poles: at $s \simeq (4 - i14)\mu_\pi^2$ and $s \simeq (70 - i34)\mu_\pi^2$ (see Fig. 3.33). The second pole, at $s = (70 - i34)\mu_\pi^2$, is located beyond the region under consideration, $|s| < 1 \text{ GeV}^2$ (nevertheless, let us underline that the K -matrix amplitude [33, 42] has a set of poles just in the region of the second pole of the N/D -


 Fig. 3.33 Complex- s plane and singularities of the N/D -amplitude.

amplitude). The pole near the threshold, at

$$s \simeq (4 - i14)\mu_\pi^2, \quad (3.282)$$

is what we discuss. The N/D -amplitude has no poles at $Re\sqrt{s} \sim 600 - 900$ MeV despite the phase shift δ_0^0 reaches 90° here.

The data do not fix the N/D -amplitude rigidly. The position of the low-mass pole can be varied in the region $Re s \sim (0 - 4)\mu_\pi^2$, and there are simultaneous variations of the scattering length in the interval $a_0^0 \sim (0.21 - 0.28)\mu_\mu^{-1}$ and the Adler zero at $s \sim (0 - 1)\mu_\pi^2$.

Let us emphasise that the way we reconstruct here the dispersion relation amplitude differs from the mainstream attempts of determination of the N/D -amplitude. In the bootstrap method which is the classic N/D procedure, the pion-pion amplitude is to be determined by analyticity, unitarity and crossing symmetry that means a unique determination of the left-hand cut by the crossing channels. However, the bootstrap procedure is not realised up to now; the problems which the recent bootstrap program faces are discussed in [49] and references therein. Nevertheless, one can try to saturate the left-hand cut by known resonances in the crossing channels. Usually, it is supposed that the dominant contribution into the left-hand cut comes from the ρ -meson exchange supplemented by the $f_2(1275)$ and σ exchanges. Within this scheme, the low-energy amplitude is restored being corrected by available experimental data. A common deficiency of these approaches is the necessity of introducing form factors in the exchange interaction vertices.

In the scheme used here, the amplitude in the physical region at 450-1950 MeV is supposed to be known (the result of the K -matrix analysis)

— then a continuation of the amplitude is made from the region of 450-900 MeV to the region of smaller masses; the continuation is corrected by the data. As a result, we restore the pole near the threshold (the low-mass σ -meson) and the left-hand cut (although with a less accuracy, actually, on qualitative level).

In the approaches, which take into account the left-hand cut as a contribution of some known meson exchanges, the following low-mass pole positions were obtained:

- (i) dispersion relation approach, $s \simeq (0.2 - i22.5)\mu_\pi^2$ [50],
- (ii) meson exchange models, $s \simeq (3.0 - i17.8)\mu_\pi^2$ [51], $s \simeq (0.5 - i13.2)\mu_\pi^2$ [52], $s \simeq (2.9 - i11.8)\mu_\pi^2$ [53],
- (iii) linear σ -model, $s \simeq (2.0 - i15.5)\mu_\pi^2$ [54].

In [55, 56], the pole positions were found in the region of the higher masses, at $\text{Re } s \sim (7 - 10) \mu_\pi^2$.

Miniconclusion

We have analysed the structure of the low-mass $\pi\pi$ -amplitude in the region $M_{\pi\pi} \lesssim 900$ MeV using the dispersion relation N/D -method, which provides us with a possibility to take the left-hand singularities into consideration. The dispersion relation N/D -amplitude is sewed with that given by the K -matrix analysis performed at $M_{\pi\pi} \sim 450 - 1950$ MeV [33, 42]. The N/D -amplitude obtained this way has a pole on the second sheet of the complex- s plane near the $\pi\pi$ threshold. This pole corresponds to the low-energy σ meson.

3.11 Appendix 3.E: Cross Sections and Amplitude Discontinuities

We use the amplitudes A , connected with the S -matrix by

$$S = 1 + i(2\pi)^4 \delta^4 \left(\sum p_{in} - \sum p_{out} \right) A. \quad (3.283)$$

Here $\sum p_{in}$ and $\sum p_{out}$ are the total incoming and outgoing momenta of the particles, respectively. We take into account the factors corresponding to particle identity directly in the amplitudes. This allows us to write phase space integrals for different or identical particles in the same form. Thus, if amplitudes are constructed according to the standard Feynman rules, additional factors enter for groups of identical particles. These are $\prod_i 1/\sqrt{n_i!}$ for bosons and $\prod_i (-1)^{P_i}/\sqrt{n_i!}$ for fermions. Here n_i is the number of the

identical particles of the i th sort, $P_i = 0, 1$ is the parity for the permutation of fermions.

The connection of such amplitudes with the measured cross sections is given below.

3.11.1 Exclusive and inclusive cross sections

With the normalisation adopted for the amplitudes the differential cross section of the process $1 + 2 \rightarrow N$ particles (Fig. 3.34a) is:

$$d\sigma_{2 \rightarrow N} = \frac{1}{J} |A_{2 \rightarrow N}|^2 d\phi_N, \quad (3.284)$$

where $J = 4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}$ is the invariant flux factor; p_1, p_2 and m_1, m_2 are the four-momenta and masses of the initial particles.

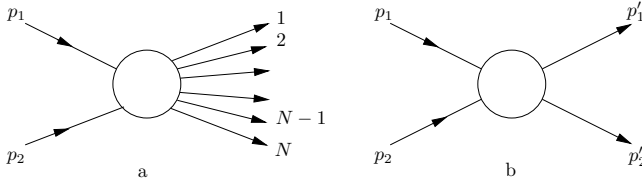


Fig. 3.34 Diagrams for (a) the N -particle production process ($2 \rightarrow N$) and (b) elastic scattering process ($2 \rightarrow 2$).

Depending on the problem we consider, we use for the phase space of N particles two versions of the definition, $d\phi_N$ and $\Phi_N(p_{in}; k_1, \dots, k_N)$:

$$d\phi_N = 2d\Phi_N(p_{in}; k_1, \dots, k_N) = (2\pi)^4 \delta^4(p_{in} - \sum_{\ell} k_{\ell}) \prod_{n=1}^N \frac{d^4 k_n}{(2\pi)^3} \delta(k_n^2 - m_n^2), \quad (3.285)$$

where $p_{in} = p_1 + p_2$. Note that the phase space element $d\Phi_2$ is used in the N/D-method (section 3.3.1).

The amplitude $A_{2 \rightarrow N}$ depends on the momenta and spins of the incoming and outgoing particles. If the colliding particles are unpolarized, the differential cross section (3.284) should be averaged over their spin projections:

$$\frac{1}{(2j_1 + 1)(2j_2 + 1)} \sum_{\mu_1, \mu_2}. \quad (3.286)$$

If the polarization properties of the outgoing particles are not measured, the cross section should be summed over their spin projections:

$$\sum_{\nu_1, \nu_2, \dots, \nu_N} . \quad (3.287)$$

The cross section (3.284) integrated over the whole phase space and summed over all spin projections leads to the total exclusive cross section of the given channel:

$$\sigma_N = \sum_{\nu_1, \dots, \nu_N} \int d\sigma_{2 \rightarrow N} . \quad (3.288)$$

A particular case of the equation (3.288) is the elastic cross section (Fig. 3.34b). At a fixed energy of the collision (or fixed $s = (p_1 + p_2)^2$), the differential elastic cross section is a function of two scattering angles. If the particles are spinless, the elastic amplitude is spin-independent, and the cross section depends on one scattering angle or on the associated variable, *e.g.* $t = (p_1 - p'_1)^2 \leq 0$:

$$\frac{d\sigma_\ell}{d(-t)} = \frac{d\sigma_\ell}{d|t|} = \frac{1}{J} |A_{2 \rightarrow N}|^2 d\phi_2 \delta(t - (p_1 - p'_1)^2) . \quad (3.289)$$

The total elastic cross section is

$$\sigma_\ell(s) = \int_{t_{min}}^0 dt \frac{d\sigma_\ell(s, t)}{d(-t)} , \quad (3.290)$$

where $t_{min} = -[s - (m_1 + m_2)^2][s - (m_1 + m_2)^2]/s$.

The sum of all possible exclusive cross sections (3.288) is the total cross section

$$\sigma_{tot} = \sum_N \sigma_N . \quad (3.291)$$

The total inelastic cross section is defined as

$$\sigma_{inel} = \sigma_{tot} - \sigma_\ell . \quad (3.292)$$

If only one secondary particle of a definite sort h is detected in the experiment, the inclusive cross section

$$1 + 2 \rightarrow h + X \quad (3.293)$$

is measured. The differential inclusive cross section of the production of the secondary h is the sum of various exclusive cross sections:

$$\frac{d\sigma}{d^3k_h}(1 + 2 \rightarrow h + X) = \sum_i n_{ih} \frac{d\sigma_i}{d^3k_h} , \quad (3.294)$$

where the sum runs over all open channels of the collision $(1 + 2)$ at fixed energy; n_{ih} is the number of secondaries of the sort h in the i th channel, while $d\sigma_i/d^3k_h$ is defined as

$$(2\pi)^3 2k_{h0} d\sigma_i / d^3k_h = \frac{1}{J} \int |A_{2 \rightarrow N_i}|^2 d\phi_{N_i-1}; \quad (3.295)$$

in the phase space element, $d\phi_{N_i-1}$, $p_{in} = p_1 + p_2 - k_h$ is taken. The inclusive cross section (3.294) is normalised according to

$$\int d^3k_h \frac{d\sigma}{d^3k_h} = \sigma(1 + 2 \rightarrow h + X) = \langle n_h \rangle \sigma_{inel}, \quad (3.296)$$

where $\langle n_h \rangle$ is the average number of secondaries of the sort h per inelastic event of the collision $(1 + 2)$.

Likewise, the multiparticle inclusive cross sections may be defined when several particles of fixed sorts are detected in the final state. For the two-particle inclusive reactions

$$1 + 2 \rightarrow h_1 + h_2 + X \quad (3.297)$$

the differential cross section

$$\begin{aligned} & \frac{d\sigma}{d^3k_{h_1} d^3k_{h_2}} (1 + 2 \rightarrow h_1 + h_2 + X) \\ &= \sum_i n_{ih_1} n_{ih_2} \cdot \frac{d\sigma_i}{d^3k_{h_1} d^3k_{h_2}} \quad (h_1 \neq h_2) \\ &= \sum_i n_{ih_1} (n_{ih_1} - 1) \frac{d\sigma_i}{d^3k_{h_1} d^3k_{h_2}} \quad (h_1 = h_2) \end{aligned} \quad (3.298)$$

is normalised according to the condition

$$\begin{aligned} & \int d^3k_{h_1} d^3k_{h_2} \frac{d\sigma}{d^3k_{h_1} d^3k_{h_2}} (1 + 2 \rightarrow h_1 + h_2 + X) \\ &= \langle n_{h_1} n_{h_2} \rangle \sigma_{inel}(12) \quad (h_1 \neq h_2) \\ &= \langle n_{h_1} (n_{h_1} - 1) \rangle \sigma_{inel}(12) \quad (h_1 = h_2). \end{aligned} \quad (3.299)$$

The difference $\langle n_{h_1} n_{h_2} \rangle - \langle n_{h_1} \rangle \langle n_{h_2} \rangle$ measures the correlation in the production of particles h_1 and h_2 ; it vanishes if they are produced independently.

3.11.2 Amplitude discontinuities and unitary condition

Cross sections of the collision processes may be expressed in terms of the amplitude discontinuities at their singular points. Two important examples are the elastic $(2 \rightarrow 2)$ and $(3 \rightarrow 3)$ amplitudes. The elastic $(2 \rightarrow 2)$

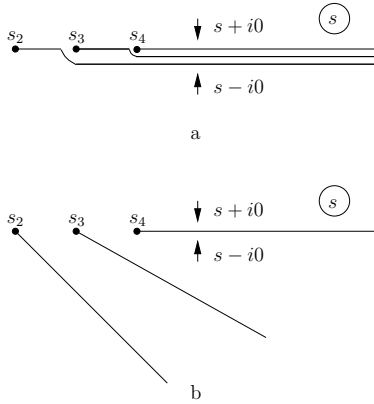


Fig. 3.35 Threshold singularities of the elastic amplitude in the complex- s plane at $s = s_n = (\sum_{i=1}^n m'_i)^2$. Here m'_i are the masses of the particles in the intermediate state; (a) cuts from the singularities are directed along the real axis; (b) cuts from the singularities s_2 and s_3 are moved to the lower half-plane.

amplitude has singularities in the physical region of s , which are connected with two-particle, three-particle, four-particle, etc. intermediate states (see Fig. 3.35).

Let us consider, *e.g.* four-particle intermediate states (Fig. 3.34a); the discontinuity of the amplitude at the four-particle threshold singularity is

$$2i \operatorname{disc}_{(4)} A(s, \dots) = A(s + i0, \dots) - A(s - i0, \dots). \quad (3.300)$$

The values $s + i0$ and $s - i0$ are shown in Fig. 3.34 by arrows. Dots stand for variables of the amplitude which are not written explicitly. The discontinuity (3.300) is:

$$\operatorname{disc}_{(4)} A(s, \dots) = \frac{1}{2} \int d\phi_4 A_{2 \rightarrow 4}(p_1, p_2, \dots) A_{4 \rightarrow 2}^+(p'_1, p'_2, \dots). \quad (3.301)$$

Both amplitudes in the integrand in the right-hand side of (3.301) are taken at the same value $(p_1 + p_2)^2 = (p'_1 + p'_2)^2 = s + i0$, *i.e.* in the physical region. For particles with spin, the right-hand side of (3.301) should be summed over the spin projections. The calculation of the discontinuities is usually called “the cutting of the diagram”; the right-hand side of (3.301) is represented graphically by the diagram in Fig. 3.36b.

The sum of all discontinuities is called the total discontinuity:

$$\begin{aligned} 2i \operatorname{disc} A(s, \dots) &= \frac{1}{2i} [A(s + i0, \dots) - A(s - i0, \dots)] = \\ &= \sum_{n \geq 2} \operatorname{disc}_{(n)} A(s, \dots). \end{aligned} \quad (3.302)$$

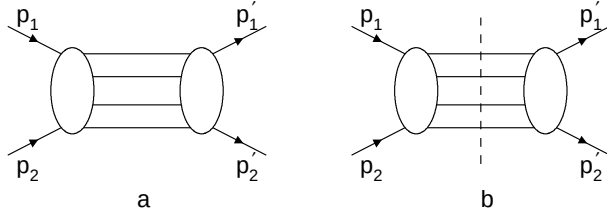


Fig. 3.36 (a) Elastic scattering with a four-particle intermediate state.
 (b) Graphical representation of the discontinuity at the four-particle threshold singularity.

The values $s+i0$ and $s-i0$ are shown in Fig. 3.35a. The total discontinuity of the amplitude is equal to its imaginary part as follows:

$$\text{disc } A = \text{Im } A = \frac{1}{2} \sum_{n \geq 2} d\phi_N A_{2 \rightarrow N}(p_1, p_2, \dots) A_{2 \rightarrow N}^*(p'_1, p'_2, \dots). \quad (3.303)$$

This equality can be obtained directly from the unitarity condition for the S-matrix:

$$SS^+ = 1. \quad (3.304)$$

The imaginary part of the elastic amplitude in the forward direction (or at $t=0$) is expressed in terms of the total cross section:

$$\text{Im } A(0) = \frac{1}{2} J \sigma_{tot}. \quad (3.305)$$

For high initial energies ($s \gg m_1^2, m_2^2$), $J = 2s$ and

$$\text{Im } A(0) \simeq s \sigma_{tot}. \quad (3.306)$$

The discontinuities of the $(3 \rightarrow 3)$ elastic amplitude (Fig. 3.37a) are determined similarly to those of the $(2 \rightarrow 2)$ amplitude. For example, the discontinuity of the $(3 \rightarrow 3)$ amplitude at the four-particle threshold is defined by the equation (3.301) with the replacements $A_{2 \rightarrow 4} \rightarrow A_{3 \rightarrow 4}$, and $A_{4 \rightarrow 2} \rightarrow A_{4 \rightarrow 3}$ (see Fig. 3.37b). Thus, the total discontinuity of $A_{3 \rightarrow 3}$ at $p_1 = p'_1$, $p_2 = p'_2$ and $k = k'$ (see Fig. 3.37a) is expressed in terms of the inclusive cross section of the production of the particle h with momentum k :

$$\frac{2}{J} \text{disc } A_{3 \rightarrow 3} = (2\pi)^3 2k_0 \frac{d\sigma}{d^3k} (1 + 2 \rightarrow h + X). \quad (3.307)$$

The discontinuities of more complicated amplitudes ($n \rightarrow n$) may be connected with the inclusive cross section of $(n-2)$ particle production in a similar manner.

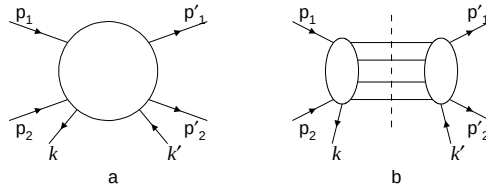


Fig. 3.37 (a) $(3 \rightarrow 3)$ elastic amplitude and (b) cut $(3 \rightarrow 3)$ diagram.

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Chapter 4

Baryon–Baryon and Baryon–Antibaryon Systems

In this chapter certain topics of the description of processes initiated by two fermions are discussed. We present the calculations in scrupulous details, keeping in mind that only this way one can provide a fundamental understanding of the technique.

In terms of the K -matrix and dispersion relation approaches we consider fermion–fermion and fermion–antifermion scattering amplitudes for isosinglet baryons, such as $\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}$ and $\Lambda\Lambda \rightarrow \Lambda\Lambda$, and for nucleons ($I = 1/2$) $NN \rightarrow NN$ and $N\bar{N} \rightarrow N\bar{N}$. The technique of expansion over angular momentum operators is given in Appendices 4.A and 4.B. In Appendix 4.C we give examples of the analysis of the reactions $NN \rightarrow NN$ in the simplest version of the dispersion relation method where the interaction is written as a sum of separable vertices. We also show here the results of calculation of the deuteron form factors as well as the deuteron disintegration processes.

We consider the production of Δ -resonances ($I = 3/2$, $J = 3/2$) in the reaction $NN \rightarrow N\Delta$. Numerical results of the analysis of this reaction carried out in terms of spectral integral technique with separable vertices are given in Appendix 4.C, while in Appendix 4.D the technique of the calculation of $N\Delta$ loop diagram is presented.

The processes of $N\bar{N}$ annihilation are also considered, namely, the production of meson resonances in the two- and three-particle final states: $N\bar{N} \rightarrow P_1P_2$ and $N\bar{N} \rightarrow P_1P_2P_3$. In Appendix 4.E, the results of fitting to data on the reactions $p\bar{p} \rightarrow \pi\pi, \eta\eta, \eta\eta'$ are presented (remind that just the analysis of these reactions proved that $f_2(2000)$ is a flavour blind state — the glueball, while the neighbouring f_2 states are not).

Recent partial wave analyses, aiming to extract the pole singularities of amplitudes and to determine resonances, should take into account the existence of other singularities: threshold ones and those which are related to

the production of resonances in the intermediate states. Threshold singularities are usually treated in terms of the K-matrix technique (for spinless particles this technique was discussed in Chapter 3). The singularities owing to resonances in the intermediate states need a more sophisticated treatment. In this chapter, when discussing the $NN \rightarrow N\Delta$ and $NN \rightarrow \Delta\Delta$ processes, we provide some examples of the rescattering processes, which give rise to strong singularities related to the triangle and box diagrams. As an example, we consider processes $NN \rightarrow N\Delta \rightarrow NN\pi \rightarrow N\Delta$ (triangle singularity) and $NN \rightarrow \Delta\Delta \rightarrow NN\pi\pi$ with a subsequent rescattering of pions (box singularity).

We present here formulae for the production of resonances with arbitrary spin, $NN \rightarrow NN_{j=n+1/2}^*$, where $n = 1, 2, 3, 4, \dots$

With the growth of the energy the resonance production region transforms gradually into that of reggeon exchange. In the intermediate region both mechanisms, resonance production and reggeon exchange, work. In this chapter we present some elements of reggeon technique for the NN -scattering amplitude.

At superhigh energies new thresholds with the production of new heavy particles may exist. In this case there appears an interesting interplay of the low-energy and high-energy physics. We consider such a possibility and investigate how the thresholds of new heavy particles stand out against a background of the light hadron scatterings (Appendix 4.F).

In Appendix 4.G we reanalyse the Schmid theorem for the triangle diagram contributions to the spectra of secondaries. Triangle singularities (as well as singularities of box-diagrams) reveal themselves in different ways in the case of pure elastic and of inelastic scatterings. We underline that, when inelasticities occur, triangle diagrams result in considerable effects in both the two-particle spectra and when averaging over other variables.

The nucleon $N(980)$ is the basic state on the (n, M^2) -trajectory. The next excited states of the nucleon are the Roper resonance $N(1440)$ and the $N(1710)$ state – the position of the poles of these states is discussed in Appendix 4.H.

4.1 Two-Baryon States and Their Scattering Amplitudes

4.1.1 Spin-1/2 wave functions

We work with baryon wave functions $\psi(p)$ and $\bar{\psi}(p) = \psi^+(p)\gamma_0$ which obey the Dirac equation

$$(\hat{p} - m)\psi(p) = 0, \quad \bar{\psi}(p)(\hat{p} - m) = 0. \quad (4.1)$$

The following γ -matrices are used:

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0 = -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \\ \gamma_0^+ &= \gamma_0, \quad \gamma^+ = -\gamma, \end{aligned} \quad (4.2)$$

and the standard Pauli matrices:

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \sigma_a\sigma_b &= I_{ab} + i\varepsilon_{abc}\sigma_c. \end{aligned} \quad (4.3)$$

The solution of the Dirac equation gives us four wave functions:

$$\begin{aligned} j = 1, 2: \quad \psi_j(p) &= \sqrt{p_0 + m} \begin{pmatrix} \varphi_j \\ \frac{(\boldsymbol{\sigma}\mathbf{p})}{p_0 + m} \varphi_j \end{pmatrix}, \\ \bar{\psi}_j(p) &= \sqrt{p_0 + m} \begin{pmatrix} \varphi_j^+, -\varphi_j^+ \frac{(\boldsymbol{\sigma}\mathbf{p})}{p_0 + m} \end{pmatrix}, \\ j = 3, 4: \quad \psi_j(-p) &= i\sqrt{p_0 + m} \begin{pmatrix} \frac{(\boldsymbol{\sigma}\mathbf{p})}{p_0 + m} \chi_j \\ \chi_j \end{pmatrix}, \\ \bar{\psi}_j(-p) &= -i\sqrt{p_0 + m} \begin{pmatrix} \chi_j^+ \frac{(\boldsymbol{\sigma}\mathbf{p})}{p_0 + m}, -\chi_j^+ \end{pmatrix}, \end{aligned} \quad (4.4)$$

where φ_j and χ_j are two-component spinors,

$$\varphi_j = \begin{pmatrix} \varphi_{j1} \\ \varphi_{j2} \end{pmatrix}, \quad \chi_j = \begin{pmatrix} \chi_{j1} \\ \chi_{j2} \end{pmatrix}, \quad (4.5)$$

normalised as

$$\varphi_j^+ \varphi_\ell = \delta_{j\ell}, \quad \chi_j^+ \chi_\ell = \delta_{j\ell}. \quad (4.6)$$

Solutions with $j = 3, 4$ refer to antibaryons. The corresponding wave function is defined as

$$j = 3, 4: \quad \psi_j^c(p) = C\bar{\psi}_j^T(-p), \quad (4.7)$$

where the matrix C obeys the requirement

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T . \quad (4.8)$$

One can use

$$C = \gamma_2 \gamma_0 = \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} . \quad (4.9)$$

We see that

$$C^{-1} = C = C^+ , \quad (4.10)$$

and $\psi_j^c(p)$ satisfies the equation:

$$(\hat{p} - m)\psi_j^c(p) = 0 . \quad (4.11)$$

Let us present $\psi_j^c(p)$ defined by (4.7) in more detail:

$$\begin{aligned} j = 3, 4 : \quad \psi_j^c(p) &= \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} (-i)\sqrt{p_0 + m} \begin{pmatrix} \frac{(\boldsymbol{\sigma}^T \mathbf{p})}{p_0 + m} \chi_j^* \\ -\chi_j^* \end{pmatrix} \\ &= -\sqrt{p_0 + m} \begin{pmatrix} \sigma_2 \chi_j^* \\ \frac{(\boldsymbol{\sigma} \mathbf{p})}{p_0 + m} \sigma_2 \chi_j^* \end{pmatrix} = \sqrt{p_0 + m} \begin{pmatrix} \varphi_j^c \\ \frac{(\boldsymbol{\sigma} \mathbf{p})}{p_0 + m} \varphi_j^c \end{pmatrix} . \end{aligned} \quad (4.12)$$

In (4.12) we have used the commutator $-\sigma_2(\boldsymbol{\sigma}^T \mathbf{p}) = \sigma_1 p_1 \sigma_2 + \sigma_2 p_2 \sigma_2 + \sigma_3 p_3 \sigma_2 = (\boldsymbol{\sigma} \mathbf{p}) \sigma_2$. Also, we defined the spinor for the antibaryon as

$$\varphi_j^c = -i\sigma_2 \chi_j^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \chi_j^* = \begin{pmatrix} -\chi_{j2}^* \\ \chi_{j1}^* \end{pmatrix} . \quad (4.13)$$

Wave functions defined by (4.4) are normalised as follows:

$$\begin{aligned} j, \ell = 1, 2 : \quad & (\bar{\psi}_j(p) \psi_\ell(p)) = 2m \delta_{j\ell} , \\ j, \ell = 3, 4 : \quad & (\bar{\psi}_j(p) \psi_\ell(p)) = -2m \delta_{j\ell} , \end{aligned} \quad (4.14)$$

and, after summing over polarisations, they obey the completeness conditions:

$$\begin{aligned} \sum_{j=1,2} \psi_{j\alpha}(p) \bar{\psi}_{j\beta}(p) &= (\hat{p} + m)_{\alpha\beta} , \\ \sum_{j=3,4} \psi_{j\alpha}(p) \bar{\psi}_{j\beta}(p) &= -(\hat{p} + m)_{\alpha\beta} . \end{aligned} \quad (4.15)$$

As an example, let us present the calculation of the normalisation conditions (4.14) in more detail:

$$\begin{aligned} j, \ell = 1, 2 : \quad & (\bar{\psi}_j(p) \psi_\ell(p)) = (p_0 + m) \left(\varphi_j^+ \varphi_\ell - \varphi_j^+ \frac{(\boldsymbol{\sigma} \mathbf{p})}{p_0 + m} \frac{(\boldsymbol{\sigma} \mathbf{p})}{p_0 + m} \varphi_\ell \right) \\ &= \frac{p_0^2 + 2p_0 m + m^2 - \mathbf{p}^2}{p_0 + m} = \frac{2m^2 + 2p_0 m}{p_0 + m} = 2m \delta_{j\ell} , \\ j, \ell = 3, 4 : \quad & (\bar{\psi}_j(p) \psi_\ell(p)) = (p_0 + m) \left(\chi_j^+ \frac{(\boldsymbol{\sigma} \mathbf{p})}{p_0 + m} \frac{(\boldsymbol{\sigma} \mathbf{p})}{p_0 + m} \chi_\ell - \chi_j^+ \chi_\ell \right) \\ &= -2m \delta_{j\ell} . \end{aligned} \quad (4.16)$$

Sometimes it is more convenient to use the four-component spinors with a different normalisation, substituting $\psi(p) \rightarrow u(p)$:

$$\begin{aligned} (\bar{u}(p)_j u_\ell(p)) &= -(\bar{u}_j(-p) u_\ell(-p)) = \delta_{j\ell} , \\ \sum_{j=1,2} u_j(p) \bar{u}_j(p) &= \frac{m + \hat{p}}{2m} , \\ \sum_{j=3,4} u_j(-p) \bar{u}_j(-p) &= \frac{-m + \hat{p}}{2m} . \end{aligned} \quad (4.17)$$

Below we use both types of four-component spinors, $\psi(p)$ and $u(p)$.

4.1.2 Baryon–antibaryon scattering

To explain the technique used here, it is convenient to start with baryon–antibaryon systems. We shall consider here the $\Lambda\bar{\Lambda}$ and $N\bar{N}$ scattering amplitudes.

4.1.2.1 Baryons with isospin $\mathbf{I} = \mathbf{0}$

Let us see first a baryon–antibaryon scattering amplitude for an isosinglet baryon, for example, the $\Lambda\bar{\Lambda}$ scattering amplitude. There are two alternative representations of the baryon–antibaryon amplitude $\Lambda(p_1)\bar{\Lambda}(p_2) \rightarrow \Lambda(p'_1)\bar{\Lambda}(p'_2)$.

(a) Angular momentum expansion in the t -channel

Let us introduce the t -channel momentum operators (we define $t = q^2 = (p'_1 - p_1)^2$):

$$\begin{aligned} M(s, t, u) &= \sum_{\substack{S, L, L', J \\ \mu_1 \dots \mu_J}} \left(\bar{\psi}(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(q) \psi(p_1) \right) \left(\bar{\psi}(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(q) \psi(-p'_2) \right) \\ &\times A_t^{(S, L' L, J)}(q^2) . \end{aligned} \quad (4.18)$$

Here $q = p_1 - p'_1$, and $q \perp (p_1 + p'_1)$. A detailed discussion of the operators $\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}$ and their properties may be found in Appendices 4.A and 4.B.

(b) Angular momentum expansion in the s -channel

Another representation is related to the s -channel momentum operators ($s = (p_1 + p_2)^2$):

$$\begin{aligned} M(s, t, u) &= \sum_{\substack{S, L, L', J \\ \mu_1 \dots \mu_J}} \left(\bar{\psi}(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k') \psi(-p'_2) \right) \left(\bar{\psi}(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \psi(p_1) \right) \\ &\times A_s^{(S, L' L, J)}(s) . \end{aligned} \quad (4.19)$$

The notations are as follows:

$$P = p_1 + p_2 = p'_1 + p'_2, \quad k = \frac{1}{2}(p_1 - p_2), \quad k' = \frac{1}{2}(p'_1 - p'_2),$$

$$g_{\nu\mu}^\perp = g_{\nu\mu} - \frac{P_\nu P_\mu}{P^2}, \quad k_{\perp\mu} = k_\nu g_{\nu\mu}^\perp. \quad (4.20)$$

Note that we consider the case of equal masses, $k_\perp = k$ and $k'_\perp = k'$. Sometimes, when necessary, we use a more detailed notation, namely:

$$g_{\nu\mu}^\perp = g_{\nu\mu} - P_\nu P_\mu / P^2 \equiv g_{\mu\nu}^{\perp P}. \quad (4.21)$$

The amplitude representations (4.18) and (4.19) are illustrated by Figs. 4.1a and 4.1b, respectively. The equation (4.18) suits the consideration of the t -channel meson or reggeon exchanges, while the formula (4.19) is convenient for the s -channel partial wave analysis. The representations (4.18) and (4.19) are related to each other by the Fierz transformation and the corresponding reexpansion of momentum operators.

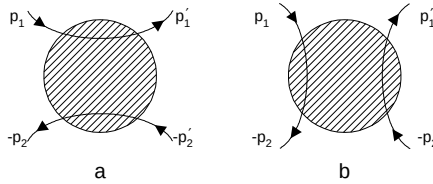


Fig. 4.1 Graphical representation of the $N\bar{N}$ scattering amplitude for a) equation (4.18) and b) equation (4.19).

In (4.18) and (4.19) we use the operators $\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}$. The general form of these operators is given in Appendix 4.B, while here, to be more illustrative, we write some of them which describe the low-lying states.

For the $\Lambda\bar{\Lambda}$ system (formula (4.19)) we present the s -channel operators $\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k)$.

For $L = 0$ the operators read:

$$J^P = 0^-(S = 0, L = 0, J = 0) : \hat{Q}^{000}(k) = i\gamma_5 \quad (4.22)$$

$$J^P = 1^-(S = 1, L = 0, J = 1) : \hat{Q}_\mu^{101}(k) = \Gamma_\mu(k)$$

$$= \gamma_\nu \left(g_{\nu\mu}^\perp - \frac{2k_{\perp\nu} k_{\perp\mu}}{m(\sqrt{s} + 2m)} \right),$$

for $L = 1$:

$$\begin{aligned}
 J^P = 0^+(S = 1, L = 1, J = 0) : \hat{Q}^{110}(k) &= mI , \\
 J^P = 1^+(S = 1, L = 1, J = 1) : \hat{Q}_\mu^{111}(k) &= \sqrt{\frac{3}{2}} \frac{1}{s} i\varepsilon_{\gamma P k \mu} , \\
 J^P = 2^+(S = 1, L = 1, J = 2) : \hat{Q}_{\mu_1 \mu_2}^{112}(k) &= \sqrt{\frac{4}{3s}} \left[k_{\perp \mu_1} \Gamma_{\mu_2}(k) \right. \\
 &\quad \left. + \Gamma_{\mu_1}(k) k_{\perp \mu_2} - \frac{2}{3} g_{\mu_1 \mu_2} (k_{\perp} \Gamma(k)) \right] , \\
 J^P = 1^+(S = 0, L = 1, J = 1) : \hat{Q}_\mu^{011}(k) &= \sqrt{\frac{3}{s}} i\gamma_5 k_{\perp \mu} .
 \end{aligned} \tag{4.23}$$

In (4.23) a short notation is used: $\varepsilon_{\gamma P k \mu} \equiv \varepsilon_{\alpha_1 \alpha_2 \alpha_3 \mu} \gamma_{\alpha_1} P_{\alpha_2} k_{\alpha_3}$.

The operators for the states with $J^P = 1^-$ and $J^P = 2^-$ (angular momenta $L = 2$ and $L = 3$) are written as follows:

$$\begin{aligned}
 J^P = 1^-(S = 1, L = 2, J = 1) : \hat{Q}_\mu^{121}(k) &= \frac{3}{\sqrt{2}s} \left[k_{\perp \mu} (k_{\perp} \Gamma(k)) \right. \\
 &\quad \left. - \frac{1}{3} k_{\perp}^2 \Gamma_{\mu}(k) \right] , \\
 J^P = 2^-(S = 1, L = 3, J = 2) : \hat{Q}_{\mu_1 \mu_2}^{132}(k) &= \frac{5}{\sqrt{2}s^{3/2}} \left[k_{\perp \mu_1} k_{\perp \mu_2} (k_{\perp} \Gamma(k)) \right. \\
 &\quad \left. - \frac{1}{5} k_{\perp}^2 \left(g_{\mu_1 \mu_2}^{\perp} (k_{\perp} \Gamma(k)) + \Gamma_{\mu_1}(k) k_{\perp \mu_2} + k_{\perp \mu_1} \Gamma_{\mu_2}(k) \right) \right] .
 \end{aligned} \tag{4.24}$$

The operators $\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k)$ are given in Appendix 4.B, for their definition we use the angular momentum operators $X_{\mu_1 \dots \mu_L}^{(L)}(k)$ (see [1]) which for $L = 0, 1, 2, 3$ read:

$$\begin{aligned}
 X^{(0)}(k) &= 1 , \quad X_{\mu}^{(1)}(k) = k_{\perp \mu} , \\
 X_{\mu_1 \mu_2}^{(2)}(k) &= \frac{3}{2} \left(k_{\perp \mu_1} k_{\perp \mu_2} - \frac{1}{3} k_{\perp}^2 g_{\mu_1 \mu_2}^{\perp} \right) , \\
 X_{\mu_1 \mu_2 \mu_3}^{(3)}(k) &= \frac{5}{2} \left[k_{\perp \mu_1} k_{\perp \mu_2} k_{\perp \mu_3} \right. \\
 &\quad \left. - \frac{1}{5} k_{\perp}^2 (g_{\mu_1 \mu_2}^{\perp} k_{\perp \mu_3} + g_{\mu_2 \mu_3}^{\perp} k_{\perp \mu_1} + g_{\mu_1 \mu_3}^{\perp} k_{\perp \mu_2}) \right] .
 \end{aligned} \tag{4.25}$$

The properties of $X_{\mu_1 \mu_2 \dots \mu_J}^{(J)}(k)$ are formulated in Appendix 4.A. Using the covariant representation of angular momentum operators $X_{\mu_1 \dots \mu_J}^{(J)}(k)$, we can construct the general form of the operators $\hat{Q}_{\mu_1 \dots \mu_J}^{(S, L, J)}$; this is done in Appendix 4.B.

But right now let us write a series similar to formula (4.19) for nucleons $N = (p, n)$ which form an isodoublet.

4.1.2.2 Nucleon-antinucleon scattering amplitude

The nucleon is an isodoublet, $I = 1/2$, with the components:

(i) proton: $p \rightarrow (I = 1/2, I_3 = 1/2)$, (ii) neutron: $n \rightarrow (I = 1/2, I_3 = -1/2)$.

The systems $p\bar{n}$ and $n\bar{p}$ have an isospin $I = 1$, and the s -channel expansions of scattering amplitudes are determined by formulae similar to those for $\Lambda\bar{\Lambda}$, Eq. (4.19). The systems $p\bar{p}$ and $n\bar{n}$ are superpositions of two states with $I = 0$ and $I = 1$, and the amplitudes read:

$$\begin{aligned}
 p(p_1)\bar{n}(p_2) \rightarrow p(p'_1)\bar{n}(p'_2) \quad (I = 1) : & \left(C_{\frac{1}{2}\frac{1}{2}, \frac{1}{2}\frac{1}{2}}^{11} \right)^2 M_1(s, t, u) = M_1(s, t, u), \\
 p(p_1)\bar{p}(p_2) \rightarrow p(p'_1)\bar{p}(p'_2) \quad (I = 0, 1) : & \left(C_{\frac{1}{2}\frac{1}{2}, \frac{1}{2}-\frac{1}{2}}^{10} \right)^2 M_1(s, t, u) \\
 & + \left(C_{\frac{1}{2}\frac{1}{2}, \frac{1}{2}-\frac{1}{2}}^{00} \right)^2 M_0(s, t, u) \\
 & = \frac{1}{2} M_1(s, t, u) + \frac{1}{2} M_0(s, t, u), \\
 p(p_1)\bar{p}(p_2) \rightarrow n(p'_1)\bar{n}(p'_2) \quad (I = 0, 1) : & C_{\frac{1}{2}\frac{1}{2}, \frac{1}{2}-\frac{1}{2}}^{10} C_{\frac{1}{2}-\frac{1}{2}, \frac{1}{2}\frac{1}{2}}^{10} M_1(s, t, u) \\
 & + C_{\frac{1}{2}\frac{1}{2}, \frac{1}{2}-\frac{1}{2}}^{00} C_{\frac{1}{2}-\frac{1}{2}, \frac{1}{2}\frac{1}{2}}^{00} M_0(s, t, u) \\
 & = \frac{1}{2} M_1(s, t, u) - \frac{1}{2} M_0(s, t, u). \quad (4.26)
 \end{aligned}$$

Note that when writing $N\bar{N}$ (or NN) scattering amplitudes, one can use alternative techniques of isotopic Pauli matrices ($I/\sqrt{2}$, $\tau/\sqrt{2}$) or Clebsch–Gordan coefficients. In (4.26) we use the Clebsch–Gordan coefficients keeping in mind that in what follows the production of states with $I > 1/2$ is also considered, and in this case the Clebsch–Gordan technique is appropriate.

For $M_I(s, t, u)$ the s -channel operator expansion gives:

$$\begin{aligned}
 I = 0 : \quad M_0(s, t, u) &= \sum_{\substack{S, L, L', J \\ \mu_1 \dots \mu_J}} \left(\bar{\psi}(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k') \psi(-p'_2) \right) \\
 &\quad \times \left(\bar{\psi}(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \psi(p_1) \right) A_0^{(S, L' L, J)}(s), \\
 I = 1 : \quad M_1(s, t, u) &= \sum_{\substack{S, L, L', J \\ \mu_1 \dots \mu_J}} \left(\bar{\psi}(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k') \psi(-p'_2) \right) \\
 &\quad \times \left(\bar{\psi}(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \psi(p_1) \right) A_1^{(S, L' L, J)}(s).
 \end{aligned} \quad (4.27)$$

In (4.27) the summation is carried out over all states, namely:

$$S = 0, J = L; \quad S = 1, J = L - 1, L, L + 1. \quad (4.28)$$

Let us remind that the momentum operators $\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k)$ for these states are given in (4.22), (4.23) and (4.24).

4.1.3 Baryon-baryon scattering

Consider now three types of the baryon-baryon scattering amplitudes:

(i) $p\Lambda \rightarrow p\Lambda$, (ii) $\Lambda\Lambda \rightarrow \Lambda\Lambda$ and (iii) $NN \rightarrow NN$.

4.1.3.1 The $p\Lambda \rightarrow p\Lambda$ scattering amplitude

It is convenient to represent the amplitude $p\Lambda \rightarrow p\Lambda$ using exactly the same technique as for the s -channel fermion-antifermion system (see Eqs. (4.19), (4.23) and (4.25)). This representation is possible if for the $p\Lambda \rightarrow p\Lambda$ scattering we declare $\bar{\Lambda}$ to be a fermion and Λ an antifermion, then

$$M_{N\Lambda \rightarrow N\Lambda}(s, t, u) = \sum_{\substack{S, L, L', J \\ \mu_1 \dots \mu_J}} \left(\bar{\psi}_N(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k') \psi_{\Lambda}^c(-p'_2) \right) \quad (4.29)$$

$$\times \left(\bar{\psi}_{\Lambda}^c(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \psi_N(p_1) \right) A_{N\Lambda \rightarrow N\Lambda}^{(S, L' L, J)}(s).$$

The operators $\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k)$ with $J = 0, 1, 2$ are given in (4.22), (4.23) and (4.24), but one should take into account that for particles with different masses the operator of a pure $S = 1$ state, $\Gamma_{\alpha}(k_{\perp})$, is equal to:

$$\Gamma_{\alpha}(k_{\perp}) = \gamma_{\beta} \left(g_{\alpha\beta}^{\perp} - \frac{4s k_{\perp\alpha} k_{\perp\beta}}{(m_N + m_{\Lambda})(\sqrt{s} + m_N + m_{\Lambda})(s - (m_N - m_{\Lambda})^2)} \right). \quad (4.30)$$

To be illustrative, let us present the initial-state terms from (4.29) with $L = 0$ in a non-relativistic limit.

(i) The S -wave terms in the non-relativistic limit.

We consider the initial-state terms with $L = 0$ using Eq. (4.29) in the c.m. system ($\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{k}$ and $\mathbf{p}'_1 = -\mathbf{p}'_2 = \mathbf{k}'$). For $L = 0$ we have the following operators in the non-relativistic approach:

$$Q^{000}(k) = i\gamma_5 = -i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad Q^{101}(k) = \Gamma_{\mu}(k_{\perp}) \simeq \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}. \quad (4.31)$$

In the c.m. system

$$j, j' = 1, 2: \quad \psi_{Nj}(p_1) \simeq \sqrt{2m_N} \begin{pmatrix} \varphi_{Nj} \\ \frac{(\boldsymbol{\sigma}\mathbf{k})}{2m_N} \varphi_{Nj} \end{pmatrix},$$

$$\bar{\psi}_{Nj'}(p'_1) \simeq \sqrt{2m_N} \begin{pmatrix} \varphi_{Nj'}^+, -\varphi_{Nj'}^+ \frac{(\boldsymbol{\sigma}\mathbf{k}')}{2m_N} \end{pmatrix},$$

$$\ell, \ell' = 3, 4: \quad \psi_{\Lambda\ell'}^c(-p'_2) \simeq i\sqrt{2m_{\Lambda}} \begin{pmatrix} -\frac{(\boldsymbol{\sigma}\mathbf{k}')}{2m_{\Lambda}} \chi_{\Lambda\ell'}^c \\ \chi_{\Lambda\ell'}^c \end{pmatrix},$$

$$\bar{\psi}_{\Lambda\ell}^c(-p_2) \simeq -i\sqrt{2m_{\Lambda}} \begin{pmatrix} \chi_{\Lambda\ell}^{c+} - \frac{(\boldsymbol{\sigma}\mathbf{k})}{2m_{\Lambda}} \\ -\chi_{\Lambda\ell}^{c+} \end{pmatrix}, \quad (4.32)$$

where φ_{Nj} and $\chi_{\Lambda\ell}^c$ are spinors.

For the waves with $J = 0, 1$ we have:

$$L = 0, J = 0 :$$

$$\begin{aligned} & \left(\bar{\psi}_N(p'_1) \hat{Q}^{000}(k') \psi_{\Lambda}^c(-p'_2) \right) \left(\bar{\psi}_{\Lambda}^c(-p_2) \hat{Q}^{000}(k) \psi_N(p_1) \right) A_{N\Lambda \rightarrow N\Lambda}^{(0,00,0)}(s) \\ & \simeq \sqrt{4m_N m_{\Lambda}} \left(\varphi_{Nj'}^+ \chi_{\Lambda\ell'}^c \right) \left(\chi_{\Lambda\ell}^+ \varphi_{Nj} \right) \sqrt{4m_N m_{\Lambda}} A_{N\Lambda \rightarrow N\Lambda}^{(0,00,0)}(s), \end{aligned}$$

$$L = 0, J = 1 :$$

$$\begin{aligned} & \left(\bar{\psi}_N(p'_1) \hat{Q}_{\mu}^{101}(k') \psi_{\Lambda}^c(-p'_2) \right) \left(\bar{\psi}_{\Lambda}^c(-p_2) \hat{Q}_{\mu}^{101}(k) \psi_N(p_1) \right) A_{N\Lambda \rightarrow N\Lambda}^{(1,00,1)}(s) \\ & \simeq i\sqrt{4m_N m_{\Lambda}} \left(\varphi_{Nj'}^+ \boldsymbol{\sigma} \chi_{\Lambda\ell'}^c \right) \left(\chi_{\Lambda\ell}^+ \boldsymbol{\sigma} \varphi_{Nj} \right) i\sqrt{4m_N m_{\Lambda}} A_{N\Lambda \rightarrow N\Lambda}^{(1,00,1)}(s). \quad (4.33) \end{aligned}$$

For nucleons and Λ we write:

$$\begin{aligned} \varphi_{Nj} &= \begin{pmatrix} \varphi_{\uparrow}(Nj) \\ \varphi_{\downarrow}(Nj) \end{pmatrix}, & \varphi_{Nj}^+ &= (\varphi_{\uparrow}^*(Nj), \varphi_{\downarrow}^*(Nj)), \\ \chi_{\Lambda\ell}^c &= i\sigma_2 \begin{pmatrix} \varphi_{\uparrow}(\Lambda\ell) \\ \varphi_{\downarrow}(\Lambda\ell) \end{pmatrix} = \begin{pmatrix} \varphi_{\downarrow}(\Lambda\ell) \\ -\varphi_{\uparrow}(\Lambda\ell) \end{pmatrix}, & \chi_{\Lambda\ell}^{c+} &= (\varphi_{\downarrow}^*(\Lambda\ell), -\varphi_{\uparrow}^*(\Lambda\ell)). \end{aligned} \quad (4.34)$$

One can use the spinors with real components in the following representation:

$$\begin{aligned} \varphi_{N1} &= \begin{pmatrix} \varphi_{\uparrow}(N) \\ 0 \end{pmatrix}, & \varphi_{N2} &= \begin{pmatrix} 0 \\ \varphi_{\downarrow}(N) \end{pmatrix}, \\ \chi_{\Lambda1}^c &= \begin{pmatrix} \varphi_{\downarrow}(\Lambda) \\ 0 \end{pmatrix}, & \chi_{\Lambda2}^c &= \begin{pmatrix} 0 \\ -\varphi_{\uparrow}(\Lambda) \end{pmatrix}. \end{aligned} \quad (4.35)$$

Within this definition, we can rewrite (4.33) in terms of the traditional technique which uses the Clebsch–Gordan coefficients.

We have for $J = 0$:

$$\begin{aligned} \left(\chi_{\Lambda\ell}^{c+} \frac{I}{\sqrt{2}} \varphi_{Nj} \right) &= \left(\varphi_{Nj}^+ \frac{I}{\sqrt{2}} \chi_{\Lambda\ell}^c \right) = \frac{1}{\sqrt{2}} (\varphi_{\uparrow}(Nj) \varphi_{\downarrow}(\Lambda\ell) - \varphi_{\downarrow}(Nj) \varphi_{\uparrow}(\Lambda\ell)) \\ &= \sum_{\alpha} C_{\frac{1}{2}\alpha, \frac{1}{2}-\alpha}^{00} \varphi_{\alpha}(Nj) \varphi_{-\alpha}(\Lambda\ell), \end{aligned} \quad (4.36)$$

and for $J = 1, J_3 = 0$:

$$\begin{aligned} \left(\chi_{\Lambda\ell}^{c+} \frac{\sigma_3}{\sqrt{2}} \varphi_{Nj} \right) &= \left(\varphi_{Nj}^+ \frac{\sigma_3}{\sqrt{2}} \chi_{\Lambda\ell}^c \right) = \frac{1}{\sqrt{2}} (\varphi_{\uparrow}(Nj) \varphi_{\downarrow}(\Lambda\ell) + \varphi_{\downarrow}(Nj) \varphi_{\uparrow}(\Lambda\ell)) \\ &= \sum_{\alpha} C_{\frac{1}{2}\alpha, \frac{1}{2}-\alpha}^{10} \varphi_{\alpha}(Nj) \varphi_{-\alpha}(\Lambda\ell). \end{aligned} \quad (4.37)$$

We have considered here the S -wave state: the D -wave is removed by the second term in the right-hand side of (4.30). But the $(J = 1)$ state may contain the D -wave as well.

(ii) The D -wave component in the operator $\gamma_\mu^\perp$.

Below we demonstrate that the operator $\gamma_\mu^\perp$ contains the D -wave. The equations (4.32) and (4.33) allow us to see easily the existence of the D -wave admixture in the operator $\gamma_\mu^\perp$. Replacing the operator $\hat{Q}_\mu^{101}(k) \rightarrow \gamma_\mu^\perp$ in (4.33), one has the following next-to-leading term in the $(J = 1)$ -wave:

$$\begin{aligned}
 & -\sqrt{4m_N m_\Lambda} \left(\varphi_{Nj'}^+ \frac{(\boldsymbol{\sigma} \mathbf{k}')}{2m_N} \boldsymbol{\sigma} \frac{(\boldsymbol{\sigma} \mathbf{k}')}{2m_\Lambda} \chi_{\Lambda\ell'}^c \right) \\
 & \times \left(\chi_{\Lambda\ell}^{c+} \frac{(\boldsymbol{\sigma} \mathbf{k})}{2m_\Lambda} \boldsymbol{\sigma} \frac{(\boldsymbol{\sigma} \mathbf{k})}{2m_N} \varphi_{Nj} \right) \sqrt{4m_N m_\Lambda} A_{N\Lambda \rightarrow N\Lambda}^{(1,00,1)}(s). \quad (4.38)
 \end{aligned}$$

The momentum operators in (4.38) may be represented as follows:

$$\frac{(\boldsymbol{\sigma} \mathbf{k})}{2m_\Lambda} \boldsymbol{\sigma} \frac{(\boldsymbol{\sigma} \mathbf{k})}{2m_N} \simeq \frac{\mathbf{k}(\boldsymbol{\sigma} \mathbf{k})}{2m_\Lambda m_N} + \boldsymbol{\sigma} O\left(\frac{\mathbf{k}^2}{m_\Lambda m_N}\right), \quad (4.39)$$

where the first term in the right-hand side refers to the D -wave, while the second one is a correction to the S -wave term. In the operator $\Gamma_\alpha(k_\perp)$, see (4.30), the D -wave admixture is cancelled by the second term: $-[4sk_\perp \alpha(k_\perp \gamma)] / [(m_N + m_\Lambda)(\sqrt{s} + m_N + m_\Lambda)(s - (m_N - m_\Lambda)^2)]$.

4.1.3.2 Amplitude for $\Lambda\Lambda \rightarrow \Lambda\Lambda$ scattering

We represent the amplitude $\Lambda\Lambda \rightarrow \Lambda\Lambda$ using the same technique as was applied to the reaction $p\Lambda \rightarrow p\Lambda$. Thus we declare one Λ hyperon to be a fermion and the second one to be an antifermion. One can distinguish between them, for example, in the c.m. system labelling a particle flying away in the backward hemisphere as an “antifermion”. Then the s -channel expansion of the $\Lambda\Lambda \rightarrow \Lambda\Lambda$ scattering amplitude reads:

$$\begin{aligned}
 M_{\Lambda\Lambda \rightarrow \Lambda\Lambda}(s, t, u) = & \sum_{\substack{S, L, L', J \\ \mu_1 \dots \mu_J}} \left(\bar{\psi}_\Lambda(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k') \psi_\Lambda^c(-p'_2) \right) \\
 & \times \left(\bar{\psi}_\Lambda^c(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \psi_\Lambda(p_1) \right) A_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S, L'L, J)}(s). \quad (4.40)
 \end{aligned}$$

In this reaction the selection rules for quantum numbers (Fermi statistics) should be taken into account. In (4.40) the following states contribute:

$$\begin{aligned}
 S = 1 : & \quad (L = 1; J = 0, 1, 2), (L = 3; J = 2, 3, 4), \dots \\
 S = 0 : & \quad (L = 0; J = 0), (L = 2; J = 2), \dots \quad (4.41)
 \end{aligned}$$

The operators $Q_{\mu_1 \dots \mu_J}^{SLJ}(k)$ are presented in Appendix 4.B.

4.1.3.3 Nucleon–nucleon scattering amplitude

The nucleon is an isodoublet with the components $p \rightarrow (I = 1/2, I_3 = 1/2)$ and $n \rightarrow (I = 1/2, I_3 = -1/2)$. The systems pp and nn have a total isospin $I = 1$, and the s -channel expansions of the scattering amplitudes are determined by formulae similar to those for $\Lambda\Lambda$, Eq. (4.40). The systems pp and nn are superpositions of two states with total isospins $I = 0$ and $I = 1$. The amplitudes read:

$$\begin{aligned}
 p(p_1) p(p_2) \rightarrow p(p'_1) p(p'_2) \quad (I=1) : & \left(C_{\frac{1}{2}\frac{1}{2}, \frac{1}{2}\frac{1}{2}}^{11} \right)^2 M_1(s, t, u) = M_1(s, t, u), \\
 p(p_1) n(p_2) \rightarrow p(p'_1) n(p'_2) \quad (I=0, 1) : & \left(C_{\frac{1}{2}\frac{1}{2}, \frac{1}{2}-\frac{1}{2}}^{10} \right)^2 M_1(s, t, u) \\
 & + \left(C_{\frac{1}{2}\frac{1}{2}, \frac{1}{2}-\frac{1}{2}}^{00} \right)^2 M_0(s, t, u) = \\
 & = \frac{1}{2} M_1(s, t, u) + \frac{1}{2} M_0(s, t, u), \quad (4.42) \\
 n(p_1) n(p_2) \rightarrow n(p'_1) n(p'_2) \quad (I=1) : & \left(C_{\frac{1}{2}-\frac{1}{2}, \frac{1}{2}-\frac{1}{2}}^{1-1} \right)^2 M_1(s, t, u) = M_1(s, t, u).
 \end{aligned}$$

The s -channel operator expansion gives for $M_I(s, t, u)$:

$$\begin{aligned}
 I = 0 : M_0(s, t, u) = & \sum_{\substack{S, L, L', J \\ \mu_1 \dots \mu_J}} \left(\bar{\psi}_p(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k') \psi_n(-p'_2) \right) \\
 & \times \left(\bar{\psi}_n^c(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \psi_p(p_1) \right) A_0^{(S, L', L, J)}(s), \\
 S = 1 : & \quad (L = 0; J = 1), \quad (L = 2; J = 1, 2, 3), \quad \dots \\
 S = 0 : & \quad (L = 1; J = 1), \quad (L = 3; J = 3), \quad \dots \quad (4.43)
 \end{aligned}$$

and

$$\begin{aligned}
 I = 1 : M_1(s, t, u) = & \sum_{\substack{S, L, L', J \\ \mu_1 \dots \mu_J}} \left(\bar{\psi}_p(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k') \psi_n^c(-p'_2) \right) \\
 & \times \left(\bar{\psi}_n^c(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \psi(p_1) \right) A_1^{(S, L', L, J)}(s), \\
 S = 1 : & \quad (L = 1; J = 0, 1, 2), \quad (L = 3; J = 2, 3, 4), \quad \dots \\
 S = 0 : & \quad (L = 0; J = 0), \quad (L = 2; J = 2), \quad \dots \quad (4.44)
 \end{aligned}$$

The selection rules obey the Fermi statistics.

Analogous partial wave expansions can be written for the reactions $pp \rightarrow pp$ and $nn \rightarrow nn$ ($I = 1$). Here, as for $\Lambda\Lambda \rightarrow \Lambda\Lambda$, we declare one nucleon to be a fermion and the second one to be an antifermion, and in the c.m. system we distinguish between them labelling a particle flying away in the backward hemisphere as an “antifermion”.

4.1.4 Unitarity conditions and K -matrix representations of the baryon–antibaryon and baryon–baryon scattering amplitudes

Here the unitarity condition is formulated, and it gives us the K -matrix representations of the baryon–antibaryon and baryon–baryon scattering amplitudes assuming that inelastic processes are switched off (for example, because the energy is not large enough). The generalisation of the K -matrix representations for the case of inelastic channels being switched on is performed in a standard way, see Chapter 3. We do not discuss here the parametrisation of the K -matrix elements, they are similar to those in Chapter 3.

In fermion–fermion scattering reactions we deal with one-channel ($J = L$) and two-channel ($J = L \pm 1$) amplitudes.

4.1.4.1 $\Lambda\bar{\Lambda}$ scattering

(i) Partial wave amplitudes for $J = L$.

For the amplitude $\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}$ of Eq. (4.19) the s -channel unitarity condition for $J = L$ reads (we have redenoted $A_s^{(S,LL,J)}(s) \rightarrow A_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S,LL,J)}(s)$):

$$\begin{aligned}
 & \sum_{\mu_1 \dots \mu_J} \left(\bar{\psi}(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k') \psi(-p'_2) \right) \left(\bar{\psi}(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \psi(p_1) \right) \\
 & \times \text{Im} A_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S,LL,J)}(s) \\
 & = \int d\Phi_2(p''_1, p''_2) \sum_{j, \ell} \sum_{\mu_1 \dots \mu_J} \left(\bar{\psi}(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k') \psi(-p'_2) \right) \\
 & \quad \times \left(\bar{\psi}_\ell(-p''_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k'') \psi_j(p''_1) \right) A_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S,LL,J)}(s) \\
 & \times \sum_{\mu'_1 \dots \mu'_J} \left[\left(\bar{\psi}(p_1) \hat{Q}_{\mu'_1 \dots \mu'_J}^{SLJ}(k) \psi(-p_2) \right) \right. \\
 & \quad \left. \times \left(\bar{\psi}_\ell(-p''_2) \hat{Q}_{\mu'_1 \dots \mu'_J}^{SLJ}(k'') \psi_j(p''_1) A_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S,LL,J)}(s) \right) \right]^+ . \quad (4.45)
 \end{aligned}$$

Therefore, we have:

$$\text{Im} A_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S,LL,J)}(s) = \rho_{\Lambda\bar{\Lambda}}^{(SLJ)}(s) A_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S,LL,J)*}(s) A_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S,LL,J)}(s), \quad (4.46)$$

where

$$\begin{aligned}
 \rho_{\Lambda\bar{\Lambda}}^{(SLJ)}(s) &= \frac{1}{2J+1} \int d\Phi_2(p''_1, p''_2) \text{Sp} \left(\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k'') (-\hat{p}''_2 + m_\Lambda) \right. \\
 & \quad \left. \times \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k'') (\hat{p}''_1 + m_\Lambda) \right) . \quad (4.47)
 \end{aligned}$$

The projection operator $O_{\mu_1'' \dots \mu_J''}^{\mu_1 \dots \mu_J}$ was introduced in [1], and the phase space is determined in a standard way:

$$O_{\mu_1'' \dots \mu_J''}^{\mu_1 \dots \mu_J} \rho_{\Lambda\Lambda}^{(SLJ)}(s) = \int d\Phi_2(p_1'', p_2'') \text{Sp} \left(\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k'') (-\hat{p}_2'' + m_\Lambda) \right. \\ \left. \times \hat{Q}_{\mu_1'' \dots \mu_J''}^{SLJ}(k'') (\hat{p}_1'' + m_\Lambda) \right), \quad (4.48)$$

$$d\Phi_2(p_1, p_2) = \frac{1}{2} (2\pi)^4 \delta^{(4)}(P - p_1 - p_2) \frac{d^3 p_1}{(2\pi)^3 2p_{10}} \frac{d^3 p_2}{(2\pi)^3 2p_{20}}. \quad (4.49)$$

The projection operators for $J = 0, 1, 2$ are equal to:

$$J = 0 : O = 1, \\ J = 1 : O_\nu^\mu = g_{\mu\nu}^\perp, \\ J = 2 : O_{\nu_1 \nu_2}^{\mu_1 \mu_2} = \frac{1}{2} \left(g_{\mu_1 \nu_1}^\perp g_{\mu_2 \nu_2}^\perp + g_{\mu_1 \nu_2}^\perp g_{\mu_2 \nu_1}^\perp - \frac{2}{3} g_{\mu_1 \mu_2}^\perp g_{\nu_1 \nu_2}^\perp \right). \quad (4.50)$$

The projection operator $O_{\mu_1'' \dots \mu_J''}^{\mu_1 \dots \mu_J}$ obeys the convolution rule

$$O_{\mu_1 \dots \mu_J}^{\mu_1 \dots \mu_J} = 2J + 1, \quad (4.51)$$

see Appendix 4.B for more detail, that gives us for the phase space (4.47).

The unitarity condition (4.46) results in the following K -matrix representation of the amplitude $\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}$:

$$A_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S, LL, J)}(s) = \frac{K_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S, LL, J)}(s)}{1 - i\rho_{\Lambda\bar{\Lambda}}^{(SLJ)}(s) K_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S, LL, J)}(s)}. \quad (4.52)$$

(ii) Partial wave amplitudes for $S = 1$ and $J = L \pm 1$.

For $L = J \pm 1$ we have four partial wave amplitudes, which form a 2×2 matrix

$$\hat{A}_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, L=J\pm 1, J)}(s) = \begin{vmatrix} A_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J-1 \rightarrow J-1, J)}(s), & A_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J-1 \rightarrow J+1, J)}(s) \\ A_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J+1 \rightarrow J-1, J)}(s), & A_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J+1 \rightarrow J+1, J)}(s) \end{vmatrix}, \quad (4.53)$$

and it can be presented as the following K -matrix:

$$\hat{A}_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, L=J\pm 1, J)}(s) = \hat{K}_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, L=J\pm 1, J)}(s) \\ \times \left[I - i\hat{\rho}_{\Lambda\bar{\Lambda}}^{(S=1, L=J\pm 1, J)}(s) \hat{K}_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, L=J\pm 1, J)}(s) \right]^{-1}. \quad (4.54)$$

Here

$$\hat{K}_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, L=J\pm 1, J)}(s) = \begin{vmatrix} K_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J-1 \rightarrow J-1, J)}(s), & K_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J-1 \rightarrow J+1, J)}(s) \\ K_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J+1 \rightarrow J-1, J)}(s), & K_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J+1 \rightarrow J+1, J)}(s) \end{vmatrix}, \\ \hat{\rho}_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, L=J\pm 1, J)}(s) = \begin{vmatrix} \rho_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J-1 \rightarrow J-1, J)}(s), & \rho_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J-1 \rightarrow J+1, J)}(s) \\ \rho_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J+1 \rightarrow J-1, J)}(s), & \rho_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J+1 \rightarrow J+1, J)}(s) \end{vmatrix}, \quad (4.55)$$

with

$$\rho_{\Lambda\bar{\Lambda}}^{(S, L \rightarrow L', J)}(s) = \frac{1}{2J+1} \int d\Phi_2(p_1'', p_2'') \text{Sp} \left(\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k'')(-\hat{p}_2'' + m_\Lambda) \right. \\ \left. \times \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k'')(\hat{p}_1'' + m_\Lambda) \right). \quad (4.56)$$

Note that the matrices $\hat{\rho}_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, L=J \pm 1, J)}(s)$ and $\hat{K}_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, L=J \pm 1, J)}(s)$ are symmetrical:

$$\rho_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J-1 \rightarrow J+1, J)}(s) = \rho_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J+1 \rightarrow J-1, J)}(s) \text{ and } K_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J-1 \rightarrow J+1, J)}(s) = K_{\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}}^{(S=1, J+1 \rightarrow J-1, J)}(s).$$

4.1.4.2 $\Lambda\Lambda$ scattering

As before, in $\Lambda\Lambda$ scattering one should distinguish between the cases $J = L$ and $J = L \pm 1$.

(i) Partial wave amplitudes $\Lambda\Lambda \rightarrow \Lambda\Lambda$ for $J = L$.

For the amplitude of the $\Lambda\Lambda \rightarrow \Lambda\Lambda$ reaction, with $J = L$, the s -channel unitarity condition reads:

$$\sum_{\mu_1 \dots \mu_J} \left(\bar{\psi}(p_1') \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k') \psi^c(-p_2') \right) \left(\bar{\psi}^c(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \psi(p_1) \right) \\ \times \text{Im} A_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S, LL, J)}(s) \\ = \int \frac{1}{2} d\Phi_2(p_1'', p_2'') \sum_{j, \ell} \sum_{\mu_1 \dots \mu_J} \left(\bar{\psi}(p_1') \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k') \psi^c(-p_2') \right) \\ \times \left(\bar{\psi}_\ell^c(-p_2'') \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k'') \psi_j(p_1'') \right) A_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S, LL, J)}(s) \\ \times \sum_{\mu_1'' \dots \mu_J''} \left[\left(\bar{\psi}(p_1) \hat{Q}_{\mu_1'' \dots \mu_J''}^{SLJ}(k) \psi^c(-p_2) \right) \right. \\ \left. \times \left(\bar{\psi}_\ell^c(-p_2'') \hat{Q}_{\mu_1'' \dots \mu_J''}^{SLJ}(k'') \psi_j(p_1'') A_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S, LL, J)}(s) \right) \right]^+. \quad (4.57)$$

In (4.57) the integrand is written with the identity factor $1/2$, thus keeping for $d\Phi_2(p_1'', p_2'')$ the definition (4.49). We have

$$\text{Im} A_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S, LL, J)}(s) = \frac{1}{2} \rho_{\Lambda\Lambda}^{(SLJ)}(s) A_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S, LL, J)*}(s) A_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S, LL, J)}(s), \quad (4.58)$$

where

$$O_{\mu_1'' \dots \mu_J''}^{\mu_1 \dots \mu_J} \rho_{\Lambda\Lambda}^{(SLJ)}(s) = \int d\Phi_2(p_1'', p_2'') \text{Sp} \left(\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k'')(-\hat{p}_2'' + m_\Lambda) \right. \\ \left. \times \hat{Q}_{\mu_1'' \dots \mu_J''}^{SLJ}(k'')(\hat{p}_1'' + m_\Lambda) \right), \\ \rho_{\Lambda\Lambda}^{(SLJ)}(s) = \frac{1}{2J+1} \int d\Phi_2(p_1'', p_2'') \text{Sp} \left(\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k'')(-\hat{p}_2'' + m_\Lambda) \right. \\ \left. \times \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k'')(\hat{p}_1'' + m_\Lambda) \right). \quad (4.59)$$

So we use the same definition of $\rho_{\Lambda\Lambda}^{(SLJ)}(s)$ and $\rho_{\Lambda\bar{\Lambda}}^{(SLJ)}(s)$, see (4.47). The unitarity condition (4.58) results in the following K -matrix for the amplitude $\Lambda\Lambda \rightarrow \Lambda\Lambda$:

$$A_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S,LL,J)}(s) = \frac{K_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S,LL,J)}(s)}{1 - i\frac{1}{2}\rho_{\Lambda\Lambda}^{(SLJ)}(s)K_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S,LL,J)}(s)} \quad (4.60)$$

Note that the denominator in (4.60) contains the identity factor $1/2$.

(ii) Partial wave amplitudes for $S = 1$ and $J = L \pm 1$.

The formulae for $\Lambda\Lambda \rightarrow \Lambda\Lambda$ are similar to those written for $\Lambda\bar{\Lambda} \rightarrow \Lambda\bar{\Lambda}$, the only difference is the appearance of the identity factor $1/2$ in front of the phase spaces. We have four partial wave amplitudes which form the 2×2 matrix:

$$\widehat{A}_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,L=J\pm 1,J)}(s) = \begin{vmatrix} A_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J-1 \rightarrow J-1,J)}(s), & A_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J-1 \rightarrow J+1,J)}(s) \\ A_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J+1 \rightarrow J-1,J)}(s), & A_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J+1 \rightarrow J+1,J)}(s) \end{vmatrix}. \quad (4.61)$$

They can be represented in the K -matrix form as follows:

$$\begin{aligned} \widehat{A}_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,L=J\pm 1,J)}(s) &= \widehat{K}_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,L=J\pm 1,J)}(s) \\ &\times \left[I - \frac{i}{2}\widehat{\rho}_{\Lambda\Lambda}^{(S=1,L=J\pm 1,J)}(s)\widehat{K}_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,L=J\pm 1,J)}(s) \right]^{-1}, \end{aligned} \quad (4.62)$$

with the definitions

$$\begin{aligned} \widehat{K}_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,L=J\pm 1,J)}(s) &= \begin{vmatrix} K_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J-1 \rightarrow J-1,J)}(s), & K_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J-1 \rightarrow J+1,J)}(s) \\ K_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J+1 \rightarrow J-1,J)}(s), & K_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J+1 \rightarrow J+1,J)}(s) \end{vmatrix}, \\ \widehat{\rho}_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,L=J\pm 1,J)}(s) &= \begin{vmatrix} \rho_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J-1 \rightarrow J-1,J)}(s), & \rho_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J-1 \rightarrow J+1,J)}(s) \\ \rho_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J+1 \rightarrow J-1,J)}(s), & \rho_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J+1 \rightarrow J+1,J)}(s) \end{vmatrix}, \end{aligned} \quad (4.63)$$

and

$$\begin{aligned} \rho_{\Lambda\Lambda}^{(S,L \rightarrow L',J)}(s) &= \frac{1}{2J+1} \int d\Phi_2(p_1'', p_2'') \text{Sp} \left(\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k'') (-\hat{p}_2'' + m_\Lambda) \right. \\ &\quad \left. \times \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k'') (\hat{p}_1'' + m_\Lambda) \right). \end{aligned} \quad (4.64)$$

Let us emphasise again that we introduced the phase spaces for $\Lambda\Lambda$ and $\Lambda\bar{\Lambda}$ which coincide one with another: $\rho_{\Lambda\Lambda}^{(S,L \rightarrow L',J)}(s) = \rho_{\Lambda\bar{\Lambda}}^{(S,L \rightarrow L',J)}(s)$.

The matrices $\widehat{\rho}_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,L=J\pm 1,J)}(s)$ and $\widehat{K}_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,L=J\pm 1,J)}(s)$ are symmetrical: $\rho_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J-1 \rightarrow J+1,J)}(s) = \rho_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J+1 \rightarrow J-1,J)}(s)$ and $K_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J-1 \rightarrow J+1,J)}(s) = K_{\Lambda\Lambda \rightarrow \Lambda\Lambda}^{(S=1,J+1 \rightarrow J-1,J)}(s)$.

4.1.4.3 The K -matrix representation for nucleon-antinucleon scattering amplitude

The K -matrix representation for the nucleon-antinucleon scattering amplitude is written exactly in the same way as in the $\Lambda\bar{\Lambda}$ case. The only novelty compared to the $\Lambda\bar{\Lambda}$ case is that the $N\bar{N}$ scattering is determined by two isotopic amplitudes with $I = 0, 1$:

$$\begin{aligned} p\bar{n} \rightarrow p\bar{n} \ (I = 1) &: \frac{1}{2}M_1(s, t, u) + \frac{1}{2}M_0(s, t, u), \\ p\bar{p} \rightarrow n\bar{n} \ (I = 0, 1) &: \frac{1}{2}M_1(s, t, u) - \frac{1}{2}M_0(s, t, u). \end{aligned} \quad (4.65)$$

Being expanded over the s -channel operators $\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \otimes \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k')$, these amplitudes are represented in terms of the partial wave amplitudes $A_0^{(S, L' L, J)}(s)$ and $A_1^{(S, L' L, J)}(s)$. The unitarity condition for these amplitudes results in the K -matrix representation.

As before, one should distinguish between the cases $J = L$ and $J = L \pm 1$.

(i) Partial wave amplitudes $N\bar{N} \rightarrow N\bar{N}$ for $J = L$.

For the amplitude $A_I^{(S, LL, J)}(s)$ with $I = 0, 1$, in the case of $J = L$ the s -channel unitarity condition is

$$\begin{aligned} \text{Im } A_I^{(S, LL, J)}(s) &= \rho_{N\bar{N}}^{(S, LL, J)}(s) A_I^{(S, LL, J)*}(s) A_I^{(S, LL, J)}(s), \\ \rho_{N\bar{N}}^{(S, LL', J)}(s) &= \frac{1}{2J+1} \int d\Phi_2(p_1, p_2) \text{Sp} \left(\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) (-\hat{p}_2 + m_N) \right. \\ &\quad \left. \times \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k) (\hat{p}_1 + m_N) \right). \end{aligned} \quad (4.66)$$

The unitarity condition (4.66) gives us the following K -matrix representation:

$$A_I^{(S, LL, J)}(s) = \frac{K_I^{(S, LL, J)}(s)}{1 - i \rho_{N\bar{N}}^{(S, LL, J)}(s) K_I^{(S, LL, J)}(s)}. \quad (4.67)$$

(ii) Partial wave amplitudes for $S = 1$ and $J = L \pm 1$.

Four partial wave amplitudes form the 2×2 matrix:

$$\hat{A}_I^{(S=1, L=L \pm 1, J)}(s) = \begin{vmatrix} A_I^{(S=1, J-1 \rightarrow J-1, J)}(s), & A_I^{(S=1, J-1 \rightarrow J+1, J)}(s) \\ A_I^{(S=1, J+1 \rightarrow J-1, J)}(s), & A_I^{(S=1, J+1 \rightarrow J+1, J)}(s) \end{vmatrix}. \quad (4.68)$$

The K -matrix representation has the form

$$\begin{aligned} \hat{A}_I^{(S=1, L=L \pm 1, J)}(s) &= \hat{K}_I^{(S=1, L=L \pm 1, J)}(s) \times \\ &\times \left[I - i \hat{\rho}_{N\bar{N}}^{(S=1, L=L \pm 1, J)}(s) \hat{K}_I^{(S=1, L=L \pm 1, J)}(s) \right]^{-1}, \end{aligned} \quad (4.69)$$

with the following definitions:

$$\begin{aligned}\widehat{K}_I^{(S=1, L=J\pm 1, J)}(s) &= \begin{vmatrix} K_I^{(S=1, J-1 \rightarrow J-1, J)}(s), K_I^{(S=1, J-1 \rightarrow J+1, J)}(s) \\ K_I^{(S=1, J+1 \rightarrow J-1, J)}(s), K_I^{(S=1, J+1 \rightarrow J+1, J)}(s) \end{vmatrix}, \\ \widehat{\rho}_{N\bar{N}}^{(S=1, L=J\pm 1, J)}(s) &= \begin{vmatrix} \rho_{N\bar{N}}^{(S=1, J-1 \rightarrow J-1, J)}(s), \rho_{N\bar{N}}^{(S=1, J-1 \rightarrow J+1, J)}(s) \\ \rho_{N\bar{N}}^{(S=1, J+1 \rightarrow J-1, J)}(s), \rho_{N\bar{N}}^{(S=1, J+1 \rightarrow J+1, J)}(s) \end{vmatrix}. \quad (4.70)\end{aligned}$$

The function $\rho_{N\bar{N}}^{(S, L \rightarrow L', J)}(s)$ is determined by (4.56), with the obvious substitution $m_\Lambda \rightarrow m_N$.

The matrices $\widehat{\rho}_I^{(S=1, L=J\pm 1, J)}(s)$ and $\widehat{K}_I^{(S=1, L=J\pm 1, J)}(s)$ are symmetrical: $\rho_{N\bar{N}}^{(S=1, J-1 \rightarrow J+1, J)}(s) = \rho_{N\bar{N}}^{(S=1, J+1 \rightarrow J-1, J)}(s)$ and $K_I^{(S=1, J-1 \rightarrow J+1, J)}(s) = K_I^{(S=1, J+1 \rightarrow J-1, J)}(s)$.

4.1.4.4 The K -matrix representation for the nucleon–nucleon scattering amplitude

The systems pp and nn are in a pure $I = 1$ state, while pn is a superposition of two states with total isospins $I = 0$ and $I = 1$. The amplitudes are

$$\begin{aligned}pp \rightarrow pp, \quad nn \rightarrow nn \quad (I = 1) &: \quad M_1(s, t, u), \\ pn \rightarrow pn \quad (I = 0, 1) &: \quad \frac{1}{2}M_1(s, t, u) + \frac{1}{2}M_0(s, t, u). \quad (4.71)\end{aligned}$$

The expansion with respect to the s -channel operators $\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \otimes \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k')$ provides us with the representation of these amplitudes in terms of partial wave amplitudes $A_0^{(S, L' L, J)}(s)$ and $A_1^{(S, L' L, J)}(s)$.

In this expansion one should take into account the selection rules:

$$\begin{aligned}I = 0, S = 1 &: \quad (L = 0; J = 1), \quad (L = 2; J = 1, 2, 3), \quad \dots \\ I = 0, S = 0 &: \quad (L = 1; J = 1), \quad (L = 3; J = 3), \quad \dots \\ I = 1, S = 1 &: \quad (L = 1; J = 0, 1, 2), \quad (L = 3; J = 2, 3, 4), \quad \dots \\ I = 1, S = 0 &: \quad (L = 0; J = 0), \quad (L = 2; J = 2), \quad \dots \quad (4.72)\end{aligned}$$

As before, there is a one-channel amplitude for $J = L$ and a two-channel one for $J = L \pm 1$.

(i) Partial wave amplitudes $NN \rightarrow NN$ for $J = L$.

For the amplitude $A_I^{(S, LL, J)}(s)$ with $I = 0, 1$, in the case of $J = L$ the s -channel unitarity condition can be written as:

$$\begin{aligned}\text{Im } A_I^{(S, LL, J)}(s) &= \frac{1}{2} \rho_{NN}^{(S, LL, J)}(s) A_I^{(S, LL, J)*}(s) A_I^{(S, LL, J)}(s), \\ \rho_{NN}^{(S, LL', J)}(s) &= \frac{1}{2J+1} \int d\Phi_2(p_1, p_2) \text{Sp} \left(\hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) (-\hat{p}_2 + m_N) \right. \\ &\quad \left. \times \hat{Q}_{\mu_1 \dots \mu_J}^{SL'J}(k) (\hat{p}_1 + m_N) \right). \quad (4.73)\end{aligned}$$

The unitarity condition (4.73) gives the following K -matrix representation:

$$A_I^{(S,LL,J)}(s) = \frac{K_I^{(S,LL,J)}(s)}{1 - \frac{i}{2} \rho_{NN}^{(S,LL,J)}(s) K_I^{(S,LL,J)}(s)}. \quad (4.74)$$

(ii) Partial wave amplitudes for $S = 1$ and $J = L \pm 1$.

In this case four partial wave amplitudes form the 2×2 matrix:

$$\hat{A}_I^{(S=1, L=J \pm 1, J)}(s) = \begin{vmatrix} A_I^{(S=1, J-1 \rightarrow J-1, J)}(s), & A_I^{(S=1, J-1 \rightarrow J+1, J)}(s) \\ A_I^{(S=1, J+1 \rightarrow J-1, J)}(s), & A_I^{(S=1, J+1 \rightarrow J+1, J)}(s) \end{vmatrix}. \quad (4.75)$$

The K -matrix representation reads

$$\hat{A}_I^{(S=1, L=J \pm 1, J)}(s) = \hat{K}_I^{(S=1, L=J \pm 1, J)}(s) \times \left[I - \frac{i}{2} \hat{\rho}_{NN}^{(S=1, L=J \pm 1, J)}(s) \hat{K}_I^{(S=1, L=J \pm 1, J)}(s) \right]^{-1}, \quad (4.76)$$

with the following definitions:

$$\hat{K}_I^{(S=1, L=J \pm 1, J)}(s) = \begin{vmatrix} K_I^{(S=1, J-1 \rightarrow J-1, J)}(s), & K_I^{(S=1, J-1 \rightarrow J+1, J)}(s) \\ K_I^{(S=1, J+1 \rightarrow J-1, J)}(s), & K_I^{(S=1, J+1 \rightarrow J+1, J)}(s) \end{vmatrix},$$

$$\hat{\rho}_{NN}^{(S=1, L=J \pm 1, J)}(s) = \begin{vmatrix} \rho_{NN}^{(S=1, J-1 \rightarrow J-1, J)}(s), & \rho_{NN}^{(S=1, J-1 \rightarrow J+1, J)}(s) \\ \rho_{NN}^{(S=1, J+1 \rightarrow J-1, J)}(s), & \rho_{NN}^{(S=1, J+1 \rightarrow J+1, J)}(s) \end{vmatrix}. \quad (4.77)$$

The matrices $\rho_{NN}^{(S=1, J-1 \rightarrow J+1, J)}(s)$ and $\rho_{NN}^{(S=1, J+1 \rightarrow J-1, J)}(s)$ (see definition (4.47)) are symmetrical as well as the K -matrix elements: $K_I^{(S=1, J-1 \rightarrow J+1, J)}(s) = K_I^{(S=1, J+1 \rightarrow J-1, J)}(s)$.

Let us note that the definitions of the phase spaces for NN and $N\bar{N}$ systems coincide: $\rho_{NN}^{(S, L \rightarrow L', J)}(s) = \rho_{N\bar{N}}^{(S, L \rightarrow L', J)}(s)$. In the unitarity condition (and in the K -matrix representation) the identity of particles in the NN systems is taken into account by the factor $1/2$.

4.1.5 Nucleon-nucleon scattering amplitude in the dispersion relation technique with separable vertices

The angular momentum operator expansion allows us to consider the fermion-fermion scattering amplitudes in the framework of the dispersion relation (or spectral integral) technique with the comparatively simple and straightforward method of separable vertices. We shall consider here the NN scattering amplitude at low and intermediate energies (below the production of Δ -resonance) in the framework of the separable vertex technique. As a first step, we investigate the case $S = 0$, $L = 0$, after which a generalisation to arbitrary S and L is performed.

4.1.5.1 The $S = 0$, $L = 0$ partial wave amplitudes

We consider here the $^1S_0$ amplitude

$$M_{I=1}(^1S_0) = (\bar{\psi}(p'_1) i\gamma_5 \psi^c(-p'_2)) (\bar{\psi}^c(-p_2) i\gamma_5 \psi(p_1)) \\ \times A_{I=1}^{(S=0, L=L'=0, J=0)}(s), \quad (4.78)$$

which obeys the unitarity condition given by (4.73).

Omitting indices and redefining

$$A_{I=1}^{(S=0, L=L'=0, J=0)}(s) \equiv A(s), \quad \frac{1}{2} \rho_{NN}^{(S=0, L=L'=0, J=0)}(s) \equiv \rho(s), \quad (4.79)$$

we have:

$$\text{Im } A(s) = \rho(s) A^*(s) A(s), \quad \rho(s) = \frac{s}{16\pi} \sqrt{\frac{s - 4m^2}{s}}. \quad (4.80)$$

We work with separable interactions. This was discussed for spinless particles in Sections 3.3.5 and 3.3.6. The interaction block is presented here similarly, as a product of separable vertices, but two more steps are needed:

- (i) we have to develop a calculation method for fermions,
- (ii) we should generalise the method by introducing a set of vertex functions required for the description of experimental data.

Correspondingly, we write for the interaction block:

$$(\bar{\psi}(p'_1) i\gamma_5 \psi^c(-p'_2)) \sum_j G_j(\tilde{s}') G^j(s) (\bar{\psi}^c(-p_2) i\gamma_5 \psi(p_1)) . \quad (4.81)$$

Here, as in Sections 3.3.5 and 3.3.6, we allow the left G_j and right G^j vertex functions to be different (this does not violate the T-invariance of scattering amplitudes). In a graphical form, the partial amplitude is written as the following set of loop diagrams:

$$A(s) = \sum_i \text{diagram}_i + \sum_{i,j} \text{diagram}_{i,j} + \dots$$

$\begin{array}{ccccc} \text{diagram}_i & \text{diagram}_j & \text{diagram}_{i,j} & \text{diagram}_k & \text{diagram}_l \\ G & G & G & B & G \end{array}$

In Section 3.3.5 we introduced a partial amplitude depending on two variables $A(\tilde{s}', s)$, while the physical amplitude is defined as $A(s) = A(s, s)$. The solution of the equation for $A(\tilde{s}', s)$ suggests the use of not a full amplitude $A(\tilde{s}', s)$ but that with the removed vertex of outgoing particles. We denote these amplitudes as $a^j(s)$:

$$A(\tilde{s}', s) = \sum_j G_j(\tilde{s}') a^j(s) . \quad (4.82)$$

The amplitudes $a^j(s)$ are given by the set of diagrams:

$$a_i = \begin{array}{c} \text{diagram 1} \\ G \end{array} + \sum_i \begin{array}{c} \text{diagram 2} \\ G \quad B \end{array} + \dots$$

The amplitude a^j satisfies the following equation:

$$\begin{aligned} a^j(s) &= \sum_{j'} a^{j'}(s) B_{j'}^j(s) + G^j(s) , \\ B_{j'}^j(s) &= \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{G_{j'}(s') \rho(s') G^j(s')}{s' - s} . \end{aligned} \quad (4.83)$$

The equation (4.83) can be rewritten in the matrix form:

$$\hat{a}(s) = \hat{B}(s) \hat{a}(s) + \hat{g}(s), \quad (4.84)$$

where

$$\hat{a}(s) = \begin{pmatrix} a_1(s) \\ a_2(s) \\ \vdots \end{pmatrix}, \quad \hat{g}(s) = \begin{pmatrix} G_1(s) \\ G_2(s) \\ \vdots \end{pmatrix}, \quad \hat{B}(s) = \begin{pmatrix} B_1^1(s) & B_1^2(s) & \vdots \\ B_2^1(s) & B_2^2(s) & \vdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \quad (4.85)$$

Thus, we have the following expression for the partial amplitude:

$$A(s) = \hat{g}^T(s) \frac{1}{I - \hat{B}(s)} \hat{g}(s), \quad (4.86)$$

where $\hat{g}^T(s) = [G^1(s), G^2(s), \dots]$. The amplitude $A(s)$ is connected with the partial S -matrix by the relation

$$S(s) = I + 2\rho(s)A(s), \quad (4.87)$$

satisfying the unitarity condition:

$$S(s)S^+(s) = I. \quad (4.88)$$

The partial S -matrix can be represented via the scattering phase δ determined from the elastic scattering as follows:

$$S(s) = \exp [2i\delta(^1S_0)]. \quad (4.89)$$

4.1.5.2 Generalisation for $S = 0$ and arbitrary $L = J$

The partial wave amplitude for $S = 0$ and arbitrary $L = J$ can be written quite similarly to $S = 0$ and $J = 0$. Instead of (4.81), we have the following interaction block:

$$\begin{aligned} & \left(\bar{\psi}(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k') \psi^c(-p'_2) \right) \\ & \times \sum_j G_j^{SLJ}(\tilde{s}') G_{SLJ}^j(s) \left(\bar{\psi}^c(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \psi(p_1) \right). \end{aligned} \quad (4.90)$$

As before, the left G_j^{SLJ} and right G_{SLJ}^j vertex functions can be different. For the sake of brevity, we introduce $A_{IJ}(s)$ and $\rho^{(J)}(s)$ for $S = 0, L = J$:

$$A_{IJ}(s) = A_I^{(0,JJ,J)}(s), \quad \rho^{(J)}(s) = \frac{1}{2} \rho_{NN}^{(0,JJ,J)}(s), \quad (4.91)$$

with $\rho_{NN}^{(S,LL',J)}(s)$ being determined by (4.73). So, omitting indices $S = 0$ and $L = J$ in vertex $G_j^{SLJ}(s) \rightarrow G_j^{IJ}(s)$, we can represent $A_{IJ}(\tilde{s}, s)$ as follows:

$$A_{IJ}(\tilde{s}, s) = \sum_j G_j^{IJ}(\tilde{s}) a_{IJ}^j(s). \quad (4.92)$$

The amplitudes a_{IJ}^j satisfy the following equations:

$$\begin{aligned} a_{IJ}^j(s) &= \sum_{j'} a_{IJ}^{j'}(s) B_{j'}^j(IJ; s) + G_{IJ}^j(s), \\ B_{j'}^j(IJ; s) &= \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{G_{j'}^{IJ}(s') \rho^{(J)}(s') G_{IJ}^j(s')}{s' - s}. \end{aligned} \quad (4.93)$$

The equation (4.93) can be rewritten in the matrix form:

$$\hat{a}_{IJ}(s) = \hat{B}_{IJ}(s) \hat{a}_{IJ}(s) + \hat{g}_{IJ}(s), \quad (4.94)$$

with

$$\begin{aligned} \hat{a}^{IJ}(s) &= \begin{vmatrix} a_1^{IJ}(s) \\ a_2^{IJ}(s) \\ \cdot \\ \cdot \\ \cdot \end{vmatrix}, \quad \hat{g}_{IJ}(s) = \begin{vmatrix} G_1^{IJ}(s) \\ G_2^{IJ}(s) \\ \cdot \\ \cdot \\ \cdot \end{vmatrix}, \\ \hat{B}(IJ; s) &= \begin{vmatrix} B_1^1(IJ; s) & B_1^2(IJ; s) & \cdot \\ B_2^1(IJ; s) & B_2^2(IJ; s) & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}. \end{aligned} \quad (4.95)$$

Thus we have the following expression for the partial wave amplitude:

$$A_{IJ}(s) = \hat{g}_{IJ}^T(s) \frac{1}{I - \hat{B}(IJ; s)} \hat{g}_{IJ}(s), \quad (4.96)$$

with $\hat{g}_{IJ}^T(s) = |G_{IJ}^1(s), G_{IJ}^2(s), \dots|$. The amplitude $A_{IJ}(s)$ is related to the partial S -matrix as follows:

$$S_{IJ}(s) = I + 2\rho^{(J)}(s)A_{IJ}(s). \quad (4.97)$$

In the energy region of elastic scattering, the partial S -matrix can be represented through the scattering phase δ as follows:

$$S_{IJ}(s) = \exp[2i\delta(IJ)]. \quad (4.98)$$

4.1.5.3 Two-channel amplitude with $S = 1$ and $J = L \pm 1$

In this case we have four partial wave amplitudes which can be written in the form of a 2×2 matrix shown in (4.75). Let us use here more compact notations:

$$\widehat{A}_I^{(L=J\pm 1, J)}(s) = \begin{vmatrix} A_I^{(J-1 \rightarrow J-1)}(s), A_I^{(J-1 \rightarrow J+1)}(s) \\ A_I^{(J+1 \rightarrow J-1)}(s), A_I^{(J+1 \rightarrow J+1)}(s) \end{vmatrix}, \quad (4.99)$$

where

$$\begin{aligned} A_I^{(J-1 \rightarrow J-1)}(s) &= \left(\psi(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{1J-1J}(k') \psi^c(-p'_2) \right) \\ &\quad \times A_{11}(s) \left(\psi(p_1) \hat{Q}_{\mu_1 \dots \mu_J}^{1J-1J}(k) \psi^c(-p_2) \right), \\ A_I^{(J-1 \rightarrow J+1)}(s) &= \left(\psi(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{1J-1J}(k') \psi^c(-p'_2) \right) \\ &\quad \times A_{12}(s) \left(\psi(p_1) \hat{Q}_{\mu_1 \dots \mu_J}^{1J+1J}(k) \psi^c(-p_2) \right), \\ A_I^{(J+1 \rightarrow J-1)}(s) &= \left(\psi(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{1J+1J}(k') \psi^c(-p'_2) \right) \\ &\quad \times A_{21}(s) \left(\psi(p_1) \hat{Q}_{\mu_1 \dots \mu_J}^{1J-1J}(k) \psi^c(-p_2) \right), \\ A_I^{(J+1 \rightarrow J+1)}(s) &= \left(\psi(p'_1) \hat{Q}_{\mu_1 \dots \mu_J}^{1J+1J}(k') \psi^c(-p'_2) \right) \\ &\quad \times A_{22}(s) \left(\psi(p_1) \hat{Q}_{\mu_1 \dots \mu_J}^{1J+1J}(k) \psi^c(-p_2) \right). \end{aligned} \quad (4.100)$$

We introduce the 2×2 matrix amplitude which depends on two variables, $\tilde{s}$ and s :

$$\hat{A}(\tilde{s}, s) = \begin{vmatrix} A_{11}(\tilde{s}, s), A_{12}(\tilde{s}, s) \\ A_{21}(\tilde{s}, s), A_{22}(\tilde{s}, s) \end{vmatrix}, \quad (4.101)$$

while physical amplitudes in (4.99), (4.100) are determined as follows:

$$A_{j\ell}(s) = A_{j\ell}(s, s) . \quad (4.102)$$

As the first step, consider the interaction as a one-vertex block (similarly to the case discussed in Section 3.3.5). Then the vertex matrix reads:

$$\hat{V} = \begin{vmatrix} G^1(\tilde{s}) \cdot G_1(s), & G^{t_1}(\tilde{s}) \cdot G_{t_2}(s) \\ G_{t_2}(\tilde{s}) \cdot G^{t_1}(s), & G^2(\tilde{s}) \cdot G_2(s) \end{vmatrix} .$$

Thus we have only six vertices:

$$G(j; s) \rightarrow G^1(s), G_1(s), G^{t_1}(s), G_{t_2}(s), G^2(s), G_2(s). \quad (4.104)$$

Let us remind that different left and right vertices do not violate the time inversion of the amplitudes.

Below, in (4.109) and (4.110) we generalise the treatment by introducing a set of vertices for each transition.

As before, we introduce the amplitudes $a_{j\ell}(s)$ which depend only on s :

$$\begin{aligned} A_{11}(\tilde{s}, s) &= G^1(\tilde{s})a_{11}(s) + G^{t_1}(\tilde{s})a_{t_2 1}(s), \\ A_{12}(\tilde{s}, s) &= G^{t_1}(\tilde{s})a_{t_2 2}(s) + G^1(\tilde{s})a_{12}(s), \\ A_{21}(\tilde{s}, s) &= G_{t_2}(\tilde{s})a_{t_1 1}(s) + G^2(\tilde{s})a_{21}(s), \\ A_{22}(\tilde{s}, s) &= G^2(\tilde{s})a_{22}(s) + G_{t_2}(\tilde{s})a_{t_1 2}(s). \end{aligned} \quad (4.105)$$

This definition is illustrated by Fig. 4.2.

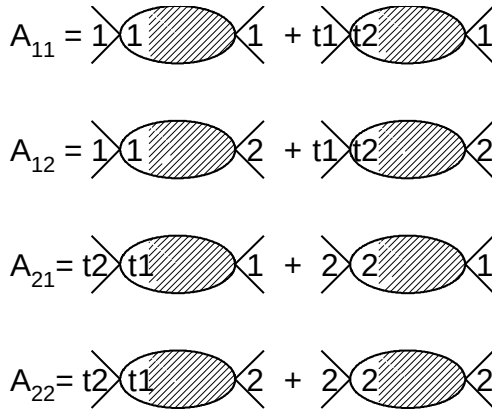


Fig. 4.2 Determination of the amplitude $a_{j\ell}(s)$.

The amplitudes $a_{j\ell}(s)$ obey the equations:

$$\begin{aligned}
 a_{11}(s) &= G_1(s) + B_{111}(s)a_{11}(s) + B_{11t_1}(s)a_{t_21}(s), \\
 a_{t_21}(s) &= B_{t_222}(s)a_{21}(s) + B_{t_22t_2}(s)a_{t_11}(s), \\
 a_{t_22}(s) &= G_{t_2}(s) + B_{t_222}(s)a_{22}(s) + B_{t_22t_2}(s)a_{t_12}(s), \\
 a_{12}(s) &= B_{111}(s)a_{12}(s) + B_{11t_1}(s)a_{t_22}(s), \\
 a_{t_11}(s) &= G^{t_1}(s) + B_{t_111}(s)a_{11}(s) + B_{t_11t_1}(s)a_{t_21}(s), \\
 a_{21}(s) &= B_{222}(s)a_{21}(s) + B_{22t_2}(s)a_{t_11}(s), \\
 a_{22}(s) &= G_2(s) + B_{222}(s)a_{22}(s) + B_{22t_2}(s)a_{t_12}(s), \\
 a_{t_12}(s) &= B_{t_111}(s)a_{12}(s) + B_{t_11t_1}(s)a_{t_22}(s).
 \end{aligned} \tag{4.106}$$

Equations (4.106) are shown in Fig. 4.3 in a graphical form, where the loop diagrams $B_{ja\ell}(s)$ are defined as follows:

$$B_{ja\ell}(s) = \int_{4m_a^2}^{\infty} \frac{ds'}{\pi} \frac{G(j; s') \rho_a(s') G(\ell; s')}{s' - s}. \tag{4.107}$$

Six vertices $G(j; s')$ are introduced in (4.104) and the phase factor is defined as $\rho_a(s) = \frac{1}{2} \rho_{NN}^{(S, LL', J)}(s)$, with $\rho_{NN}^{(S, LL', J)}(s)$ given in (4.73).

If we use the above-written formulae for fitting the scattering data, the vertices $G(j; s)$ are free parameters. These vertices have left-hand side singularities and can be written as integrals along the corresponding left-hand cuts:

$$G(j; s) = \int_{-\infty}^{s_L} \frac{ds'}{\pi} \frac{\text{disc } G(j; s')}{s - s'}, \tag{4.108}$$

where $s_L = 4m^2 - \mu^2$ is the position of the nearest singularity related to the pion t -channel exchange.

A generalisation of the above formulae for the case when each interaction is described by several vertices is performed in the standard way. Instead of the one-vertex matrix (4.103), we use the manifold one:

$$\hat{V} = \begin{vmatrix} \sum_{n_1} G_{n_1}^1(\tilde{s}) \cdot G_1^{m_1}(s), & \sum_{n_t} G_{n_t}^{t_1}(\tilde{s}) \cdot G_{t_2}^{n_t}(s) \\ \sum_{n_t} G_{t_2}^{n_t}(\tilde{s}) \cdot G_{n_t}^{t_1}(s), & \sum_{n_2} G_{n_2}^2(\tilde{s}) \cdot G_2^{n_2}(s) \end{vmatrix}.$$

To fit the data in the physical region ($s > 4m^2$), it is convenient to represent the integral for $G_{n_j}^j(s)$ as a sum of pole terms, with the poles located at $s < s_L$:

$$G_{n_j}^j(s) = \int_{-\infty}^{s_L} \frac{ds'}{\pi} \frac{\text{disc } G_{n_j}^j(s')}{s - s'} \rightarrow f^j(s) \sum_{n_j} \frac{\gamma_{n_j}^j}{s - s_{n_j}^j}. \tag{4.110}$$

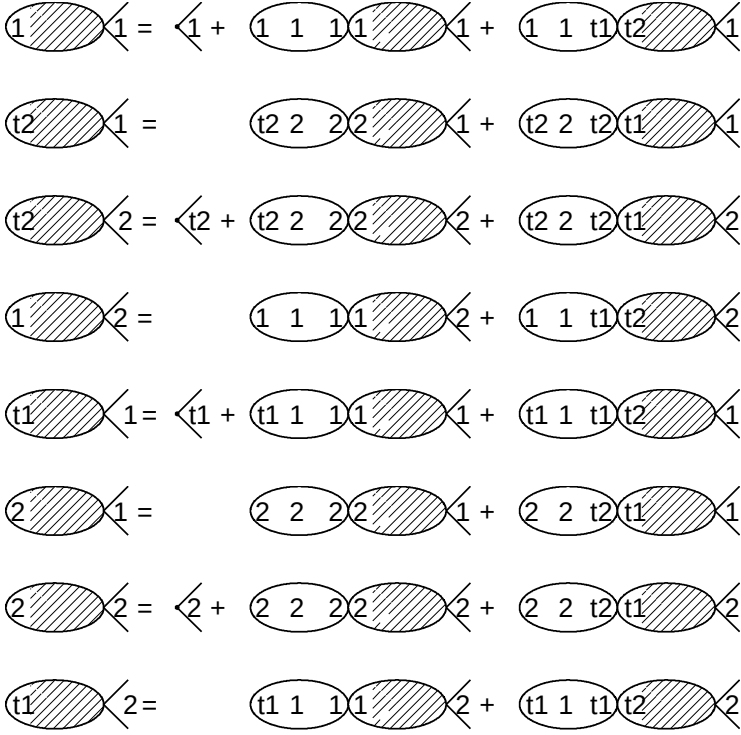


Fig. 4.3 Graphical representation of the spectral integral equations for $a_{j\ell}(s)$.

It is also convenient to choose $f^j(s)$ in such a form that the integrand of the loop diagram (4.107) at $s < 4m^2$ has only pole singularities:

$$G_{n_j}^j(s) \rho_a(s) G_{n_\ell}^\ell(s) \sim \sum_{n_j} \frac{\gamma_{n_j}^j}{s - s_{n_j}^j} \cdot \sum_{n_\ell} \frac{\gamma_{n_\ell}^\ell}{s - s_{n_\ell}^\ell} \quad \text{at } s < 4m^2. \quad (4.111)$$

In Appendix 4.C we demonstrate examples of fitting to the NN scattering data within such a method, following the results of [2, 3, 4].

4.1.6 Comments on the spectral integral equation

So far we discussed the fitting to experimental data in one- or two-channels of the two-particle reactions with the purpose to find resonances and their residues. In most cases a correct determination of pole terms in the amplitude is a difficult task because of the presence of threshold singularities, hence the determination of poles needs, in fact, the reconstruction of the

whole analytical amplitude. For the reconstruction of the analytical amplitude in the physical region, there is no essential difference whether one uses the K -matrix technique or spectral integral equation. That is because in both cases, working with the amplitude in the physical region, we account for threshold singularities, and in both cases left singularities are on the edge of the studied region. In both techniques, we perform approximate evaluation of left singularities, though in different ways.

Had we been able to carry out the bootstrap procedure, that is, had we known about the interaction in the crossing channels, we could definitely take into account the contribution from left singularities. The left singularities being known, in the framework of the dispersion relation method the resonances and their vertices can be singled out with a high accuracy. But until now this is not so, and both methods look equivalent from the point of view of the search for resonances.

It might seem that in this conclusion we do not try to use the characteristics of the dispersion relation method, which in the mid 60's gave us serious hope for the realisation of the bootstrap procedure. We mean by this that in most cases the properties of particles which form the forces due to the t -channel exchange, are known: we refer to the pion, ρ , ω , σ mesons, and so on. But actually the knowledge of particle masses and certain vertices is not sufficient for the adequate restoration of left singularities. Namely, referring to the t -channel particle exchanges, we also need to know the form factors of these particles in a broad region of the momentum transfers.

However, as a matter of fact, the situation is even more complicated — not only there exist unknown form factors in the exchange interactions but at moderately large $|s|$ in the left-hand side singularities a noticeable contribution comes from resonances in the crossing channels, with masses of the order of 1.0 – 2.0 GeV. Among them there are high spin resonances, for example, with $J = 2$, which are located precisely in this mass region: after accounting for these resonances, we obtain the N -function which increases when moving along the left cut to large negative s . This growth is power-like: it is just the well-known contribution of non-reggeised particles with a high spin. Therefore, a reggeisation is needed which can “kill” the rapidly increasing terms and transform them into the decreasing ones.

But this way we face a number of new problems.

There are two possible scenarios of calculations.

In the first version, one should work from the beginning with contributions of the reggeised exchanges on the left cut. But this means that the low-energy region should be described by reggeised amplitudes — till now we

do not have such a self-consistent approach. In the second, and more pragmatic, version, the left cut is divided into two pieces: the first one is for the exchanges of non-reggeised well-known particles (though with poorly known form factors), while for the second piece of the cut the reggeon exchanges are used. However, not much is known about reggeon exchanges. Indeed, we do not know the behaviour of form factors in unphysical region, neither we know definitely the daughter trajectories or their couplings. Summarising, this is a hopeless situation, with a countless number of free parameters, when one may get by accident the desirable result and think that it is the true one.

There is a method allowing us to reduce the contribution of the large negative s , namely, to perform a sufficiently large number of subtractions. Still, a subtraction is the imposed constraint for the amplitude in the physical region. After performing one or two subtractions, one can achieve a freedom to include into calculated amplitude the wanted features. Moreover, at small negative masses the problems do not disappear with the treatment of the exchange form factors and hypotheses imposed on their behaviour.

It is clear that, to solve the problems of the determination of the left-cut contribution, one needs a trustworthy bootstrap method. But at present there is no such procedure.

From this point of view, the K -matrix procedure or the spectral integration method with separable vertices look though roughly straightforward but the most trustworthy ones. Let us emphasise again that this procedure aims at the as precise as possible reconstruction of the analytical amplitudes in the physical region. As the next step, it suggests a continuation of these amplitudes to the left cut. An analytical continuation can be carried out in the K -matrix approach under the ansatz of the behaviour of “smooth terms” in the K -matrix elements or, in the spectral integral method, by the choice of vertices.

Suppose that general constraints (analyticity and unitarity) in the right-hand side of the s -plane are correctly taken into account, the accuracy of these methods (we mean the K -matrix or the spectral integral approaches) is restricted by the accuracy of the experiment only.

Therefore it is necessary to reconstruct the left-hand cut using the information on the amplitudes in crossing channels. However, the spectral integration method is not unique: there is another similar approach based on the Feynman integral representation of the amplitude. We mean the Bethe–Salpeter equation. Still, we believe that just the spectral integration

method but not the Bethe-Salpeter equation is adequate to the problems to be solved, namely, the systematisation of hadrons as composite systems of constituents and the description of physical processes involving these composite systems. The reason is as follows.

As was already said in Chapter 3, the Bethe-Salpeter equation in its general form (the sum of ladder diagrams) does not fix the number of constituents of the composite system: actually, we have a many-component composite state, where, apart from the main two or three constituents, there are also states with additional particles which participate in the formation of interaction forces (the cutting through t -channel particles, see discussion in Section 3.3.4). At the same time, this is not a unique deficiency of the Bethe-Salpeter equation. Problems appear when we start to consider particles with spin.

In this case the four-momentum squares, k_i^2 , appear in the numerators related to intermediate states. In the spectral integration $k_i^2 = m_i^2$ (remind that the integration is carried out over the total energy which is not conserved), but in the framework of the Feynman technique $k_i^2 \neq m_i^2$, hence one may write

$$k_i^2 = (k_i^2 - m_i^2) + m_i^2. \quad (4.112)$$

The first term in the right-hand side cancels the denominator of the constituent propagator creating the so-called "animal-like" diagram. For example, the self-energy diagram turns into:

$$\text{a} \quad \longrightarrow \quad \text{b} + \text{c} \quad (4.113)$$

The term (4.113a) is the Feynman diagram, while (4.113b) corresponds to the first term in the right-hand side of (4.112) (one propagator, say, $(k_2^2 - m^2)$ is cancelled) and (4.113c) is related to the diagram with $k_2^2 = m^2$. Hence, a composite system is not only a two-constituent state but it contains the so-called "penguin" diagram (4.113b) as well. So, while the problem of a many-particle state may be solved by using instantaneous forces, the problem of "animal-like" contributions still exists.

The existence of "animal-like" diagrams is rather essential for the description of electromagnetic processes with composite systems. Indeed, let the electromagnetic field interact with the first constituent. Then, similarly to (4.113), there exists the following diagrammatical representation of the

Feynman diagram (4.114a):

$$\text{a} \rightarrow \text{b} + \text{c} \quad (4.114)$$

The equation (4.114) means that, when using wave functions of the Bethe–Salpeter equation, it is necessary to include the diagram of (4.114b) to obtain a gauge invariant solution. Obviously, the term (4.114b) is beyond the additive quark model.

As was noted above, in the spectral integration technique there are neither “animal-like” diagrams nor diagrams of the (4.114b)-type: we deal with the diagrams (4.114c) of the additive quark model only, which are gauge invariant. The wave functions obtained in the spectral integration method, under the ansatz of separable interaction (see Chapter 3.3.6), immediately provide us with the correct normalisation of the charge form factor: $F(0) = 1$.

The method of the deuteron form factor calculations, based on the reconstruction of the deuteron wave function obtained from the $np \rightarrow np$ scattering, was used in [3]. In Appendix 4.C we show the results of such calculations of the form factors. We also demonstrate the results for the reaction of deuteron disintegration $\gamma d \rightarrow pn$ carried out in [4].

4.2 Inelastic Processes in $N\bar{N}$ Collisions: Production of Mesons

In this section we consider the production of mesons in $N\bar{N}$ collisions. To be precise, we give formulae for two important cases: the production of two and three pseudoscalar mesons in the $p\bar{p}$ annihilation. This type of reactions was studied in a set of papers, for example, in [5, 6, 7, 8]. Here we give a general expression for the angular momentum expansion in the $p\bar{p} \rightarrow P_1 P_2$ reaction (the pseudoscalar meson is denoted as P_a); a combined analysis of the reactions $p\bar{p} \rightarrow \pi\pi, \eta\eta, \eta\eta'$ is presented in Appendix 4.E. Another type of reaction we consider is the production of a resonance in the final state with its subsequent decay. As an example, we investigate the production of tensor resonance $p\bar{p} \rightarrow f_2 P_1 \rightarrow P_1 P_2 P_3$.

4.2.1 Reaction $p\bar{p} \rightarrow$ two pseudoscalar mesons

In the $p\bar{p}$ annihilation we have two isospin states, $I = 0$ and $I = 1$. Correspondingly, for the amplitude $p(p_1)\bar{p}(p_2) \rightarrow P_1(k_1)P_2(k_2)$ we write:

$$C_{1/2\,1/2\,,\,1/2\,-1/2}^{10} M_{p\bar{p} \rightarrow P_1 P_2}^{(1)}(s, t, u) \\ + C_{1/2\,1/2\,,\,1/2\,-1/2}^{00} M_{p\bar{p} \rightarrow P_1 P_2}^{(0)}(s, t, u). \quad (4.115)$$

We use the following notation for total momenta of incoming and outgoing particles: $P = p_1 + p_2 = k_1 + k_2$. The relative momenta are:

$$p_{\perp\mu} = \frac{1}{2}(p_{1\mu} - p_{2\mu}) = g_{\mu\nu}^{\perp P} p_{1\nu} = -g_{\mu\nu}^{\perp P} p_{2\nu}, \\ k_{\perp\mu} = g_{\mu\nu}^{\perp P} k_{1\nu} = -g_{\mu\nu}^{\perp P} k_{2\nu}. \quad (4.116)$$

Using this notation, the s -channel operator expansion gives us for $M_{p\bar{p} \rightarrow P_1 P_2}^{(I)}(s, t, u)$:

$$M_{p\bar{p} \rightarrow P_1 P_2}^{(I)}(s, t, u) = \sum_{\substack{S, L, J \\ \mu_1 \dots \mu_J}} X_{\mu_1 \dots \mu_J}^{(J)}(k_{\perp}) A_I^{(S, L, J)}(s) \times \\ \times \left(\bar{\psi}(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(p_{\perp}) \psi(p_1) \right). \quad (4.117)$$

In (4.117) the summation is carried out over all states, namely:

$$S = 0, J = L; \quad S = 1, J = L - 1, L, L + 1. \quad (4.118)$$

Searching for resonances with large masses, one can take into account in a rough approximation only the pole terms in the amplitude $A_I^{(S, L, J)}(s)$. This is equivalent to the representation of the amplitude in the form

$$A_I^{(S, L, J)}(s) = \sum_n \frac{g_{p\bar{p} \rightarrow R(n)}^{(I; S, L, J)} g_{R(n) \rightarrow P_1 P_2}^{(I; S, L, J)}}{s - m_{R(n)}^2 - i m_{R(n)} \Gamma_{R(n)}(s)} + f^{(I; S, L, J)}(s). \quad (4.119)$$

For narrow resonances one can put $\Gamma_{R(n)}(s) \rightarrow \Gamma_{R(n)}(m_{R(n)}^2)$. But in other cases, say, in the presence of the threshold singularity in the vicinity of the resonance, the s -dependence in $\Gamma_{R(n)}(s)$ should be kept.

The results of combined analysis of the reactions $p\bar{p} \rightarrow \pi\pi, \eta\eta, \eta\eta'$, using expansions (4.117) and (4.119), are presented in Appendix 4.E. As was shown in Chapter 2 (Section 2.6.1.5), just the study of the reactions $p\bar{p} \rightarrow \pi\pi, \eta\eta, \eta\eta'$ proved that the broad state $f_2(2000)$ is the lowest tensor glueball.

4.2.2 Reaction $p\bar{p} \rightarrow f_2 P_3 \rightarrow P_1 P_2 P_3$

In papers [6, 7, 8, 9] the reaction $p\bar{p} \rightarrow f_2 \pi \rightarrow \eta \eta \pi$ was studied. It is just these reactions where the $f_2(2000)$ was observed. We would like to bring these reactions to the attention of the reader because they provide a good example for the application of the angular momentum operator technique to the three-particle reactions.

In these reactions the initial and final states are shown which determine the possible transitions in the reaction $p\bar{p} \rightarrow f_2 \pi$.

$p\bar{p}$ -system			$f_2 P$ -system	
L_{in}	S_{in}	J^{PC}	L	J^{PC}
0	0	0^{-+}	0	2^{-+}
	1	1^{--}		
1	0	1^{+-}	1	1^{++}
	1	0^{++}		2^{++}
		1^{++}		3^{++}
		2^{++}		
2	0	2^{-+}	2	0^{-+}
	1	1^{--}		1^{-+}
		2^{--}		2^{-+}
		3^{--}		3^{-+}
				4^{-+}
3	0	3^{+-}	3	1^{++}
	1	2^{++}		2^{++}
		3^{++}		3^{++}
		4^{++}		4^{++}
				5^{++}

(4.120)

Recall that only transitions with the same J^{PC} are possible. The $p\bar{p}$ system can have $I = 0, 1$, thus defining the isotopic spin of P in the final state $f_2 P$.

Let us introduce the momenta of initial and final states:

$$\begin{aligned}
 p &= p_1 + p_2, \quad p^\perp = \frac{1}{2}(p_1 - p_2), \quad (pp^\perp) = 0, \\
 p &= k = k_{f1} + k_{f2} + k_3, \quad k_f = k_{f1} + k_{f2}, \quad (k^\perp k_f) = 0, \quad (4.121)
 \end{aligned}$$

where p_1 and p_2 are proton and antiproton momenta, respectively; k_3 is the pion momentum and k_{f1} , k_{f2} refer to η -mesons; $k^\perp$ is the relative momentum of η -mesons. It should be underlined that all relative momenta are as follows:

$$\begin{aligned}
 p_\mu^\perp &= g_{\mu\nu}^\perp p_{1\nu} = -g_{\mu\nu}^\perp p_{2\nu}, \\
 k_\mu^\perp &= g_{\mu\nu}^\perp k_{f\nu} = -g_{\mu\nu}^\perp k_{3\nu}, \\
 k_\mu^{\perp f} &= g_{\mu\nu}^{\perp k_f} k_{f1\nu} = -g_{\mu\nu}^{\perp k_f} k_{f2\nu}.
 \end{aligned} \quad (4.122)$$

Within this notation, the s -channel operator expansion gives:

$$M_{p\bar{p} \rightarrow f_2 P_3 \rightarrow P_1 P_2 P_3}^{(I)}(s, t, u) = \sum_{\substack{S_{in}, L_{in}, L, J \\ \mu_1 \dots \mu_J}} Q_{\mu_1 \dots \mu_J}^{(f_2 P; L, J)} A_I^{(S_{in}, L_{in}, L, J)}(s) \\ \times \left(\bar{\psi}(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{S_{in} L_{in} J}(p_\perp) \psi(p_1) \right). \quad (4.123)$$

In (4.121) the summation is carried out over all states with the allowed transitions.

The operators $Q_{\mu_1 \dots \mu_J}^{(f_2 P; L, J)}$ for the $f_2 P$ system read:

$$\begin{aligned} J = L - 2 : Q_{\mu_1 \dots \mu_J}^{(f_2 P; L, J=L-2)} &= k_{\alpha'_1}^{\perp f} k_{\alpha'_2}^{\perp f} O_{\alpha_1 \alpha_2}^{\alpha'_1 \alpha'_2}(\perp k_f) \\ &\times X_{\alpha_1 \alpha_2 \mu'_3 \dots \mu'_L}^{(L)}(k^\perp) O_{\mu_1 \dots \mu_J}^{\mu'_3 \dots \mu'_L}(\perp k), \\ J = L - 1 : Q_{\mu_1 \dots \mu_J}^{(f_2 P; L, J=L-1)} &= i k_{\alpha'_1}^{\perp f} k_{\alpha'_2}^{\perp f} O_{\alpha_1 \alpha_2}^{\alpha'_1 \alpha'_2}(\perp k_f) \varepsilon_{p \alpha_1 \mu'_1 \mu'_2} \\ &\times X_{\mu'_1 \alpha_2 \mu'_3 \dots \mu'_L}^{(L)}(k^\perp) O_{\mu_1 \dots \mu_J}^{\mu'_2 \mu'_3 \dots \mu'_L}(\perp k), \\ J = L : Q_{\mu_1 \dots \mu_J}^{(f_2 P; L, J=L)} &= k_{\alpha'_1}^{\perp f} k_{\alpha'_2}^{\perp f} O_{\alpha_1 \alpha_2}^{\alpha'_1 \alpha'_2}(\perp k_f) \\ &\times X_{\alpha_2 \mu'_2 \dots \mu'_L}^{(L)}(k^\perp) O_{\mu_1 \mu_2 \dots \mu_J}^{\alpha'_1 \mu'_2 \dots \mu'_L}(\perp k), \\ J = L + 1 : Q_{\mu_1 \dots \mu_J}^{(f_2 P; L, J=L+1)} &= i k_{\alpha'_1}^{\perp f} k_{\alpha'_2}^{\perp f} O_{\alpha_1 \alpha_2}^{\alpha'_1 \alpha'_2}(\perp k_f) \varepsilon_{p \alpha_1 \mu'_1 \mu'_1''} \\ &\times X_{\mu'_1 \mu'_2 \dots \mu'_L}^{(L)}(k^\perp) O_{\mu_1 \dots \mu_J}^{\alpha_2 \mu'_1 \mu'_2 \dots \mu'_J}(\perp k), \\ J = L + 2 : Q_{\mu_1 \dots \mu_J}^{(f_2 P; L, J=L+2)} &= k_{\alpha'_1}^{\perp f} k_{\alpha'_2}^{\perp f} O_{\alpha_1 \alpha_2}^{\alpha'_1 \alpha'_2}(\perp k_f) \\ &\times X_{\mu'_1 \mu'_2 \dots \mu'_L}^{(L)}(k^\perp) O_{\mu_1 \mu_2 \dots \mu_J}^{\alpha_1 \alpha_2 \mu'_1 \dots \mu'_L}(\perp k). \end{aligned} \quad (4.124)$$

In the region of large masses one can work in the approximation which takes into account only the pole terms in the amplitude $A_I^{(S_{in}, L_{in}, L, J)}(s)$. So we represent the amplitude in the form:

$$A_I^{(S_{in}, L_{in}, L, J)}(s) = \sum_n \frac{g_{p\bar{p} \rightarrow R(n)}^{(I; S_{in}, L_{in}, J)}(s) g_{R(n) \rightarrow f_2 P_3}^{(I; S_{in}, L_{in}, J)}(s)}{s - m_{R(n)}^2 - i m_{R(n)} \Gamma_{R(n)}(s)} \\ \times \frac{g_{f_2 \rightarrow P_1 P_2}^{(f_2)}(s_f)}{s_f - m_{f_2}^2 - i m_{f_2} \Gamma_{f_2}(s_f)} + f^{(S_{in}, L_{in}, L, J)}(s, s_f). \quad (4.125)$$

Here $s_f = k_f^2$. For a narrow resonance one can put $\Gamma_{R(n)}(s) \rightarrow \Gamma_{R(n)}(m_{R(n)}^2)$. But in other cases, say, in the presence of the threshold singularity in the vicinity of the resonance, the s -dependence in $\Gamma_{R(n)}(s)$ should be kept.

The detailed description of results obtained in the analysis of the reaction $p\bar{p} \rightarrow f_2 \pi \rightarrow \eta \eta \pi$ is given in [6].

4.3 Inelastic Processes in NN Collisions: the Production of Δ -Resonances

In the NN collisions the inelastic processes are switched on with the increase of energy that is mainly due to the reactions $NN \rightarrow \Delta N$ and $NN \rightarrow \Delta\Delta$. Here we present the corresponding formalism for writing the amplitudes and discuss certain characteristic features related to the non-stability of the Δ .

4.3.1 *Spin- $\frac{3}{2}$ wave functions*

To describe Δ and $\bar{\Delta}$, we use the wave functions $\psi_\mu(p)$ and $\bar{\psi}_\mu(p) = \psi_\mu^\dagger(p)\gamma_0$ which satisfy the following constraints:

$$\begin{aligned} (\hat{p} - m)\psi_\mu(p) &= 0, & \bar{\psi}_\mu(p)(\hat{p} - m) &= 0, \\ p_\mu\psi_\mu(p) &= 0, & \gamma_\mu\psi_\mu(p) &= 0. \end{aligned} \quad (4.126)$$

Here $\psi_\mu(p)$ is a four-component spinor and μ is a four-vector index. Sometimes, to underline spin variables, we use the notation $\psi_\mu(p; a)$ for the spin- $\frac{3}{2}$ wave functions.

4.3.1.1 *Wave function for Δ*

The equation (4.126) gives four wave functions for the Δ :

$$\begin{aligned} a = 1, 2: \quad \psi_\mu(p; a) &= \sqrt{p_0 + m} \left(\frac{\varphi_{\mu\perp}(a)}{\frac{(\sigma\mathbf{p})}{p_0 + m}} \varphi_{\mu\perp}(a) \right), \\ \bar{\psi}_\mu(p; a) &= \sqrt{p_0 + m} \left(\varphi_{\mu\perp}^+(a), -\varphi_{\mu\perp}^+(a) \frac{(\sigma\mathbf{p})}{p_0 + m} \right), \end{aligned} \quad (4.127)$$

where the spinors $\varphi_{\mu\perp}(a)$ are determined to be perpendicular to p_μ :

$$\varphi_{\mu\perp}(a) = g_{\mu\mu'}^\perp p \varphi_{\mu'}(a), \quad g_{\mu\mu'}^\perp = g_{\mu\mu'} - p_\mu p_{\mu'} / p^2. \quad (4.128)$$

The requirement $\gamma_\mu\psi_\mu(p; a)$ results in the following constraints for $\varphi_{\mu\perp}$:

$$\begin{aligned} m\varphi_{0\perp}(a) &= (\mathbf{p}\boldsymbol{\varphi}_\perp(a)), \\ m(p_0 + m)(\sigma\boldsymbol{\varphi}_\perp(a)) + (\mathbf{p}\boldsymbol{\sigma})(\mathbf{p}\boldsymbol{\varphi}_\perp(a)) &= 0. \end{aligned} \quad (4.129)$$

In the limit $p \rightarrow 0$ (the Δ at rest), we have:

$$\begin{aligned} m\varphi_{0\perp}(a) &= 0, \\ (\sigma\boldsymbol{\varphi}_\perp(a)) &= 0, \end{aligned} \quad (4.130)$$

thus keeping for Δ four independent spin components $\mu_z = 3/2, 1/2, -1/2, -3/2$ related to the spin $S = 3/2$ and removing the components with $S = 1/2$.

The completeness conditions for the spin- $\frac{3}{2}$ wave functions can be written as follows:

$$\begin{aligned} \sum_{a=1,2} \psi_\mu(p; a) \bar{\psi}_\nu(p; a) &= (\hat{p} + m) \left(-g_{\mu\nu}^\perp + \frac{1}{3} \gamma_\mu^\perp \gamma_\nu^\perp \right) \\ &= (\hat{p} + m) \frac{2}{3} \left(-g_{\mu\nu}^\perp + \frac{1}{2} \sigma_{\mu\nu}^\perp \right), \quad (4.131) \end{aligned}$$

where $g_{\mu\nu}^\perp \equiv g_{\mu\nu}^{\perp p}$ and $\gamma_\mu^\perp = g_{\mu\mu'}^{\perp p} \gamma_{\mu'}$. The factor $(\hat{p} + m)$ commutes with $(g_{\mu\nu}^\perp - \frac{1}{3} \gamma_\mu^\perp \gamma_\nu^\perp)$ in (4.131) because $\hat{p} \gamma_\mu^\perp \gamma_\nu^\perp = \gamma_\mu^\perp \gamma_\nu^\perp \hat{p}$. The matrix $\sigma_{\mu\nu}^\perp$ is determined in a standard way, $\sigma_{\mu\nu}^\perp = \frac{1}{2} (\gamma_\mu^\perp \gamma_\nu^\perp - \gamma_\nu^\perp \gamma_\mu^\perp)$. The completeness condition in the form (4.131) was used in [4, 10].

4.3.1.2 Wave function for $\bar{\Delta}$

The anti-delta, $\bar{\Delta}$, is determined by the following four wave functions:

$$\begin{aligned} b = 3, 4: \quad \psi_\mu(-p; b) &= i\sqrt{p_0 + m} \begin{pmatrix} \frac{(\boldsymbol{\sigma} \mathbf{p})}{p_0 + m} \chi_{\mu\perp}(b) \\ \chi_{\mu\perp}(b) \end{pmatrix}, \\ \bar{\psi}_\mu(-p; b) &= -i\sqrt{p_0 + m} \begin{pmatrix} \chi_{\mu\perp}^+(b) \frac{(\boldsymbol{\sigma} \mathbf{p})}{p_0 + m} \\ -\chi_{\mu\perp}^+(b) \end{pmatrix}, \quad (4.132) \end{aligned}$$

where in the system at rest ($p \rightarrow 0$) the spinors $\chi_{\mu\perp}(b)$ obey the relations:

$$m\chi_{0\perp}(b) = 0, \quad (\boldsymbol{\sigma} \boldsymbol{\chi}_\perp(b)) = 0, \quad (4.133)$$

that take away the spin- $\frac{1}{2}$ components.

The completeness conditions for spin- $\frac{3}{2}$ wave functions with $b = 3, 4$ are

$$\begin{aligned} \sum_{b=3,4} \psi_\mu(-p; b) \bar{\psi}_\nu(-p) &= -(\hat{p} + m) \left(-g_{\mu\nu}^\perp + \frac{1}{3} \gamma_\mu^\perp \gamma_\nu^\perp \right) \\ &= -(\hat{p} + m) \frac{2}{3} \left(-g_{\mu\nu}^\perp + \frac{1}{2} \sigma_{\mu\nu}^\perp \right). \quad (4.134) \end{aligned}$$

The equation (4.132) can be rewritten in the form of (4.127) using the charge conjugation matrix C which was introduced for spin- $\frac{1}{2}$ particles, $C = \gamma_2 \gamma_0$. It satisfies the relations $C^{-1} \gamma_\mu C = -\gamma_\mu^T$ and $C^{-1} = C = C^+$. We write:

$$b = 3, 4: \quad \psi_\mu^c(p; b) = C \bar{\psi}_\mu^T(-p; b). \quad (4.135)$$

The wave functions $\psi_\mu^c(p; b)$ with $b = 3, 4$ obey the equation:

$$(\hat{p} - m)\psi_\mu^c(p; b) = 0. \quad (4.136)$$

In the explicit form the charge conjugated wave functions read:

$$\begin{aligned} b = 3, 4 : \psi_\mu^c(p; b) &= -\sqrt{p_0 + m} \begin{pmatrix} \sigma_2 \chi_{\mu\perp}^*(b) \\ \frac{(\boldsymbol{\sigma} \mathbf{p})}{p_0 + m} \sigma_2 \chi_{\mu\perp}^*(b) \end{pmatrix} \\ &= \sqrt{p_0 + m} \begin{pmatrix} \varphi_{\mu\perp}^c(b) \\ \frac{(\boldsymbol{\sigma} \mathbf{p})}{p_0 + m} \varphi_{\mu\perp}^c(b) \end{pmatrix}, \end{aligned} \quad (4.137)$$

with $\varphi_{\mu\perp}^c(b) = -\sigma_2 \chi_{\mu\perp}^*(b)$.

4.3.2 Processes $NN \rightarrow N\Delta \rightarrow NN\pi$. Triangle singularity

When the production of Δ is considered in the three-body process $NN \rightarrow NN\pi$, one faces, due to the decay $\Delta \rightarrow N\pi$, a number of problems induced by the three-body interactions. Our consideration is focused mainly on the discussion of singularities of the partial wave amplitudes related to the final state, namely, the poles owing to the production of Δ and triangle diagram singularities, which appear as a result of the rescattering processes with Δ in the intermediate state. The existence of the triangle-diagram singularities, which may be located near the physical region of the three-particle production reaction, was observed in [11, 12, 13]. In the reaction $NN \rightarrow N\Delta \rightarrow NN\pi$, these singularities are of the types:

- (i) $\ln(s_{\pi N} - s_{\pi N}^{\text{tr}})$ for the πN -rescattering, and
- (ii) $\ln(s_{NN} - s_{NN}^{\text{tr}})$ for the NN -rescattering where $s_{\pi N}$ and s_{NN} are invariant energies squared of the produced particles (see Figs. 4.4b,c).

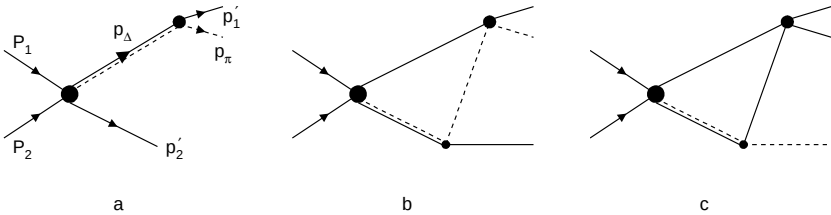


Fig. 4.4 The pole diagram with the production of Δ -isobar (a) and triangle diagrams with rescatterings of the products of the Δ decay (b,c).

4.3.2.1 Pole singularity in the $NN \rightarrow N\Delta \rightarrow NN\pi$ reaction

The amplitude for the production and the decay of the Δ -isobar, $NN \rightarrow \Delta N \rightarrow NN\pi$ (see Fig. 4.4a), reads:

$$\begin{aligned} A_{NN \rightarrow \Delta N \rightarrow (N\pi)N} &= C[NN \rightarrow \Delta N \rightarrow (N\pi)N] \times \\ &\times \sum_{\substack{S, S', L, L', J \\ \mu_1 \dots \mu_J}} \left(\bar{\psi}_c(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \psi(p_1) \right) G_{NN \rightarrow N\Delta}^{(S, S', L, L', J)}(s) \\ &\times \left(\bar{\psi}(p'_1) g_{\Delta} p_{\pi\mu}^{\perp p_{\Delta}} \frac{\Delta_{\mu\nu}(p_{\Delta})}{m_{\Delta}^2 - p_{\Delta}^2 - im_{\Delta}\Gamma_{\Delta}} \hat{Q}_{\nu\mu_1 \dots \mu_J}^{S' L' J}(N\Delta; p_2'^{\perp}) \psi_c(-p'_2) \right). \end{aligned} \quad (4.138)$$

The factor $C[NN \rightarrow \Delta N \rightarrow (N\pi)N]$ is related to the isotopic Clebsch-Gordan coefficients for the corresponding reaction. As previously, here $\bar{\psi}_c(-p_2)$ and $\psi_c(-p'_2)$ refer to the incoming and outgoing nucleons with the momenta p_2 and p'_2 ; the momentum of the produced pion is denoted as p_{π} . The numerator of the spin-3/2 fermion propagator, $\Delta_{\mu\nu}(p_{\Delta})$ (here $p_{\Delta} = p'_1 + p_{\pi}$), is determined by the completeness condition (4.131) (see [14] and Chapter 3):

$$\Delta_{\mu\nu}(p_{\Delta}) = (\hat{p}_{\Delta} + m_{\Delta})(-g_{\mu\nu}^{\perp p_{\Delta}} + \frac{1}{3}\gamma_{\mu}^{\perp p_{\Delta}}\gamma_{\nu}^{\perp p_{\Delta}}), \quad (4.139)$$

Relative momenta in (4.138) are equal to:

$$p_{2\mu}^{\perp} = g_{\mu\mu'}^{\perp p} p_{2\mu'}', \quad p_{\pi\mu}^{\perp p_{\Delta}} = g_{\mu\mu'}^{\perp p_{\Delta}} p_{\pi\mu'}', \quad (4.140)$$

with $p = p_1 + p_2 = p'_1 + p'_2 + p_{\pi}$.

The moment operator $\hat{Q}_{\nu\mu_1 \dots \mu_J}^{S' L' J}(N\Delta; p_2'^{\perp})$ depends on the spin of outgoing particles: $S' = \mathbf{3}/2 + \mathbf{1}/2 = 1, 2$ and the angular momentum of the $N\Delta$ -system L' .

4.3.2.2 The decay width of Δ

The decay width of Δ is determined by the loop diagram of Fig. 4.5. Namely, we expand the propagator of Δ in a series over Γ_{Δ} :

$$\begin{aligned} \frac{\Delta_{\mu\nu}(p_{\Delta})}{m_{\Delta}^2 - p_{\Delta}^2 - im_{\Delta}\Gamma_{\Delta}} &\simeq \frac{\Delta_{\mu\nu}(p_{\Delta})}{m_{\Delta}^2 - p_{\Delta}^2} \left[1 + \frac{im_{\Delta}\Gamma_{\Delta}}{m_{\Delta}^2 - p_{\Delta}^2} \right] \\ &= \frac{\Delta_{\mu\nu}(p_{\Delta})}{m_{\Delta}^2 - p_{\Delta}^2} + \frac{\Delta_{\mu\nu'}(p_{\Delta}) - g_{\nu'\nu''}^{\perp} i m_{\Delta} \Gamma_{\Delta}}{m_{\Delta}^2 - p_{\Delta}^2} \frac{\Delta_{\nu''\nu}(p_{\Delta})}{m_{\Delta}^2 - p_{\Delta}^2} \\ &= \frac{\Delta_{\mu\nu}(p_{\Delta})}{m_{\Delta}^2 - p_{\Delta}^2} + \frac{\Delta_{\mu\nu'}(p_{\Delta})}{m_{\Delta}^2 - p_{\Delta}^2} i \text{Im} B_{\nu'\nu''}(p_{\Delta}^2) \frac{\Delta_{\nu''\nu}(p_{\Delta})}{m_{\Delta}^2 - p_{\Delta}^2}. \end{aligned} \quad (4.141)$$

Here $\text{Im} B_{\nu'\nu''}(p_{\Delta}^2)$ is the imaginary part of the loop diagram Fig. 4.5.

Fig. 4.5 Loop diagram $\Delta^{++} \rightarrow p\pi^+ \rightarrow \Delta^{++}$.

In the calculation of (4.141), we have used that $(-g_{\mu\nu}^\perp + \frac{1}{3}\gamma_\mu^\perp\gamma_\nu^\perp)(-g_{\nu'\nu}^\perp + \frac{1}{3}\gamma_{\nu'}^\perp\gamma_\nu^\perp) = -(-g_{\mu\nu}^\perp + \frac{1}{3}\gamma_\mu^\perp\gamma_\nu^\perp)$ and $(m_\Delta + \hat{p}_\Delta)^2 = 2m_\Delta(m_\Delta + \hat{p}_\Delta)$.

Determining the transition vertex $\Delta^{++} \rightarrow p\pi^+$ as $k_\nu^\perp g_\Delta$ (here $k^\perp \equiv k^\perp p_\Delta$), we have:

$$\begin{aligned} \text{Im } B_{\nu\nu'}(p_\Delta^2) &= \text{Im} \int \frac{d^4 k}{i(2\pi)^4} \\ &\times k_\nu^\perp g_\Delta \frac{\hat{p}_\Delta - \hat{k} + m_N}{(k^2 - m_\pi^2 + i0)((p_\Delta - k)^2 - m_\Delta^2 + i0)} k_{\nu'}^\perp g_\Delta \\ &= \int \frac{d^4 k}{(2\pi)^4} 4\pi^2 m_\Delta \Theta(k_0) \delta(k^2 - m_\pi^2) \\ &\times \Theta(p_{\Delta 0} - k_0) \delta((p_\Delta - k)^2 - m_N^2) g_\Delta^2 k_\nu^\perp k_{\nu'}^\perp. \end{aligned} \quad (4.142)$$

Replacing in (4.142)

$$k_\nu^\perp k_{\nu'}^\perp \rightarrow \frac{1}{3} g_{\nu\nu'}^\perp k^{\perp 2}, \quad (4.143)$$

we obtain:

$$\text{Im } B_{\nu\nu'}(p_\Delta^2) = g_\Delta^2 \frac{m_N |\mathbf{k}_\perp|^3}{12\pi \sqrt{p_\Delta^2}} (-g_{\nu\nu'}^\perp), \quad (4.144)$$

with

$$|\mathbf{k}_\perp|^2 = \frac{1}{4p_\Delta^2} [p_\Delta^2 - (m_N + m_\pi)^2] [p_\Delta^2 - (m_N - m_\pi)^2], \quad (4.145)$$

that results in

$$m_\Delta \Gamma_\Delta = g_\Delta^2 \frac{m_N^2 |\mathbf{k}_\perp|^3}{6\pi \sqrt{p_\Delta^2}}. \quad (4.146)$$

The formula (4.146) is not a unique expression used for the width of the Δ . One can either generalise it by introducing the energy dependence in the decay coupling $g_\Delta \rightarrow g_\Delta(p_\Delta^2)$ or simplify it by putting $p_\Delta^2 \rightarrow m_\Delta^2$.

4.3.2.3 Triangle-diagram amplitude with pion-nucleon rescattering: the logarithmic singularity

In the amplitudes with the production of three-particle states, the unitarity condition is fulfilled because of the final state rescatterings. Some rescatterings result in strong singularities where the amplitude tends to infinity.

The triangle diagram with Δ in the intermediate state gives us an example of this type of process: it has a logarithmic singularity which under the condition $\sqrt{s} \sim m_N + m_\Delta$ can appear near the physical region. Because of that, we consider the amplitude $pp \rightarrow N\Delta$ with the S -wave $N\Delta$ system. The isospin of the $N\Delta$ is equal to $I = 1$, and we have the following quantum numbers for the final state with $L' = 0$:

$$I = 1, \quad J^P = 1^+, 2^+. \quad (4.147)$$

The initial pp system ($I = 1$) contains the states

$$\begin{aligned} S = 0 : \quad & L = 0, 2, 4, \dots \quad J^P = 0^+, 2^+, 4^+, \dots \\ S = 1 : \quad & L = 1, 3, \dots \quad J^P = 0^-, 1^-, 2^-, 4^-, \dots \end{aligned} \quad (4.148)$$

Therefore, we consider the transition $pp(S = 0, L = 2, J^P = 2^+) \rightarrow N\Delta(S' = 2, L' = 0, J^P = 2^+)$.

The corresponding pole amplitude (4.138) reads:

$$\begin{aligned} A_{NN \rightarrow NN\pi}^{pole} &= C^{(p)}[NN \rightarrow \Delta N \rightarrow (N\pi)N]G_{pp \rightarrow N\Delta}^{(S=0, S'=2, L=2, L'=0, J=2)}(s) \\ &\times \left(\bar{\psi}(p'_1) g_\Delta p'^{\perp p_\Delta}_{\pi\mu} \frac{\Delta_{\mu\nu}(p_\Delta)}{m_\Delta^2 - p_\Delta^2 - im_\Delta \Gamma_\Delta} \gamma_{\nu'} \psi_c(-p'_2) \right) \\ &\times \left(\bar{\psi}_c(-p_2) i\gamma_5 X_{\nu\nu'}^{(2)}(k) \psi(p_1) \right). \end{aligned} \quad (4.149)$$

As in (4.138), the factor $C^{(p)}[NN \rightarrow \Delta N \rightarrow (N\pi)N]$ refers to the isotopic Clebsch-Gordan coefficients, and $p'^{\perp p_\Delta}_{\pi\mu}$ is given in (4.140).

For the sake of simplicity, we use $\gamma_{\nu'}$ in (4.149) as a spin factor for the production of ΔN , namely: $\Gamma_{\nu'}(k'_\perp) \rightarrow \gamma_{\nu'}$. Still, using the definition (4.22), one can easily rewrite (4.149) in a more rigorous form.

Taking into account the pion rescattering, $\pi N \rightarrow \Delta \rightarrow \pi N$, one has for the triangle-diagram amplitude:

$$\begin{aligned} A_{NN \rightarrow NN\pi}^{triangle} &= C^{(tr)}[NN \rightarrow \Delta N \rightarrow (N\pi)N]G_{pp \rightarrow N\Delta}^{(S=0, S'=2, L=2, L'=0, J=2)}(s) \\ &\times \left(\bar{\psi}(p'_1) \int \frac{d^4 k_\pi}{i(2\pi)^4} \right. \\ &\times \frac{1}{m_\pi^2 - k_\pi^2 - i0} g_\Delta k'^{\perp p'_\Delta}_{\pi\mu'} \frac{\Delta_{\mu'\nu}(p'_\Delta)}{m_\Delta^2 - p_\Delta'^2 - im_\Delta \Gamma_\Delta} \gamma_{\nu'} \frac{-\hat{p}_2'' + m}{m^2 - p_2''^2 - i0} g_\Delta k'^{\perp p_\Delta}_{\pi\mu''} \\ &\times \left. \frac{\Delta_{\mu''\nu''}(-p_\Delta)}{m_\Delta^2 - p_\Delta^2 - im_\Delta \Gamma_\Delta} g_\Delta p'^{\perp p_\Delta}_{\pi\mu''} \psi_c(-p'_2) \right) \left(\bar{\psi}_c(-p_2) i\gamma_5 X_{\nu\nu'}^{(2)}(k) \psi(p_1) \right). \end{aligned} \quad (4.150)$$

Here

$$k_{\pi\mu'}^{\perp p'_\Delta} = g_{\mu'\alpha}^{\perp p'_\Delta} k_{\pi\alpha} , \quad k_{\pi\mu''}^{\perp p_\Delta} = g_{\mu''\alpha}^{\perp p_\Delta} k_{\pi\alpha} , \quad p_{\pi\nu''}^{\perp p_\Delta} = g_{\nu''\alpha}^{\perp p_\Delta} p_{\pi\alpha} , \quad (4.151)$$

and

$$p'_\Delta = p'_1 + k_\pi , \quad p_\Delta = p'_2 + p_\pi = p''_2 + k_\pi , \quad P = p'_\Delta + p''_2 . \quad (4.152)$$

One can simplify (4.150) fixing the numerator in the singular point which corresponds to

$$m_\Delta^2 = p_\Delta'^2 , \quad m^2 = p_2''^2 , \quad m_\pi^2 = k_\pi^2 . \quad (4.153)$$

The equation (4.150) can be written as

$$\begin{aligned} A_{NN \rightarrow NN\pi}^{triangle} &= C^{(tr)} [NN \rightarrow \Delta N \rightarrow (N\pi)N] G_{pp \rightarrow N\Delta}^{(S=0, S'=2, L=2, L'=0, J=2)}(s) \\ &\times \left(\bar{\psi}(p'_1) g_\Delta k_{\pi\mu'}^{\perp p'_\Delta} (\text{tr}) \Delta_{\mu'\nu}(p'_\Delta(\text{tr})) \gamma_{\nu'} (-\hat{p}_2''(\text{tr}) + m) g_\Delta k_{\pi\mu''}^{\perp p_\Delta} (\text{tr}) \right. \\ &\times \frac{\Delta_{\mu''\nu''}(-p_\Delta)}{m_\Delta^2 - p_\Delta^2 - im_\Delta \Gamma_\Delta} g_\Delta p_{\pi\nu''}^{\perp p_\Delta} \psi_c(-p'_2) \Big) \\ &\times \left(\bar{\psi}_c(-p_2) i\gamma_5 X_{\nu\nu'}^{(2)}(k) \psi(p_1) \right) A_{\text{tr}}(p_\Delta^2) , \end{aligned} \quad (4.154)$$

where

$$A_{\text{tr}}(p_\Delta^2) = \int \frac{d^4 k_\pi}{i(2\pi)^4} \frac{1}{m_\pi^2 - k_\pi^2 - i0} \frac{1}{m_\Delta^2 - p_\Delta'^2 - im_\Delta \Gamma_\Delta} \frac{1}{m^2 - p_2''^2 - i0} \quad (4.155)$$

is the triangle diagram amplitude for spinless particles. In (4.154) the momenta

$$k_{1\mu'}^{\perp p'_\Delta}(\text{tr}), \quad p'_\Delta(\text{tr}), \quad p_2''(\text{tr}), \quad k_{2\mu''}^{\perp p_\Delta}(\text{tr}) \quad (4.156)$$

obey the constraints (4.153).

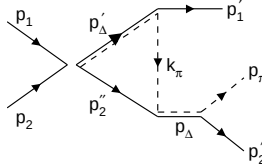


Fig. 4.6 Triangle diagram.

In Appendix 4.F the triangle diagram calculations are presented. First, we calculate the triangle-diagram integral which enters equation (4.154):

$$A_{\text{triangle}}^{\text{spinless}}(W^2, s) = \int \frac{d^4 k_\pi}{i(2\pi)^4} \frac{1}{m_\pi^2 - k_\pi^2 - i0} \times \frac{1}{m_\Delta^2 - (p - p_\Delta + k_\pi)^2 - im_\Delta \Gamma_\Delta} \frac{1}{m_N^2 - (p_\Delta - k_\pi)^2 - i0}. \quad (4.157)$$

The notations of momenta are illustrated by Fig. 4.6. Here

$$p = p_1 + p_2, \quad p^2 = W^2, \quad p_\Delta^2 = s. \quad (4.158)$$

The physical region is determined by the interval:

$$(m_N + m_\pi)^2 \leq s \leq (W - m_N)^2. \quad (4.159)$$

In Fig. 4.7 (left column), the triangle-diagram amplitude $A_{\text{triangle}}^{\text{spinless}}(W^2, s)$ given by (4.157) is shown in the physical region (4.159).

In the right column the positions of the logarithmic singularities on the second sheet of the complex- s plane are shown. The physical region of the reaction is also drawn (thick solid line): it is located on the lower edge of the cut related to the threshold singularity (thin solid line). The positions of logarithmic singularities are as follows:

$$s_{(\pm)}^{(tr)} = m_\pi^2 + m_N^2 + \frac{(W^2 - M_\Delta^2 - m_N^2)(M_\Delta^2 + m_\pi^2 - m_N^2)}{2M_\Delta^2} \pm \left[(m_\pi^2 - (M_\Delta - m_N)^2)(m_\pi^2 - (M_\Delta + m_N)^2) \times (W^2 - (M_\Delta - m_N)^2)(W^2 - (M_\Delta + m_N)^2) \right]^{1/2}, \quad (4.160)$$

where $M_\Delta^2 = m_\Delta^2 - im_\Delta \Gamma_\Delta$. The singularities $s_{(-)}^{(tr)}$ (black circles) and $s_{(+)}^{(tr)}$ (black squares) are located on the second sheet of the complex- s plane, see Fig. 4.7. When $s_{(+)}^{(tr)}$ dives onto the third sheet, its position is shown by an open square.

In the left column of Fig. 4.7, the real and imaginary parts of the amplitude (4.157) at different total energies W are shown by solid and dashed curves, respectively.

4.3.3 The $NN \rightarrow \Delta\Delta \rightarrow NN\pi\pi$ process. Box singularity.

The amplitude of the $NN \rightarrow \Delta\Delta \rightarrow NN\pi\pi$ process with the rescattering of particles in the final state contains so-called box diagrams. The box diagrams give stronger singularities, of the $(s - s_0)^{-1/2}$ -type [15, 16].

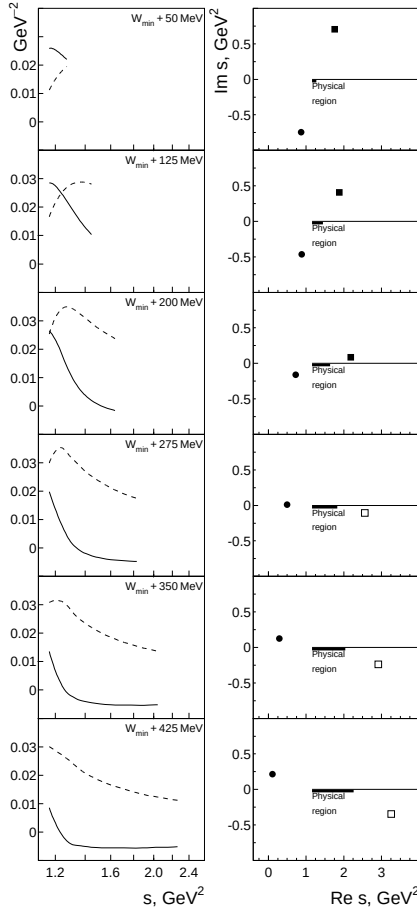


Fig. 4.7 Triangle diagram amplitude. In the left panel real and imaginary parts of the amplitude in the physical region are shown as functions of s (energy squared of the πN system) by solid and dashed curves, correspondingly. The initial energy, W , is shown on the top of each panel. In the right columns singularity positions, $s_{(-)}^{(tr)}$ (black circles) and $s_{(+)}^{(tr)}$ (black squares), see (4.160), are shown on the 2nd sheet of the complex- s plane. When $s_{(+)}^{(tr)}$ dives onto the 3rd sheet, its position is shown by the open square.

Here we present the box-diagram singular amplitudes for the reaction $NN \rightarrow \Delta\Delta \rightarrow NN\pi\pi$ taking into account the spin structure that allows us to include these singular amplitudes into the partial wave analysis.

Let us introduce the notations for the two-pole and box diagrams in the reactions $NN \rightarrow \Delta\Delta \rightarrow NN\pi\pi$.

Initial state momenta are denoted as

$$P_1 + P_2 = P, \quad P^2 = W^2, \quad \frac{1}{2}(P_1 - P_2) = q, \quad (4.161)$$

while the final state momenta are

$$\begin{aligned} (p_1 + p_3)^2 &= s_{13}, & (p_1 + p_3 + p_2)^2 &= s_4, \\ p_1 + p_3 &= k_1, & k_1^\perp &= \frac{1}{2}(p_1 - p_3)^\perp k_1 = p_1^\perp k_1 = -p_3^\perp k_1, \\ (p_2 + p_4)^2 &= s_{24}, & (p_2 + p_4 + p_1)^2 &= s_3, \\ p_2 + p_4 &= k_2, & k_2^\perp &= \frac{1}{2}(p_2 - p_4)^\perp k_2 = p_2^\perp k_2 = -p_4^\perp k_2, \\ (p_1 + p_2)^2 &= s, & p_1 + p_2 &= p. \end{aligned} \quad (4.162)$$

The symbol $\perp k_i$ stands for the component of a vector which is perpendicular to k_i . For example,

$$p_{a\mu}^\perp k_i = p_{a\mu} - k_{i\mu} \frac{(k_i p_a)}{k_i^2}. \quad (4.163)$$

Notations of the momenta are shown in Fig. 4.8.

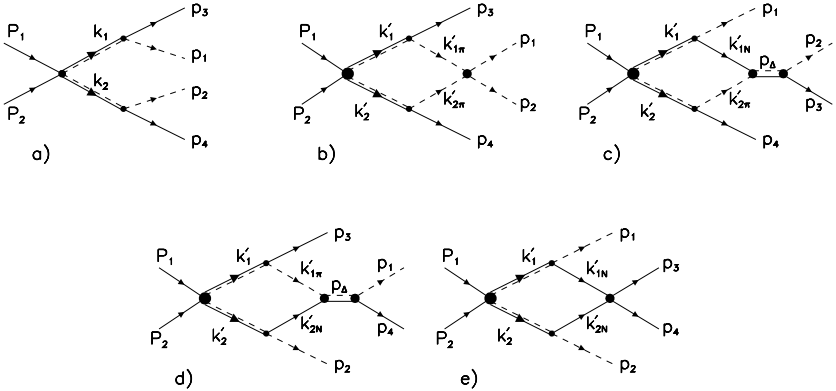


Fig. 4.8 Two-pole diagram (a) and box diagrams with pion-pion (b), pion-nucleon (c,d) and nucleon-nucleon (e) rescatterings.

Box-diagram singularities are located near the physical region at $W \sim 2m_\Delta$. Correspondingly, we consider the $\Delta\Delta$ production in the S -wave. This means that initial nucleons (to be definite, we consider the pp system) can be in S and D states only.

(i) $\Delta\Delta$ production from the initial S -wave state

The amplitude for the production and decay of two Δ -isobars, $NN \rightarrow$

$\Delta\Delta \rightarrow NN\pi\pi$ (we omit charge indices and Clebsch–Gordan coefficients) reads:

$$\begin{aligned}
 A_{NN \rightarrow \Delta\Delta \rightarrow (N\pi)(N\pi)} &= (\bar{\psi}_c(-P_2)\psi(P_1)) \cdot G_{NN \rightarrow \Delta\Delta}(W) \times \\
 &\times \left(\bar{\psi}(p_3)g_\Delta k_{1\mu}^\perp \frac{\Delta_{\mu\nu'}(k_1)}{m_\Delta^2 - s_{13} - i m_\Delta \Gamma_\Delta} \frac{\Delta_{\nu'\nu}(-k_2)}{m_\Delta^2 - s_{24} - i m_\Delta \Gamma_\Delta} \right. \\
 &\times (-)k_{2\nu}^\perp g_\Delta \psi_c(-p_4) \Big). \tag{4.164}
 \end{aligned}$$

Here $\bar{\psi}_c(-P_2)$ and $\psi_c(-p_4)$ correspond to the incoming and outgoing nucleons with momenta P_2 and p_4 : $\psi_c(p) = C\bar{\psi}^T(-p)$, with $C = \gamma_2\gamma_0$. The numerator of the spin-3/2 fermion propagator is written in the form used in Section 4.3: $\Delta_{\mu\nu}(k) = (\hat{k} + m_\Delta)(-g_{\mu\nu}^\perp + \gamma_\mu^\perp \gamma_\nu^\perp/3)$, $\gamma_\mu^\perp = g_{\mu\nu}^\perp \gamma_\nu$ and $g_{\mu\nu}^\perp = g_{\mu\nu} - k_\mu k_\nu/m_\Delta^2$. The decay vertex g_Δ determines the width of Δ , see Section 4.3.2.

(ii) Box-diagram amplitude with pion–pion rescattering

The box-diagram amplitude with pion–pion rescattering, see Fig. 4.8b, in the Feynman technique reads:

$$\begin{aligned}
 A_{NN \rightarrow \Delta\Delta \rightarrow NN + (\pi\pi \rightarrow \pi\pi)_S} &= (\bar{\psi}_c(-P_2)\psi(P_1)) \cdot G_{NN \rightarrow \Delta\Delta}^{(L=0)}(W) A_{\pi\pi \rightarrow \pi\pi}^{S-wave}(s) \\
 &\times \int \frac{d^4 k'}{i(2\pi)^4} \frac{1}{(m_\pi^2 - k_{1\pi}^2 - i0)(m_\pi^2 - k_{2\pi}^2 - i0)} \\
 &\times \frac{(\bar{\psi}(p_3)g_\Delta k_{1\mu}^\perp \Delta_{\mu\nu'}(k'_1) \Delta_{\nu'\nu}(-k'_2)(-k_{2\nu}^\perp g_\Delta \psi_c(-p_4))}{(m_\Delta^2 - s'_{13} - i m_\Delta \Gamma_\Delta)(m_\Delta^2 - s'_{24} - i m_\Delta \Gamma_\Delta)}. \tag{4.165}
 \end{aligned}$$

Here we take into account the low-energy $\pi\pi$ interaction in S wave only. The K -matrix representation of the $\pi\pi$ scattering amplitude, $A_{\pi\pi \rightarrow \pi\pi}^{S-wave}(s_{12})$, reads (see Chapter 3 for more detail):

$$A_{\pi\pi \rightarrow \pi\pi}^{S-wave}(s_{12}) = \frac{K(s_{12})}{1 - i\rho(s_{12})K(s_{12})}, \quad \rho(s_{12}) = \frac{1}{16\pi} \sqrt{\frac{s_{12} - 4m_\pi^2}{s_{12}}}. \tag{4.166}$$

Of course, there is no problem with the account for pion–pion rescattering in other waves, for example, in the P -wave either.

The approximation we use in our calculation of the box diagram (4.165) is related to the extraction of leading singular terms in the amplitude. To this aim, we fix the numerator of the integrand in the propagator poles as follows:

$$k_1'^2 \rightarrow m_\Delta^2, \quad k_2'^2 \rightarrow m_\Delta^2, \quad k_{1\pi}'^2 \rightarrow m_\pi^2, \quad k_{2\pi}'^2 \rightarrow m_\pi^2. \tag{4.167}$$

This leads to the following substitution in (4.165):

$$\begin{aligned}
 k_{1\mu}^\perp &\rightarrow k_{1\mu}^\perp(\text{box}) = -p_3^{\perp k_1(\text{box})}, \quad k_1' \rightarrow k_1(\text{box}), \\
 k_{2\nu}^\perp &\rightarrow k_{2\nu}^\perp(\text{box}) = -p_4^{\perp k_2(\text{box})}, \quad k_2' \rightarrow k_2(\text{box}). \tag{4.168}
 \end{aligned}$$

For example, in the c.m. system the momenta $k_a(\text{box})$ have the components

$$\begin{aligned} k_1(\text{box}) &= (W/2, 0, 0, \sqrt{W^2/4 - m_\Delta^2}), \\ k_2(\text{box}) &= (W/2, 0, 0, -\sqrt{W^2/4 - m_\Delta^2}), \end{aligned} \quad (4.169)$$

where we use the notation $k = (k_0, k_x, k_y, k_z)$. Under the constraints (4.167) the numerator of the integrand does not depend on integration variables, and it can be written separately for the leading singular term:

$$\begin{aligned} A_{NN \rightarrow \Delta\Delta \rightarrow NN + (\pi\pi \rightarrow \pi\pi)_S}^{(\text{leading term})} &= (\bar{\psi}_c(-P_2)\psi(P_1)) G_{NN \rightarrow \Delta\Delta}^{(L=0)}(W) A_{\pi\pi \rightarrow \pi\pi}^{S\text{-wave}}(s_{12}) \\ &\times \left(\bar{\psi}(p_3) g_\Delta p_{3\mu}^{\perp k_1(\text{box})} \Delta_{\mu\nu'}(k_1(\text{box})) \Delta_{\nu'\nu} k_2(\text{box}) p_{4\nu}^{\perp k_2(\text{box})} g_\Delta \psi_c(-p_4) \right) \\ &\times \int \frac{d^4 k'}{i(2\pi)^4} \frac{1}{(m_\pi^2 - (\frac{1}{2}p + k')^2 - i0)(m_\pi^2 - (\frac{1}{2}p - k')^2 - i0)} \\ &\times \frac{1}{(m_\Delta^2 - (\frac{1}{2}p + k' + p_3)^2 - im_\Delta\Gamma_\Delta)(m_\Delta^2 - (\frac{1}{2}p - k' + p_4)^2 - im_\Delta\Gamma_\Delta)}. \end{aligned} \quad (4.170)$$

Here

$$\frac{1}{2}p + k' = k'_{1\pi}, \quad \frac{1}{2}p - k' = k'_{2\pi}, \quad p_1 + p_2 = p. \quad (4.171)$$

One can calculate in a standard way the box-diagram integral which enters (4.170):

$$\begin{aligned} A_{\text{box}}^{\text{spinless}}(W^2, s_3, s_4, s_{12}) &= \int \frac{d^4 k'}{i(2\pi)^4} \frac{1}{(m_\pi^2 - (\frac{1}{2}p + k')^2 - i0)(m_\pi^2 - (\frac{1}{2}p - k')^2 - i0)} \\ &\times \frac{1}{(m_\Delta^2 - (\frac{1}{2}p + k' + p_3)^2 - im_\Delta\Gamma_\Delta)(m_\Delta^2 - (\frac{1}{2}p - k' + p_4)^2 - im_\Delta\Gamma_\Delta)}. \end{aligned} \quad (4.172)$$

In Fig. 4.9 we show the results of our calculation of $A_{\text{box}}^{\text{spinless}}(W^2, s_3, s_4, s_{12})$ as a function of pion-pion energy squared s_{12} at different total energies W , under the following constraint on s_3 and s_4 (remind that $s_3 = (p - p_3)^2$, $s_4 = (p - p_4)^2$, $s_{12} = (p_1 + p_2)^2$, $W^2 = p^2$ and $s_3 = s_4 = W\sqrt{s_{12}} + m_N^2$). This constraint corresponds to the following kinematics in the c.m. system:

$$\begin{aligned} p &= (W, 0, 0, 0), \\ p_1 &= (\sqrt{m_\pi^2 + p_{1z}^2}, 0, 0, p_{1z}), & p_2 &= (\sqrt{m_\pi^2 + p_{1z}^2}, 0, 0, -p_{1z}), \\ p_3 &= (\sqrt{m_N^2 + p_{3z}^2}, 0, 0, p_{3z}), & p_4 &= (\sqrt{m_N^2 + p_{3z}^2}, 0, 0, -p_{3z}). \end{aligned} \quad (4.173)$$

Let us introduce the notation

$$M_\Delta^2 = m_\Delta^2 - im_\Delta\Gamma_\Delta. \quad (4.174)$$

Then the positions of the box-diagram singularities can be represented as follows:

$$\begin{aligned}
s_{12}^{\text{box}} &= 2m_\pi^2 + \frac{1}{2W^2}(s_3 - m_N^2)(s_4 - m_N^2) \\
&+ \frac{(2W^2 M_\Delta^2 - W^2(s_3 - m_N^2))(2W^2 M_\Delta^2 - W^2(s_4 - m_N^2))}{2W^2((W^2 - 2M_\Delta^2)^2 - 4M_\Delta^4)} \\
&- \left[\left(\frac{(s_3 - m_N^2)^2}{2W^2} - 2m_\pi^2 - \frac{(2W^2 M_\Delta^2 - W^2(s_3 - m_N^2))^2}{2W^2((W^2 - 2M_\Delta^2)^2 - 4M_\Delta^4)} \right) \right. \\
&\times \left. \left(\frac{(s_4 - m_N^2)^2}{2W^2} - 2m_\pi^2 - \frac{(2W^2 M_\Delta^2 - W^2(s_4 - m_N^2))^2}{2W^2((W^2 - 2M_\Delta^2)^2 - 4M_\Delta^4)} \right) \right]^{\frac{1}{2}}. \quad (4.175)
\end{aligned}$$

At $s_3 = s_4$ equation (4.175) reads:

$$s_{12}^{\text{box}} = 4m_\pi^2 + \frac{W^2(2M_\Delta^2 - s_3 + m_N^2)^2}{(W^2 - 2M_\Delta^2)^2 - 4M_\Delta^4}. \quad (4.176)$$

(iii) Box-diagram amplitude with pion–nucleon rescattering

In the framework of the Feynman technique the amplitude of the box-diagram with pion–nucleon rescattering (see Fig. 4.8c) in the resonance state ($I = 3/2, J = 3/2$) can be written as

$$\begin{aligned}
A_{NN \rightarrow \Delta\Delta \rightarrow N\pi + (N\pi \rightarrow N\pi)_\Delta} &= (\bar{\psi}_c(-P_2)\psi(P_1)) G_{NN \rightarrow \Delta\Delta}^{(L=0)}(W) \quad (4.177) \\
&\times \left(\bar{\psi}(p_3)g_\Delta \frac{1}{2}(p_2 - p_3)_\mu^{\perp p_\Delta} \frac{\Delta_{\mu\mu'}(p_\Delta)}{m_\Delta^2 - p_\Delta^2 - im_\Delta\Gamma_\Delta} \right. \\
&\times \int \frac{d^4 k'}{i(2\pi)^4} \frac{1}{2}(k'_{2\pi} - k'_{1N})_{\mu'}^{\perp p_\Delta} g_\Delta \frac{\hat{k}'_{1N} + m_N}{m_N^2 - k_{1N}'^2 - i0} g_\Delta \\
&\times \left. \frac{\frac{1}{2}(p_1 - k'_{1N})_{\mu'}^{\perp k'_1} \Delta_{\mu'\nu'}(k'_1) \Delta_{\nu'\nu}(-k'_2) \frac{1}{2}(-k'_{2\pi} + p_4)_\nu^{\perp k'_2}}{(m_\Delta^2 - k_{1N}'^2 - im_\Delta\Gamma_\Delta)(m_\Delta^2 - k_{2\pi}'^2 - im_\Delta\Gamma_\Delta)(m_\pi^2 - k_{2\pi}'^2 - i0)} g_\Delta \psi_c(-p_4) \right),
\end{aligned}$$

where $p_\Delta = p_2 + p_3$. By fixing the numerator of (4.177) at

$$k_{1\pi}'^2 \rightarrow m_\Delta^2, \quad k_{2\pi}'^2 \rightarrow m_\Delta^2, \quad k_{1\pi}'^2 \rightarrow m_\pi^2, \quad k_{1N}'^2 \rightarrow m_N^2, \quad (4.178)$$

we obtain the leading singular terms of the box-diagram amplitude:

$$\begin{aligned}
A_{NN \rightarrow \Delta\Delta \rightarrow N\pi + (N\pi \rightarrow N\pi)_\Delta}^{(\text{leading term})} &= (\bar{\psi}_c(-P_2)\psi(P_1)) G_{NN \rightarrow \Delta\Delta}^{(L=0)}(W) \\
&\times \left(\bar{\psi}(p_3)g_\Delta \frac{1}{2}(p_2 - p_3)_\mu^{\perp p\Delta} \frac{\Delta_{\mu\mu'}(p_\Delta)}{m_\Delta^2 - p_\Delta^2 - im_\Delta\Gamma_\Delta} \right. \\
&\times \frac{1}{2}(-k_1(\text{box}) + p_1 + k_2(\text{box}) - p_4)_{\mu'}^{\perp p\Delta} g_\Delta(\hat{k}_1(\text{box}) - \hat{p}_1 + m_N) g_\Delta \\
&\times p_{1\mu'}^{\perp k_1(\text{box})} \Delta_{\mu'\nu'}(k_1(\text{box})) \Delta_{\nu'\nu}(-k_2(\text{box})) p_{4\nu}^{\perp k_2(\text{box})} g_\Delta \psi_c(-p_4) \Big) \\
&\times \int \frac{d^4 k_\pi}{i(2\pi)^4} \frac{1}{(m_N^2 - (p_\Delta - k_\pi)^2 - i0)(m_\Delta^2 - (p_\Delta - k_\pi + p_1)^2 - im_\Delta\Gamma_\Delta)} \\
&\times \frac{1}{(m_\Delta^2 - (k_\pi + p_4)^2 - im_\Delta\Gamma_\Delta)(m_\pi^2 - k_\pi^2 - i0)}. \tag{4.179}
\end{aligned}$$

There exists another box diagram with the rescattering of another pion on the nucleon (see Fig. 4.8d: the production of Δ in the $(p_1 + p_4)^2$ -channel), the corresponding amplitude is given by an expression analogous to (4.179).

(iii) Box-diagram amplitude with nucleon-nucleon rescattering

Following the developed method, one can calculate the box-diagram amplitudes with nucleon-nucleon rescattering, see Fig. 4.8e. The corresponding singularities contribute in the region of low NN energies and should affect the pn , pp and nn spectra near their thresholds.

4.3.3.1 The $\Delta\Delta$ -production from $(NN)_D$ -state with $J^P = 2^+$

As was already pointed out, the production of $\Delta\Delta$ near the threshold in the S-wave gives also $J^P = 2^+$ (the initial pp state in the D -wave), leading to a strong box-diagram singularity in this wave. Below we present formulae for this case, they are written similarly to those with an initial pp s -wave.

(i) Two-pole diagram

In the state $J^P = 2^+$ there is a transition $(NN)_{D\text{-wave}} \rightarrow (\Delta\Delta)_{S\text{-wave}}$ which gives also the two-pole amplitude, see Fig. 4.8a:

$$\begin{aligned}
A_{(NN)_D \rightarrow (\Delta\Delta)_S \rightarrow (N\pi)(N\pi)} &= (\bar{\psi}_c(-P_2)X_{\nu'\nu''}^{(2)}(q)\psi(P_1)) \cdot G_{NN \rightarrow \Delta\Delta}^{(L=2)}(W) \\
&\times \left(\bar{\psi}(p_3)g_\Delta k_{1\mu}^\perp \frac{\Delta_{\mu\nu'}(k_1)}{m_\Delta^2 - s_{13} - im_\Delta\Gamma_\Delta} \frac{\Delta_{\nu''\nu}(-k_2)}{m_\Delta^2 - s_{24} - im_\Delta\Gamma_\Delta} (-)k_{2\nu}^\perp g_\Delta \psi_c(-p_4) \right). \tag{4.180}
\end{aligned}$$

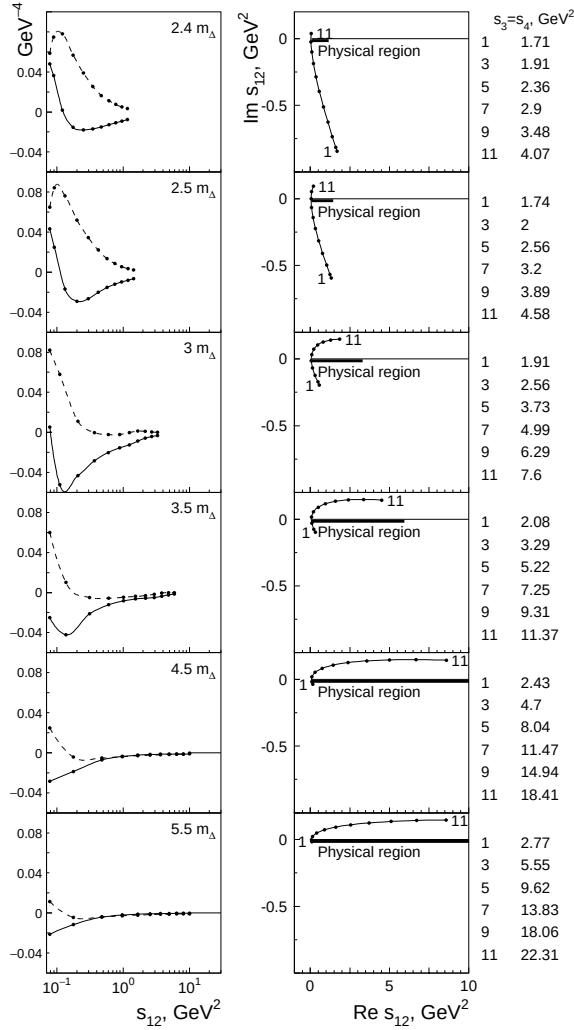


Fig. 4.9 Box diagram amplitude as a function of s_{12} under the constraint (4.174) (corresponding magnitudes of s_3 and s_4 are shown in the right column for different points labelled 1,2,3,...11). In the left column real and imaginary parts of the amplitude are shown by solid and dashed curves, respectively. Initial energies, $W = 2.4m_\Delta, 2.5m_\Delta, \dots, 5.5m_\Delta$, are shown on the top of each panel. In the right column the singularity positions, s_{12}^{box} , Eq. (4.175), are shown on the 2nd sheet of the complex- s_{12} plane.

(ii) Box-diagram amplitude with pion-pion rescattering

The box diagram with pion rescattering is shown in Fig. 4.8b, for the initial D -wave it reads:

$$\begin{aligned}
A_{NN \rightarrow \Delta \Delta \rightarrow NN + (\pi\pi \rightarrow \pi\pi)_S}^{(leading\ term)} &= \left(\bar{\psi}_c(-P_2) X_{\nu'\nu''}^{(2)}(q) \psi(P_1) \right) G_{NN \rightarrow \Delta \Delta}^{(L=2)}(W) \\
&\times A_{\pi\pi \rightarrow \pi\pi}^{S-wave}(s_{12}) \\
&\times \left(\bar{\psi}(p_3) g_{\Delta} p_{3\mu}^{\perp k_1(\text{box})} \Delta_{\mu\nu'}(k_1(\text{box})) \Delta_{\nu''\nu} k_2(\text{box}) p_{4\nu}^{\perp k_2(\text{box})} g_{\Delta} \psi_c(-p_4) \right) \\
&\times \int \frac{d^4 k'}{i(2\pi)^4} \frac{1}{(m_{\pi}^2 - (\frac{1}{2}p + k')^2 - i0)(m_{\pi}^2 - (\frac{1}{2}p - k')^2 - i0)} \\
&\times \frac{1}{(m_{\Delta}^2 - (\frac{1}{2}p + k' + p_3)^2 - im_{\Delta}\Gamma_{\Delta})(m_{\Delta}^2 - (\frac{1}{2}p - k' + p_4)^2 - im_{\Delta}\Gamma_{\Delta})}.
\end{aligned} \tag{4.181}$$

(iii) Box-diagram amplitude with pion-nucleon rescattering

The box diagram with pion-nucleon rescattering is shown on Fig. 4.8c, and for the initial D -wave it equals

$$\begin{aligned}
A_{NN \rightarrow \Delta \Delta \rightarrow N\pi + (N\pi \rightarrow N\pi)_{\Delta}}^{(leading\ term)} &= \left(\bar{\psi}_c(-P_2) X_{\nu'\nu''}^{(2)}(q) \psi(P_1) \right) G_{NN \rightarrow \Delta \Delta}^{(L=2)}(W) \\
&\times \left[\bar{\psi}(p_3) g_{\Delta} \frac{1}{2} (p_2 - p_3)_{\mu}^{\perp p_{\Delta}} \frac{\Delta_{\mu\mu'}(p_{\Delta})}{m_{\Delta}^2 - p_{\Delta}^2 - im_{\Delta}\Gamma_{\Delta}} \right. \\
&\times \frac{1}{2} (-k_1(\text{box}) + p_1 + k_2(\text{box}) - p_4)_{\mu'}^{\perp p_{\Delta}} g_{\Delta} (\hat{k}_1(\text{box}) - \hat{p}_1 + m_N) g_{\Delta} \\
&\times p_{1\mu'}^{\perp k_1(\text{box})} \Delta_{\mu'\nu'}(k_1(\text{box})) \Delta_{\nu''\nu}(-k_2(\text{box})) p_{4\nu}^{\perp k_2(\text{box})} g_{\Delta} \psi_c(-p_4) \left. \right] \\
&\times \int \frac{d^4 k_{\pi}}{i(2\pi)^4} \frac{1}{(m_N^2 - (p_{\Delta} - k_{\pi})^2 - i0)(m_{\Delta}^2 - (p_{\Delta} - k_{\pi} + p_1)^2 - im_{\Delta}\Gamma_{\Delta})} \\
&\times \frac{1}{(m_{\Delta}^2 - (k_{\pi} + p_4)^2 - im_{\Delta}\Gamma_{\Delta})(m_{\pi}^2 - k_{\pi}^2 - i0)}.
\end{aligned} \tag{4.182}$$

4.4 The $NN \rightarrow N_j^* + N \rightarrow NN\pi$ process with $j > 3/2$

We consider here the production and the decay of the resonance $N_j^* \rightarrow (N\pi)_{\ell}$, where $j = n + \frac{1}{2} > \frac{3}{2}$ and ℓ is the angular momentum of the $(N\pi)$ -

pair. The amplitude of the reaction $NN \rightarrow N_j^* + N \rightarrow (N\pi)_\ell + N$ reads:

$$A[NN \rightarrow N_j^* N \rightarrow (N\pi)_\ell N] = C[NN \rightarrow N_j^* N \rightarrow (N\pi)N] \times \sum_{\substack{S, S', L, L', J \\ \mu_1 \dots \mu_J}} \left(\bar{\psi}_c(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{SLJ}(k) \psi(p_1) \right) G_{NN \rightarrow N_j^* N}^{(S, S', L, L', J)}(s) \left(\bar{\psi}(p'_1) g_{N_j^*}^{(j, \ell)}(p_{N_j^*}^2) \right) \times \frac{\left[N_{\beta_1 \dots \beta_n}^{(j, \ell)}(p_1^\perp) \hat{F}_{\alpha_1 \dots \alpha_n}^{\beta_1 \dots \beta_n}(p_{N_j^*}) V_{\mu_1 \dots \mu_J}^{(S' L' J) \alpha_1 \dots \alpha_n}(p_2'^\perp) \right]}{m_{N_j^*}^2 - p_{N_j^*}^2 - i m_{N_j^*} \Gamma_{N_j^*}} \psi_c(-p'_2). \quad (4.183)$$

The factor $C[NN \rightarrow N_j^* N \rightarrow (N\pi)N]$ is related to the isotopic Clebsch-Gordan coefficients for the corresponding reaction. As before, $\bar{\psi}_c(-p_2)$ and $\psi_c(-p'_2)$ stand for the incoming and outgoing nucleons with the momenta p_2 and p'_2 ; the momentum of the produced pion is denoted as p_π . As usual, $k = \frac{1}{2}(p_1 - p_2)$. The relative momenta in the final state are equal to $p_{1\mu}^\perp = g_{\mu\mu'}^{\perp p_{N_j^*}} p'_{1\mu'}$, $p_{2\mu}^\perp = g_{\mu\mu'}^{\perp p} p'_{2\mu'}$, where $p = P_1 + P_2$.

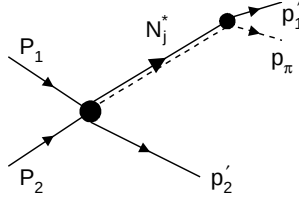


Fig. 4.10 The pole diagram with the production of the N_j^* resonance and its consequent decay $N_j^* N \rightarrow (N\pi)N$.

The numerator of the spin- j fermion propagator is denoted as $\hat{F}_{\alpha_1 \dots \alpha_n}^{\beta_1 \dots \beta_n}(p_{N_j^*})$ (remember that $j = n + \frac{1}{2}$). Following [17], we write (for the sake of simplicity, we replace below $p_{N_j^*} \rightarrow p$):

$$\begin{aligned} \hat{F}_{\alpha_1 \dots \alpha_n}^{\beta_1 \dots \beta_n}(p) &= (-1)^n \frac{n+1}{2n+1} \frac{\hat{p} + m_{N_j^*}}{2m_{N_j^*}} O_{\alpha_1 \dots \alpha_n}^{\mu_1 \dots \mu_n}(\perp p) \left(g_{\mu_1 \nu_1}^{\perp p} - \frac{n}{n+1} \sigma_{\mu_1 \nu_1}^{\perp p} \right) \\ &\times g_{\mu_2 \nu_2}^{\perp p} g_{\mu_3 \nu_3}^{\perp p} \dots g_{\mu_n \nu_n}^{\perp p} O_{\nu_1 \dots \nu_n}^{\beta_1 \dots \beta_n}(\perp p), \\ \sigma_{\mu\nu}^{\perp p} &= \frac{1}{2} (\gamma_\mu^{\perp p} \gamma_\nu^{\perp p} - \gamma_\nu^{\perp p} \gamma_\mu^{\perp p}). \end{aligned} \quad (4.184)$$

Written in the form of (4.184), the numerator of the spin- j fermion propagator is a generalised convolution of the wave functions normalised according to Eq. (4.17).

The numerator of the fermion propagator satisfies the following equation:

$$(\hat{p} - m_{N_j^*}) \hat{F}_{\alpha_1 \alpha_2 \dots \alpha_n}^{\beta_1 \beta_2 \dots \beta_n}(p) = 0 \quad (4.185)$$

under the constraints

$$\begin{aligned} p_{\alpha_1} \hat{F}_{\alpha_1 \alpha_2 \dots \alpha_n}^{\beta_1 \beta_2 \dots \beta_n}(p) &= \hat{F}_{\alpha_1 \alpha_2 \dots \alpha_n}^{\beta_1 \beta_2 \dots \beta_n}(p) p_{\beta_1} = 0, \\ \gamma_{\alpha_1} \hat{F}_{\alpha_1 \alpha_2 \dots \alpha_n}^{\beta_1 \beta_2 \dots \beta_n}(p) &= \hat{F}_{\alpha_1 \alpha_2 \dots \alpha_n}^{\beta_1 \beta_2 \dots \beta_n}(p) \gamma_{\beta_1} = 0. \end{aligned} \quad (4.186)$$

The convolution requirement for the numerator of the fermion propagators reads:

$$\hat{F}_{\alpha_1 \alpha_2 \dots \alpha_n}^{\mu_1 \mu_2 \dots \mu_n}(p) \hat{F}_{\mu_1 \mu_2 \dots \mu_n}^{\beta_1 \beta_2 \dots \beta_n}(p) = (-1)^n \hat{F}_{\alpha_1 \alpha_2 \dots \alpha_n}^{\beta_1 \beta_2 \dots \beta_n}(p). \quad (4.187)$$

In (4.183) the spin factors in the vertices for the production of $(N\pi)_\ell$ and $N_j^* N$ are denoted as

$$\begin{aligned} (N\pi)_\ell - \text{vertex} : & \quad N_{\beta_1 \dots \beta_n}^{(j, \ell)}(p_1^\perp), \\ (N_j^* N) - \text{vertex} : & \quad V_{\mu_1 \dots \mu_J}^{(S' L' J) \alpha_1 \dots \alpha_n}(p_2^\perp). \end{aligned} \quad (4.188)$$

Here L' and S' are the angular momentum and the spin ($S' = j + 1/2$ or $j - 1/2$) of the final state baryons, N_j^* and N , respectively.

When $S' = j - 1/2$ and $L' + n - J = 2m$, ($m = 0, 1, 2, \dots$)

$$\begin{aligned} V_{\mu_1 \dots \mu_J}^{(S' L' J) \alpha_1 \dots \alpha_n}(p_2^\perp) &= i\gamma_5 X_{\alpha_1 \dots \alpha_m \xi_{m+1} \dots \xi_{L'}}(p_2^\perp) \\ &\times O_{\mu_1 \dots \mu_J}^{\xi_{m+1} \dots \xi_{L'} \alpha_{m+1} \dots \alpha_n}(p). \end{aligned} \quad (4.189)$$

When $S' = j + 1/2$ and $L' + n - J = 2m + 1$, ($m = 0, 1, 2, \dots$)

$$\begin{aligned} V_{\mu_1 \dots \mu_J}^{(S' L' J) \alpha_1 \dots \alpha_n}(p_2^\perp) &= i\gamma_5 \varepsilon_{\eta \alpha_1 p_2^\perp p_{N_j^*}} X_{\alpha_2 \dots \alpha_m \xi_{m+1} \dots \xi_{L'}}(p_2^\perp) \\ &\times O_{\mu_1 \dots \mu_J}^{\xi_{m+1} \dots \xi_{L'} \eta \alpha_{m+1} \dots \alpha_n}(p). \end{aligned} \quad (4.190)$$

4.5 NN Scattering Amplitude at Moderately High Energies — the Reggeon Exchanges

With increasing energies, at $p_{lab} \sim 3 - 5$ GeV/c (or $s \sim 10$ GeV²), we enter the region of moderately high energies, where, on the one hand, resonance production is still essential and, on the other hand, reggeon exchanges start to work.

When calculating the low energy diagrams, it is convenient to operate with the four-component spinors, while at high energies the use of two-component spinors is more convenient. Here we present some elements of the reggeon calculus in cases when two-component spinors are used.

4.5.1 Reggeon-quark vertices in the two-component spinor technique

At high energies and small momentum transfers, *i.e.* in the region where the Regge description of the amplitudes can be used, the trajectory of a fast particle virtually does not change, hence, the direction of motion of the incident particle defines the axis for the spin projection.

The vertex of the hadron-reggeon interaction depends on two vectors. These are the direction of the momentum of the incident hadron, $\mathbf{n}_z$, and the hadron momentum transferred $\mathbf{q}_\perp$ which flows along the reggeon. Let us use the notations for four-vectors: $A = (A_0, \mathbf{A}_\perp, A_z)$. If so, the vectors determining the hadron-reggeon vertex are

$$n_z = (0, 0, 1), \quad q = (0, \mathbf{q}_\perp, 0). \quad (4.191)$$

The complete set of operators for two-component spinors is given by the unit matrix I and Pauli matrices $\boldsymbol{\sigma}$. Hence, the quark-reggeon vertices have to be constructed from two vectors, $\mathbf{n}_z$ and $\mathbf{q}_\perp$, and four matrices I and $\boldsymbol{\sigma}$. We can obtain two scalars,

$$I, \quad i(\boldsymbol{\sigma}[\mathbf{n}_z, \mathbf{q}_\perp]), \quad (4.192)$$

and two pseudoscalars,

$$(\boldsymbol{\sigma}\mathbf{q}_\perp), \quad i(\boldsymbol{\sigma}\mathbf{n}_z). \quad (4.193)$$

All the vertices (4.192), (4.193) are C -even; under charge conjugation, the operators transform into their Hermitian conjugates, $\boldsymbol{\sigma} \rightarrow \boldsymbol{\sigma}^+ = \boldsymbol{\sigma}$, and at the same time $i \rightarrow -i$, while the direction of the collision axis changes to the opposite, $\mathbf{n}_z \rightarrow -\mathbf{n}_z$. Hence, $\mathbf{q}_\perp \rightarrow \mathbf{q}_\perp$.

For reggeons with positive naturality (naturality means the product of the P -parity and the signature, see Chapter 3), we have to take the vertices (4.192). These are the reggeons $\mathbf{P}$, P' (or f_2), ω , ρ , ϕ , f' (or f'_2), a_2 , *etc.* For them the vertex can be written as

$$g_1^R(q_\perp^2) + \frac{i}{2m}(\boldsymbol{\sigma}[\mathbf{n}_z\mathbf{q}_\perp])g_2^R(q_\perp^2). \quad (4.194)$$

The nucleons p and n are isodoublets. Therefore, nucleon-reggeon vertices for isovector reggeons (for example, ρ and a_2) should include the operator $\boldsymbol{\tau}$ which is the Pauli matrix acting in the isotopic space, while for isoscalar reggeons the vertices include unit matrices in the isotopic space.

For π - and η -trajectories the vertices are proportional to $(\boldsymbol{\sigma}\mathbf{q}_\perp)$. These vertices differ by isotopic operators,

$$\begin{aligned} \pi\text{-trajectory} : & \quad \boldsymbol{\tau} \cdot (\boldsymbol{\sigma}\mathbf{q}_\perp)g_\pi(q_\perp^2), \\ \eta\text{-trajectory} : & \quad (\boldsymbol{\sigma}\mathbf{q}_\perp)g_\eta(q_\perp^2). \end{aligned} \quad (4.195)$$

The vertices for the a_1 - and f_1 -trajectories contain the spin factor $i(\boldsymbol{\sigma}\mathbf{n}_z)$, and they differ also only by isotopic factors,

$$\begin{aligned} a_1\text{-trajectory} &: \boldsymbol{\tau} \cdot i(\boldsymbol{\sigma}\mathbf{n}_z)g_{a_1}(q_\perp^2), \\ f_1\text{-trajectory} &: i(\boldsymbol{\sigma}\mathbf{n}_z)g_{f_1}(q_\perp^2). \end{aligned} \quad (4.196)$$

Working with the isotopic variables, it may turn out to be more convenient to use Clebsch–Gordan coefficients instead of matrices in the isotopic space. This is, for example, the case of the kaon trajectories, *i.e.* the K - and K^* -trajectories. The spin structure of the vertex corresponding to the K -trajectory coincides with that of the η -trajectory, the spin structure corresponding to the K^* -trajectory is the same as that of the ω -trajectory.

4.5.2 Four-component spinors and reggeon vertices

Here we present the transformation of four-component spinors to two-component ones.

4.5.2.1 Scalar vertex

A particle with the smallest possible spin, which may be situated on the $\mathbf{P}$ - or P' -trajectories, is the scalar meson. Because of that, let us consider the pomeron (or P' -reggeon) vertex as a vertex of a fermion with a scalar meson $\bar{\psi}(p)I\psi(p')$. Here I is the four-dimensional unit matrix ($p' = p + q_\perp$) and p is the momentum of a fast fermion flying along the z -axis, $p = (p_0, 0, p) \simeq (p + m^2/2p, 0, p)$, $p' \simeq (p + (m^2 + \mathbf{q}_\perp^2)/2p, \mathbf{q}_\perp, p)$. If so, we can write $\bar{\psi}(p)\psi(p')$ in terms of two-component spinors and go to the limit $p \rightarrow \infty$:

$$\begin{aligned} \bar{\psi}(p)\psi(p') &= \sqrt{p_0 + m} \left[\varphi^+ \varphi' - \varphi^+ \frac{(\boldsymbol{\sigma}\mathbf{p})(\boldsymbol{\sigma}\mathbf{p}')}{(p'_0 + m)(p_0 + m)} \varphi' \right] \sqrt{p'_0 + m} \\ &\simeq p\varphi^+ \left[I - \left(I - \frac{1}{p}(\boldsymbol{\sigma}\mathbf{q}_\perp)(\boldsymbol{\sigma}\mathbf{n}_z) \left(1 - \frac{2m}{p} \right) \right) \right] \varphi' \\ &= \varphi^+ \left[I + \frac{i}{2m}(\mathbf{q}_\perp[\mathbf{n}_z, \boldsymbol{\sigma}]) \right] \varphi'. \end{aligned} \quad (4.197)$$

We wrote here $\mathbf{p}' = \mathbf{p} + \mathbf{q}_\perp$ and made use of $(\boldsymbol{\sigma}\mathbf{p})(\boldsymbol{\sigma}\mathbf{p}) = p^2$ and $\sigma_a\sigma_b = i\varepsilon_{abc}\sigma_c$, when $a \neq b$; in (4.197) I is understood, of course, as a two-dimensional unit matrix, which acts on the two-component spinors φ^+ and φ' .

The fermion–pomeron vertex (4.197) contains a definite combination of two possible operators, I and $i(\mathbf{q}_\perp[\mathbf{n}_z, \boldsymbol{\sigma}])$. This is quite natural, since

in (4.197) we have considered only one of the possible vertices, the one which corresponds to a scalar meson (a scalar glueball, for example). For the vertex corresponding to a tensor glueball exchange there would be a different combination of the operators I and $i(\mathbf{q}_\perp[\mathbf{n}_z, \boldsymbol{\sigma}])$. In the general case we have to write an arbitrary superposition of these operators, as it was done in (4.194).

According to certain experimental observations based on nucleon–nucleon scatterings, the contribution of spin-dependent terms of the amplitude decreases rapidly with the growth of p_{lab} , and it is rather small at moderately high energies. This may serve us as a basis for neglecting the spin-flip contributions in (4.194), *i.e.* to accept $g_1^R \gg g_2^R$.

4.5.2.2 Vector reggeon vertex

There are two contributions to the $\bar{\psi}(p)\gamma_\mu\psi(p')$ vertex: one with a zero component $\mu = 0$ and another with a space-like one. We have

$$\begin{aligned}\bar{\psi}(p)\gamma_0\psi(p') &= \sqrt{p_0+m} \left[\varphi^+\varphi' + \varphi^+ \frac{(\boldsymbol{\sigma}\mathbf{p})(\boldsymbol{\sigma}\mathbf{p}')}{(p'_0+m)(p_0+m)} \varphi' \right] \sqrt{p'_0+m} \\ &\simeq 2p\varphi^+ I\varphi'\end{aligned}\tag{4.198}$$

and

$$\begin{aligned}\bar{\psi}(p)\boldsymbol{\gamma}\psi(p') &= \sqrt{p_0+m} \left[\varphi^+ \frac{\boldsymbol{\sigma}(\boldsymbol{\sigma}\mathbf{p}')}{p'_0+m} \varphi' + \varphi^+ \frac{(\boldsymbol{\sigma}\mathbf{p})\boldsymbol{\sigma}}{p_0+m} \varphi' \right] \sqrt{p'_0+m} \\ &\simeq 2p\mathbf{n}_z\varphi^+ I\varphi' .\end{aligned}\tag{4.199}$$

Up to the large factor p the vertex (4.198) with $\mu = 0$ has a standard form, see (4.192), but the right-hand side of (4.199) has a space-like vector contribution. However, we must remember that the nucleon–nucleon amplitude has a second vertex for the incoming particle with momentum p_2 , so that actually we have to consider the bilinear form $\bar{\psi}(p_1)\gamma_\mu\psi(p'_1) \cdot \bar{\psi}(p_2)\gamma_\mu\psi(p'_2)$ which leads to

$$\bar{\psi}(p_1)\gamma_\mu\psi(p'_1) \cdot \bar{\psi}(p_2)\gamma_\mu\psi(p'_2) \rightarrow 8p^2(\varphi_1^+ I\varphi'_1)(\varphi_2^+ I\varphi'_2).$$

Since the factor $8p^2$ renormalises the reggeon propagator, we should use

$$(\varphi^+ I\varphi')\tag{4.200}$$

as the vertex for vector reggeons.

But, working with four-component spinors, it is rather inconvenient to have in mind the existence of the second vertex all the time. Let us, rather,

separate the leading- p components just from the beginning, as suggested by [18]. In other words, we have to substitute:

$$\gamma_\mu \rightarrow \gamma_\mu n_\mu = \hat{n} \quad \text{with} \quad n = \frac{1}{2p}(1, 0, -1). \quad (4.201)$$

The four-vector n singles out the leading components, since $p_\mu n_\mu \simeq 1 + m^2/4p^2$. In addition, the factor $1/2p$ introduced in n kills in the vertex the terms increasing with the energy and leaves an s -dependence only in the reggeon propagator. We have

$$\bar{\psi}(p)\hat{n}\psi(p') \simeq 2\varphi^+ I \varphi'. \quad (4.202)$$

Hence, the operator $\hat{n}$ provides us with a spin-independent vertex for vector reggeons.

4.5.2.3 Pseudoscalar reggeon vertex

The pseudoscalar vertex $\bar{\psi}(p)\gamma_5\psi(p')$ can be rewritten in terms of two-component spinors in the following form:

$$\begin{aligned} \bar{\psi}(p)\gamma_5\psi(p') &= \sqrt{p_0 + m}\varphi^+ \left(-\frac{(\boldsymbol{\sigma}\mathbf{p}')}{p'_0 + m} + \frac{(\boldsymbol{\sigma}\mathbf{p})}{p_0 + m} \right) \varphi' \sqrt{p'_0 + m} \\ &\simeq -\varphi^+(\boldsymbol{\sigma}\mathbf{q}_\perp)\varphi'. \end{aligned} \quad (4.203)$$

The vertex (4.203) describes the coupling of the pionic reggeon with the fermion.

4.5.2.4 Pseudovector reggeon vertex

It is obvious that introducing the pseudovector vertex one has to repeat the procedure used for the vector vertex. Because of that, let us carry out a substitution analogous to (4.201), *i.e.* substitute the fermion–pseudovector reggeon vertex in the following way:

$$\bar{\psi}(p)i\gamma_5\gamma_\mu\psi(p') \rightarrow \bar{\psi}(p)i\gamma_5\hat{n}\psi(p'). \quad (4.204)$$

In the two-component form this vertex can be written as

$$\bar{\psi}(p)i\gamma_5\hat{n}\psi(p') \simeq 2\varphi^+ i(\boldsymbol{\sigma}\mathbf{n}_z)\varphi'. \quad (4.205)$$

This is, actually, the vertex of the a_1 -trajectory. Similarly to the pionic one, the leading a_1 -trajectory has an intercept close to zero.

4.6 Production of Heavy Particles in the High Energy Hadron–Hadron Collisions: Effects of New Thresholds

As was demonstrated before, the production of new particles, like resonances in the reactions $NN \rightarrow N\Delta$ or $NN \rightarrow \Delta\Delta$, leads to specific effects due to the presence of amplitude singularities near thresholds related to these production processes.

Apart from ordinary (light-quark) resonances, there exist resonances with heavy quarks. One cannot exclude the existence of even heavier, strongly interacting particles. This gives rise to a question of how the production of these rather heavy particles (or resonances) reveals itself in the standard characteristics measured in high energy collisions of, say, NN or $N\bar{N}$: we mean changes in the behaviour of σ_{tot} , σ_{el} , $\rho = \text{Im } A_{el}/\text{Re } A_{el}$.

This question has been put forward in [19, 20, 21]. The problem was initiated by the UA4 experiment, where the $p\bar{p}$ collision at $\sqrt{s} = 546$ GeV [22] was studied and an irregularity in $\rho = \text{Im } A_{el}/\text{Re } A_{el}$ was observed.

We are investigating this problem for the $p\bar{p}$ scattering using the impact parameter representation: this representation is the most suitable for the consideration of high energy scattering amplitudes at small momentum transfers and, correspondingly, for the calculation of σ_{tot} , σ_{el} , $\rho = \text{Im } A_{el}/\text{Re } A_{el}$. The K -matrix technique is, as a rule, suitable for extracting the threshold effects, we use this technique here. The appropriate method was suggested in [23].

4.6.1 *Impact parameter representation of the scattering amplitude*

The scattering amplitude in the impact parameter representation is defined as

$$\begin{aligned} A(\mathbf{q}, s) &= 2 \int d^2b e^{i\mathbf{q}\mathbf{b}} f(\mathbf{b}, s) , \\ f(\mathbf{b}, s) &= i (1 - \eta(\mathbf{b}, s) \exp[2i\delta(\mathbf{b}, s)]) . \end{aligned} \quad (4.206)$$

In the diffraction scattering region at large energies, the momentum transfer is $\mathbf{q} \perp \mathbf{p}_{in}$ thus fixing the dimension of $\mathbf{q}$.

Total, elastic and inelastic cross sections expressed in terms of δ and η

read:

$$\begin{aligned}\sigma_{tot} &= \text{Im}A(0, s) = 2 \int d^2b (1 - \eta \cos 2\delta) , \\ \sigma_{el} &= \frac{1}{16\pi} \int_0^\infty dq^2 |A(\mathbf{q}, s)|^2 = \int d^2b (1 - 2\eta \cos 2\delta + \eta^2) , \\ \sigma_{inel} &= \sigma_{tot} - \sigma_{el} = \int d^2b (1 - \eta^2) .\end{aligned}\quad (4.207)$$

4.6.1.1 Example: Elastic scattering amplitude for two spinless particles

Let us illustrate the eikonal formulae (4.206) and (4.207) by considering as an example two spinless particles when inelastic processes are supposed to be absent ($\eta = 1$). The scattering amplitude for this case is written as follows (see Chapter 3):

$$f(\mathbf{q}) = \frac{1}{2ip} \sum_{\ell} (2\ell + 1) [e^{2i\delta_{\ell}} - 1] P_{\ell}(\cos \theta) . \quad (4.208)$$

We use here the standard quantum-mechanical notation $f(\mathbf{q})$ for the scattering amplitude which differs by a factor from that written in (4.206).

At large energies, when the wave length of the particle is much less than the characteristic size of the interaction region r_0 , the number n_{ℓ} of the partial waves, giving a relevant contribution to (4.208), is large:

$$n_{\ell} \sim pr_0 \gg 1 . \quad (4.209)$$

Let us introduce the impact parameter b as

$$pb = \ell + \frac{1}{2} . \quad (4.210)$$

At a large ℓ and a small angle θ we can use the relation

$$P_{\ell}(\cos \theta) \simeq \int_0^{2\pi} \frac{d\varphi}{2\pi} \exp \left[i(2\ell + 1) \sin \frac{\theta}{2} \cos \varphi \right] = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i\mathbf{q}\mathbf{b}} , \quad (4.211)$$

where we substituted

$$(2\ell + 1) \sin \frac{\theta}{2} \cos \varphi = 2p \sin \frac{\theta}{2} \frac{\ell + 1/2}{p} \cos \varphi = \mathbf{q}\mathbf{b} . \quad (4.212)$$

We restrict ourselves to small scattering angles, for which $\mathbf{q} \simeq \mathbf{q}_{\perp}$ can be assumed. The vectors $\mathbf{q}_{\perp}$ and $\mathbf{b}$ lie in the plane perpendicular to the z -axis which coincides with the initial direction of the particle; φ is the angle

between the vectors $\mathbf{q}$ and $\mathbf{b}$. Inserting (4.211) into (4.208) and integrating instead of summing over ℓ , we obtain

$$f(\mathbf{q}) = \frac{ip}{2\pi} \int d^2b e^{i\mathbf{q}\mathbf{b}} [1 - e^{2i\delta_\ell(\mathbf{b})}] \equiv \frac{ip}{2\pi} \int d^2b e^{i\mathbf{q}\mathbf{b}} \gamma(\mathbf{b}). \quad (4.213)$$

This is the standard eikonal representation for the scattering amplitude when inelastic processes are absent; $\gamma(\mathbf{b})$ is the profile function which plays an important role in the Glauber–Sitenko formalism [24, 25]. The inclusion of inelastic processes (*i.e.* $[1 - e^{2i\delta_\ell(\mathbf{b})}] \rightarrow [1 - \eta(\mathbf{b})e^{2i\delta_\ell(\mathbf{b})}]$ in (4.213)) leads to Eq. (4.206).

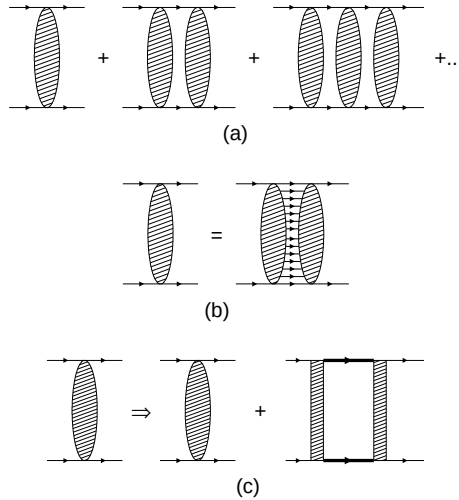


Fig. 4.11 a) High energy scattering amplitude as a set of two-particle rescatterings, b) the K-matrix block for high energy scattering amplitude: it includes multiparticle states while two-particle ones are excluded, c) the K-matrix block with the inclusion of heavy-particle intermediate states (thick lines).

4.6.1.2 *K-matrix representation of the impact parameter amplitude*

In Eqs. (4.206) and (4.207) we take into account the inelasticity parameter $\eta(b, s)$ and redefine the profile function:

$$\frac{p}{2\pi} \gamma(\mathbf{b}) = f(\mathbf{b}, s). \quad (4.214)$$

To use the K-matrix representation for $f(\mathbf{b}, s)$, we have to extract the elastic channel directly:

$$f(\mathbf{b}, s) = \frac{2K(\mathbf{b}, s)}{1 - iK(\mathbf{b}, s)}. \quad (4.215)$$

Here the phase space factors are included into the K-matrix block (just as it was done in Chapter 3). The K-matrix block contains all multiparticle states and their threshold singularities (this is illustrated by Fig. 4.11b). Because of that, the K-matrix block is a complex-valued function, but the two-particle states are excluded. Correspondingly, the right-hand side of (4.215) can be presented graphically as a set of diagrams with a different number of two-particle rescatterings by means of the block K (Fig. 4.11a).

The high energy scattering amplitude in the region of large energies and small $\mathbf{q}^2$ can be represented in the following form:

$$A(\mathbf{q}, s) = i\sigma_{tot}(s) \left(1 - i\rho(s, \mathbf{q}^2) \right) \exp[-r^2(s)\mathbf{q}^2]. \quad (4.216)$$

Below we simplify $\rho(s, \mathbf{q}^2) \rightarrow \rho(s)$; the parameters $\sigma_{tot}(s)$, $\rho(s)$, $r^2(s)$ are subjects of experimental measurements. Correspondingly, we have for $f(\mathbf{b}, s)$:

$$f(\mathbf{b}, s) = i \frac{\sigma_{tot}(s)}{8\pi r^2(s)} \left(1 - i\rho(s) \right) \exp\left[-\frac{\mathbf{b}^2}{4r^2(s)}\right], \quad (4.217)$$

and $K(\mathbf{q}, s)$:

$$K(\mathbf{b}, s) = \frac{f(\mathbf{b}, s)}{2 + if(\mathbf{b}, s)}. \quad (4.218)$$

The production of new heavy particles results in the appearance of an additional term in the K-matrix block:

$$K(\mathbf{b}, s) \rightarrow K(\mathbf{b}, s) + \alpha(\mathbf{b}, s). \quad (4.219)$$

This procedure is equivalent to that described in Section 3.5.2 for the transformation of a one-channel amplitude into the two-channel one via the replacement $K_{11} \rightarrow K_{11} + K_{12}[1 - iK_{22}]^{-1}K_{21}$. We mean here $K_{12}[1 - iK_{22}]^{-1}K_{21} \rightarrow \alpha(\mathbf{b}, s)$. The production of new heavy particles in the intermediate state is shown in Fig. 4.11c, the corresponding amplitude contains threshold singularities of different types. For a direct two-particle production it is $\sqrt{s - 4m_{heavy}^2}$, for a three-particle diffractive production

$-(s - (2m_{heavy} + \mu)^2)^2 \ln[s - (2m_{heavy} + \mu)^2]$ where μ is the light hadron mass, and so on (see Section 3.2.6 for details). Hence, we have:

$$\alpha(\mathbf{b}, s) \simeq \alpha_2(\mathbf{b}, s) \sqrt{s - 4m_{heavy}^2} \quad (4.220)$$

$$+ \alpha_3(\mathbf{b}, s) (s - (2m_{heavy} + \mu)^2)^2 \left[\frac{1}{i\pi} \ln(s - (2m_{heavy} + \mu)^2) - 1 \right] + \dots$$

Let us remind that below the threshold, $s < 4m_{heavy}^2$, the factor $\sqrt{s - 4m_{heavy}^2}$ is imaginary, while the logarithmic term at $s < (2m_{heavy} + \mu)^2$ transforms as follows: $\ln[s - (2m_{heavy} + \mu)^2] \rightarrow \ln[(2m_{heavy} + \mu)^2 - s] + i\pi$. These singularities lead to the cusps in σ_{tot} and $\rho = \text{Im } A_{el} / \text{Re } A_{el}$.

We come to the conclusion that the new particle production processes provide s -channel singularities in the scattering amplitude and the behaviour of the amplitude near the singularity is restricted by the unitarity condition: the unitarity constraint suppresses the singular behaviour of the amplitude. Such a suppression is rather strong in the central region, but it is weaker for peripheral processes. The unitarity constraints are especially important at high energies because the scattering amplitude has a maximal inelasticity in the region of small b and this region increases as $\ln s$ with the growth of s . Therefore, the effects of cusps due to opening of new thresholds require a special analysis, with the unitarity constraints taken into account.

In Appendix 4.G we present an example of such an analysis of the UA4 collaboration data at $\sqrt{s} = 546$ GeV [22], where a cusp in $\rho = \text{Im } A_{el} / \text{Re } A_{el}$ was reported.

4.7 Appendix 4.A. Angular Momentum Operators

The angular-dependent part of the wave function of a composite state is described by operators constructed for the relative momenta of particles and the metric tensor. Such operators (we denote them as $X_{\mu_1 \dots \mu_L}^{(L)}$, where L is the angular momentum) are called angular momentum operators; they correspond to irreducible representations of the Lorentz group. They satisfy the following properties:

(i) Symmetry with respect to the permutation of any two indices:

$$X_{\mu_1 \dots \mu_i \dots \mu_j \dots \mu_L}^{(L)} = X_{\mu_1 \dots \mu_j \dots \mu_i \dots \mu_L}^{(L)}. \quad (4.221)$$

(ii) Orthogonality to the total momentum of the system, $P = k_1 + k_2$:

$$P_{\mu_i} X_{\mu_1 \dots \mu_i \dots \mu_L}^{(L)} = 0. \quad (4.222)$$

(iii) Tracelessness with respect to the summation over any two indices:

$$g_{\mu_i \mu_j} X_{\mu_1 \dots \mu_i \dots \mu_j \dots \mu_L}^{(L)} = 0. \quad (4.223)$$

Let us consider a one-loop diagram describing the decay of a composite system into two spinless particles, which propagate and then form again a composite system. The decay and formation processes are described by angular momentum operators. Owing to the quantum number conservation, this amplitude must vanish for initial and final states with different spins. The S-wave operator is a scalar and can be taken as a unit operator. The P-wave operator is a vector. In the dispersion relation approach it is sufficient that the imaginary part of the loop diagram, with S- and P-wave operators as vertices, equals 0. In the case of spinless particles, this requirement entails

$$\int \frac{d\Omega}{4\pi} X_\mu^{(1)} = 0, \quad (4.224)$$

where the integral is taken over the solid angle of the relative momentum. In general, the result of such an integration is proportional to the total momentum P_μ (the only external vector):

$$\int \frac{d\Omega}{4\pi} X_\mu^{(1)} = \lambda P_\mu. \quad (4.225)$$

Convoluting this expression with P_μ and demanding $\lambda = 0$, we obtain the orthogonality condition (4.222). The orthogonality between the D- and S-waves is provided by the tracelessness condition (4.223); equations (4.222), (4.223) provide the orthogonality for all operators with different angular momenta.

The orthogonality condition (4.222) is automatically fulfilled if the operators are constructed from the relative momenta $k_\mu^\perp$ and tensor $g_{\mu\nu}^\perp$. Both of them are orthogonal to the total momentum of the system:

$$k_\mu^\perp = \frac{1}{2} g_{\mu\nu}^\perp (k_1 - k_2)_\nu, \quad g_{\mu\nu}^\perp = g_{\mu\nu} - \frac{P_\mu P_\nu}{s}. \quad (4.226)$$

In the c.m. system, where $P = (P_0, \vec{P}) = (\sqrt{s}, 0)$, the vector $k^\perp$ is space-like: $k^\perp = (0, \vec{k}, 0)$.

The operator for $L = 0$ is a scalar (for example, a unit operator), and the operator for $L = 1$ is a vector, which can be constructed from $k_\mu^\perp$ only. The orbital angular momentum operators for $L = 0$ to 3 are:

$$\begin{aligned} X^{(0)}(k^\perp) &= 1, & X_\mu^{(1)} &= k_\mu^\perp, \\ X_{\mu_1 \mu_2}^{(2)}(k^\perp) &= \frac{3}{2} \left(k_{\mu_1}^\perp k_{\mu_2}^\perp - \frac{1}{3} k_\perp^2 g_{\mu_1 \mu_2}^\perp \right), \\ X_{\mu_1 \mu_2 \mu_3}^{(3)}(k^\perp) &= \frac{5}{2} \left[k_{\mu_1}^\perp k_{\mu_2}^\perp k_{\mu_3}^\perp - \frac{k_\perp^2}{5} (g_{\mu_1 \mu_2}^\perp k_{\mu_3}^\perp + g_{\mu_1 \mu_3}^\perp k_{\mu_2}^\perp + g_{\mu_2 \mu_3}^\perp k_{\mu_1}^\perp) \right]. \end{aligned} \quad (4.227)$$

The operators $X_{\mu_1 \dots \mu_L}^{(L)}$ for $L \geq 1$ can be written in the form of a recurrency relation:

$$\begin{aligned} X_{\mu_1 \dots \mu_L}^{(L)}(k^\perp) &= k_\alpha^\perp Z_{\mu_1 \dots \mu_L}^\alpha(k^\perp), \\ Z_{\mu_1 \dots \mu_L}^\alpha(k^\perp) &= \frac{2L-1}{L^2} \left(\sum_{i=1}^L X_{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_L}^{(L-1)}(k^\perp) g_{\mu_i \alpha}^\perp \right. \\ &\quad \left. - \frac{2}{2L-1} \sum_{\substack{i,j=1 \\ i < j}}^L g_{\mu_i \mu_j}^\perp X_{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_{j-1} \mu_{j+1} \dots \mu_L}^{(L-1)}(k^\perp) \right). \end{aligned} \quad (4.228)$$

The convolution equality reads

$$X_{\mu_1 \dots \mu_L}^{(L)}(k^\perp) k_{\mu_L}^\perp = k_\perp^2 X_{\mu_1 \dots \mu_{L-1}}^{(L-1)}(k^\perp). \quad (4.229)$$

On the basis of Eq.(4.229) and taking into account the tracelessness property of $X_{\mu_1 \dots \mu_L}^{(L)}$, one can write down the orthogonality–normalisation condition for orbital angular operators

$$\begin{aligned} \int \frac{d\Omega}{4\pi} X_{\mu_1 \dots \mu_L}^{(L)}(k^\perp) X_{\mu_1' \dots \mu_L'}^{(L')}(k^\perp) &= \delta_{LL'} \alpha_L k_\perp^{2L}, \\ \alpha_L &= \prod_{l=1}^L \frac{2l-1}{l}. \end{aligned} \quad (4.230)$$

Iterating equation (4.229), one obtains the following expression for the operator $X_{\mu_1 \dots \mu_L}^{(L)}$:

$$\begin{aligned} X_{\mu_1 \dots \mu_L}^{(L)}(k^\perp) &= \alpha_L \left[k_{\mu_1}^\perp k_{\mu_2}^\perp k_{\mu_3}^\perp k_{\mu_4}^\perp \dots k_{\mu_L}^\perp \right. \\ &\quad - \frac{k_\perp^2}{2L-1} \left(g_{\mu_1 \mu_2}^\perp k_{\mu_3}^\perp k_{\mu_4}^\perp \dots k_{\mu_L}^\perp + g_{\mu_1 \mu_3}^\perp k_{\mu_2}^\perp k_{\mu_4}^\perp \dots k_{\mu_L}^\perp + \dots \right) \\ &\quad + \frac{k_\perp^4}{(2L-1)(2L-3)} \left(g_{\mu_1 \mu_2}^\perp g_{\mu_3 \mu_4}^\perp k_{\mu_5}^\perp k_{\mu_6}^\perp \dots k_{\mu_L}^\perp \right. \\ &\quad \left. + g_{\mu_1 \mu_2}^\perp g_{\mu_3 \mu_5}^\perp k_{\mu_4}^\perp k_{\mu_6}^\perp \dots k_{\mu_L}^\perp + \dots \right) + \dots \left. \right]. \end{aligned} \quad (4.231)$$

4.7.1 Projection operators and denominators of the boson propagators

The projection operator $O_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L}$ is constructed of the metric tensors $g_{\mu\nu}^\perp$. It has the properties as follows:

$$\begin{aligned} X_{\mu_1 \dots \mu_L}^{(L)} O_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L} &= X_{\nu_1 \dots \nu_L}^{(L)}, \\ O_{\alpha_1 \dots \alpha_L}^{\mu_1 \dots \mu_L} O_{\nu_1 \dots \nu_L}^{\alpha_1 \dots \alpha_L} &= O_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L}. \end{aligned} \quad (4.232)$$

Taking into account the definition of projection operators (4.232) and the properties of the X -operators (4.231), we obtain

$$k_{\mu_1} \dots k_{\mu_L} O_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L} = \frac{1}{\alpha_L} X_{\nu_1 \dots \nu_L}^{(L)}(k^\perp). \quad (4.233)$$

This equation is the basic property of the projection operator: it projects any operator with L indices onto the partial wave operator with angular momentum L .

For the lowest states,

$$\begin{aligned} O &= 1, \quad O_\nu^\mu = g_{\mu\nu}^\perp, \\ O_{\nu_1 \nu_2}^{\mu_1 \mu_2} &= \frac{1}{2} \left(g_{\mu_1 \nu_1}^\perp g_{\mu_2 \nu_2}^\perp + g_{\mu_1 \nu_2}^\perp g_{\mu_2 \nu_1}^\perp - \frac{2}{3} g_{\mu_1 \mu_2}^\perp g_{\nu_1 \nu_2}^\perp \right). \end{aligned} \quad (4.234)$$

For higher states, the operator can be calculated using the recurrent expression:

$$\begin{aligned} O_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L} &= \frac{1}{L^2} \left(\sum_{i,j=1}^L g_{\mu_i \nu_j}^\perp O_{\nu_1 \dots \nu_{j-1} \nu_{j+1} \dots \nu_L}^{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_L} \right. \\ &\quad \left. - \frac{4}{(2L-1)(2L-3)} \times \sum_{\substack{i < j \\ k < m}} g_{\mu_i \mu_j}^\perp g_{\nu_k \nu_m}^\perp O_{\nu_1 \dots \nu_{k-1} \nu_{k+1} \dots \nu_{m-1} \nu_{m+1} \dots \nu_L}^{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_{j-1} \mu_{j+1} \dots \mu_L} \right). \end{aligned} \quad (4.235)$$

The product of two X -operators integrated over a solid angle (that is equivalent to the integration over internal momenta) depends only on the external momenta and the metric tensor. Therefore, it must be proportional to the projection operator. After straightforward calculations we obtain

$$\int \frac{d\Omega}{4\pi} X_{\mu_1 \dots \mu_L}^{(L)}(k^\perp) X_{\nu_1 \dots \nu_L}^{(L)}(k^\perp) = \frac{\alpha_L k_\perp^{2L}}{2L+1} O_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L}. \quad (4.236)$$

Let us introduce the positive valued $|\vec{k}|^2$:

$$|\vec{k}|^2 = -k_\perp^2 = \frac{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}{4s}. \quad (4.237)$$

In the c.m.s. of the reaction, $\vec{k}$ is the momentum of a particle. In other systems we use this definition only in the sense of $|\vec{k}| \equiv \sqrt{-k_\perp^2}$; clearly, $|\vec{k}|^2$ is a relativistically invariant positive value. If so, equation (4.236) can be written as

$$\int \frac{d\Omega}{4\pi} X_{\mu_1 \dots \mu_L}^{(L)}(k^\perp) X_{\nu_1 \dots \nu_L}^{(L)}(k^\perp) = \frac{\alpha_L |\vec{k}|^{2L}}{2L+1} (-1)^L O_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L}. \quad (4.238)$$

The tensor part of the numerator of the boson propagator is defined by the projection operator. Let us write it as follows:

$$F_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L} = (-1)^L O_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L}, \quad (4.239)$$

with the definition of the propagator

$$\frac{F_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L}}{M^2 - s}. \quad (4.240)$$

This definition guarantees that the width of a resonance (calculated using the decay vertices) is positive.

4.7.2 Useful relations for $Z_{\mu_1 \dots \mu_n}^\alpha$ and $X_{\nu_2 \dots \nu_n}^{(n-1)}$

Here we list a few useful expressions:

$$Z_{\mu_1 \dots \mu_n}^\alpha = X_{\nu_2 \dots \nu_n}^{(n-1)} O_{\mu_1 \dots \mu_n}^{\alpha \nu_2 \dots \nu_n} \frac{2n-1}{n}, \quad (4.241)$$

$$\begin{aligned} Z_{\mu_1 \dots \mu_n}^\alpha(q)(-1)^n O_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} Z_{\nu_1 \dots \nu_n}^\beta(k) &= \frac{\alpha_n}{n^2} (-1)^n \\ &\times \left(\sqrt{k_\perp^2} \sqrt{q_\perp^2} \right)^{n-1} \left[g_{\alpha\beta}^\perp P'_n - \left(\frac{q_\alpha^\perp q_\beta^\perp}{q_\perp^2} + \frac{k_\alpha^\perp k_\beta^\perp}{k_\perp^2} \right) P_{n-1}'' \right. \\ &\left. + \frac{q_\alpha^\perp k_\beta^\perp}{\sqrt{k_\perp^2} \sqrt{q_\perp^2}} (P_{n-2}'' - 2P_{n-1}') + \frac{k_\alpha^\perp q_\beta^\perp}{\sqrt{k_\perp^2} \sqrt{q_\perp^2}} P_n'' \right], \end{aligned} \quad (4.242)$$

$$\begin{aligned} X_{\alpha\mu_1 \dots \mu_n}(q)(-1)^n O_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} X_{\beta\nu_1 \dots \nu_n}(k) &= \frac{\alpha_n}{(n+1)^2} (-1)^n \\ &\times \left(\sqrt{k_\perp^2} \sqrt{q_\perp^2} \right)^{n+1} \left[g_{\alpha\beta}^\perp P'_{n+1} - \left(\frac{q_\alpha^\perp q_\beta^\perp}{q_\perp^2} + \frac{k_\alpha^\perp k_\beta^\perp}{k_\perp^2} \right) P_{n+1}'' \right. \\ &\left. + \frac{q_\alpha^\perp k_\beta^\perp}{\sqrt{k_\perp^2} \sqrt{q_\perp^2}} (P_{n+2}'' - 2P_{n+1}') + \frac{k_\alpha^\perp q_\beta^\perp}{\sqrt{k_\perp^2} \sqrt{q_\perp^2}} P_n'' \right], \end{aligned} \quad (4.243)$$

$$\begin{aligned} Z_{\mu_1 \dots \mu_n}^\alpha(q^\perp)(-1)^n O_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} X_{\beta\nu_1 \dots \nu_n}(k) &= \frac{\alpha_{n-1}}{n(n+1)} (-1)^n \\ &\times (-k_\perp^2) \left(\sqrt{k_\perp^2} \sqrt{q_\perp^2} \right)^{n+1} \left[g_{\alpha\beta}^\perp P'_n - \frac{q_\alpha^\perp q_\beta^\perp}{q_\perp^2} P_{n-1}'' \right. \\ &\left. - \frac{k_\alpha^\perp k_\beta^\perp}{k_\perp^2} P_{n+1}'' + \frac{q_\alpha^\perp k_\beta^\perp}{\sqrt{k_\perp^2} \sqrt{q_\perp^2}} P_n'' + \frac{k_\alpha^\perp q_\beta^\perp}{\sqrt{k_\perp^2} \sqrt{q_\perp^2}} P_n'' \right]. \end{aligned} \quad (4.244)$$

Consider now a few expressions used in the one-loop diagram calculations. In our case, the operators are constructed of $X_{\alpha\mu_1 \dots \mu_n}^{(n+1)}$ and $Z_{\mu_1 \dots \mu_n}^\beta$, where α and β indices are to be convoluted with tensors. Let us start with the loop diagram with the Z -operator:

$$\int \frac{d\Omega}{4\pi} Z_{\mu_1 \dots \mu_n}^\alpha(k^\perp) T_{\alpha\beta} Z_{\nu_1 \dots \nu_n}^\beta(k^\perp) = \lambda O_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} (-1)^n. \quad (4.245)$$

For different tensors $T_{\alpha\beta}$, one has the following λ 's:

$$T_{\alpha\beta} = g_{\alpha\beta}, \quad \lambda = -\frac{\alpha_n}{n} |\vec{k}|^{2n-2}, \quad (4.246)$$

$$T_{\alpha\beta} = k_\alpha^\perp k_\beta^\perp, \quad \lambda = \frac{\alpha_n}{2n+1} |\vec{k}|^{2n}. \quad (4.247)$$

The equation (4.246) can be easily obtained using (4.241) and (4.236), while equation (4.247) can be obtained using (4.229) and (4.236). For the X operators, one has

$$\int \frac{d\Omega}{4\pi} X_{\alpha\mu_1\dots\mu_n}^{(n+1)}(k^\perp) T_{\alpha\beta} X_{\beta\nu_1\dots\nu_n}^{(n+1)}(k^\perp) = \lambda O_{\nu_1\dots\nu_n}^{\mu_1\dots\mu_n} (-1)^n, \quad (4.248)$$

where

$$\begin{aligned} T_{\alpha\beta} &= g_{\alpha\beta}, & \lambda &= -\frac{\alpha_n}{n+1} |\vec{k}|^{2n+2}, \\ T_{\alpha\beta} &= k_\alpha^\perp k_\beta^\perp, & \lambda &= \frac{\alpha_n}{2n+1} |\vec{k}|^{2n+4}. \end{aligned} \quad (4.249)$$

To derive (4.249), the properties of the projection operator

$$O_{\alpha\nu_1\dots\nu_n}^{\alpha\mu_1\dots\mu_n} = \frac{2n+3}{2n+1} O_{\nu_1\dots\nu_n}^{\mu_1\dots\mu_n} \quad (4.250)$$

and Eq. (4.229) are used. The interference term between X and Z operators is given by

$$\int \frac{d\Omega}{4\pi} X_{\alpha\mu_1\dots\mu_n}^{(n+1)}(k^\perp) T_{\alpha\beta} Z_{\nu_1\dots\nu_n}^\beta(k^\perp) = \lambda O_{\nu_1\dots\nu_n}^{\mu_1\dots\mu_n} (-1)^n, \quad (4.251)$$

with

$$\begin{aligned} T_{\alpha\beta} &= g_{\alpha\beta}, & \lambda &= 0, \\ T_{\alpha\beta} &= k_\alpha^\perp k_\beta^\perp, & \lambda &= -\frac{\alpha_n}{2n+1} |\vec{k}|^{2n+2}. \end{aligned} \quad (4.252)$$

Equation (4.252) is derived using (4.241) and the orthogonality (4.230) of the X operators.

4.8 Appendix 4.B. Vertices for Fermion–Antifermion States

Here we present a full set of operators for fermion–antifermion states. These operators are constructed of the angular momentum and spin operators. For fermion–antifermion operators we use the definition which differs from that for $\hat{Q}_{\mu_1\dots\mu_J}^{SLJ}(k)$ given in Section 4.1.2 by the dimension factor $\sim s^{L/2}$ — such a change is helpful for cumbersome loop calculations. Correspondingly, we also change the notations for these operators:

$$\begin{aligned} \hat{Q}_{\mu_1\dots\mu_J}^{S=0,L,J=L}(k) &\rightarrow V_{\mu_1\dots\mu_J}, & \hat{Q}_{\mu_1\dots\mu_J}^{S=1,L,J=L}(k) &\rightarrow V_{\mu_1\dots\mu_J}^{L=J}, \\ \hat{Q}_{\mu_1\dots\mu_J}^{S=1,L,J=L-1}(k) &\rightarrow V_{\mu_1\dots\mu_J}^{L>J}, & \hat{Q}_{\mu_1\dots\mu_J}^{S=1,L,J=L+1}(k) &\rightarrow V_{\mu_1\dots\mu_J}^{L<J}. \end{aligned} \quad (4.253)$$

4.8.1 Operators for ^1L_J states

For a singlet spin state, the total angular momentum J is equal to the orbital angular momentum L between two particles. The ground state of such a system is $^1S_0$ ($^{2S+1}L_J$) and the corresponding operator equals the spin-0 operator $i\gamma_5$. For states with orbital momentum L , the operator is constructed as a product of the spin-0 operator and the angular momentum operator $X_{\mu_1 \dots \mu_J}$:

$$V_{\mu_1 \dots \mu_J} = \sqrt{\frac{2J+1}{\alpha_J}} i\gamma_5 X_{\mu_1 \dots \mu_J}^{(J)}(k^\perp). \quad (4.254)$$

The normalisation factor introduced here simplifies the expression for the loop diagram.

4.8.2 Operators for ^3L_J states with $J=L$

The ground state in this series is $^3P_1$, so one should make a convolution of two vectors, Γ_μ and $X_\nu^{(1)}$, thus creating a $J=1$ state (a vector state). In this case, the vertex operator is equal to $\varepsilon_{\nu_1\eta\xi\gamma}\gamma_\eta k_\xi^\perp P_\gamma$. For states with higher orbital momenta, one needs to substitute $k_\xi^\perp$ by $X_{\xi\nu_2 \dots \nu_J}^{(J)}$ and perform a full symmetrisation over $\nu_1, \nu_2, \dots, \nu_J$ indices, which can be done by a convolution with the projection operator $O_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L}$. The general form of such a vertex is

$$V_{\mu_1 \dots \mu_J}^{L=J} \sim \varepsilon_{\nu_1\eta\xi\gamma}\gamma_\eta X_{\xi\nu_2 \dots \nu_J}^{(J)}(k^\perp) P_\gamma O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}. \quad (4.255)$$

Using equations (4.231) and (4.233), one has

$$\varepsilon_{\nu_1\eta\xi\gamma} X_{\xi\nu_2 \dots \nu_J}^{(J)}(k^\perp) O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J} = (2 - \frac{1}{J}) \varepsilon_{\nu_1\eta\xi\gamma} k_\xi^\perp X_{\nu_2 \dots \nu_J}^{(J-1)}(k^\perp) O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}. \quad (4.256)$$

Finally, making use of Eq. (4.241), the vertex operator can be written as:

$$V_{\mu_1 \dots \mu_J}^{L=J} = \sqrt{\frac{(2J+1)J}{(J+1)\alpha_J}} \frac{1}{\sqrt{s}} i\varepsilon_{\alpha\eta\xi\gamma}\gamma_\eta k_\xi^\perp P_\gamma Z_{\mu_1 \dots \mu_J}^\alpha(k^\perp), \quad (4.257)$$

where normalisation parameters are introduced. Note that, due to the property of antisymmetrical tensor $\varepsilon_{\alpha\eta\xi\gamma}$, the vertex given by (4.257) does not change if one replaces γ_η by a pure spin operator Γ_η .

4.8.3 Operators for ^3L_J states with $L < J$ and $L > J$

To construct operators for ^3L_J states, one should multiply the spin operator γ_α by the orbital momentum operator for $L = J+1$. So one has:

$$V_{\mu_1 \dots \mu_J}^{L < J} \sim \gamma_{\nu_1} X_{\nu_2 \dots \nu_J}^{(J-1)}(k^\perp) O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}. \quad (4.258)$$

Using (4.241), one can write the vertex operator in the form:

$$V_{\mu_1 \dots \mu_J}^{L < J} = \gamma_\alpha Z_{\mu_1 \dots \mu_J}^\alpha(k^\perp) \sqrt{\frac{J}{\alpha_J}}, \quad (4.259)$$

and for the pure spin operator:

$$\tilde{V}_{\mu_1 \dots \mu_J}^{L < J} = \Gamma_\alpha Z_{\mu_1 \dots \mu_J}^\alpha(k^\perp) \sqrt{\frac{J}{\alpha_J}}. \quad (4.260)$$

The normalisation constant is chosen to facilitate the calculation of loop diagrams containing such a vertex.

To construct such an operator for $L > J$, one should reduce the number of indices in the orbital operator by convoluting it with the spin operator:

$$V_{\mu_1 \dots \mu_J}^{L > J} = \gamma_\alpha X_{\alpha \mu_1 \dots \mu_J}(k^\perp) \sqrt{\frac{J+1}{\alpha_J}}. \quad (4.261)$$

For a pure spin state we have:

$$\tilde{V}_{\mu_1 \dots \mu_J}^{L > J} = \Gamma_\alpha X_{\alpha \mu_1 \dots \mu_J}(k^\perp) \sqrt{\frac{J+1}{\alpha_J}}. \quad (4.262)$$

4.9 Appendix 4.C. Spectral Integral Approach with Separable Vertices: Nucleon–Nucleon Scattering Amplitude $NN \rightarrow NN$, Deuteron Form Factors and Photodisintegration and the Reaction $NN \rightarrow N\Delta$

As was said above, the spectral integration technique has an advantage allowing us to describe the two-particle reactions in a relativistically invariant way. The vertices obtained within such a method permit us to perform the calculations of electromagnetic processes in a gauge invariant form. Henceforth we strictly control the content of the considered systems. Dealing with two-particle systems, we do not meet additional multiparticle virtual states (as it happens in the Bethe–Salpeter equation) and in case of the high-spin states there are no “animal-like” diagrams also inherent in the Bethe–Salpeter equation.

Still, the approach of separable vertices applied to a partial amplitude has a disadvantage: it does not provide us with a correct result when performing the analytical continuation of the amplitude to the left-hand cut. But this is a deficiency common to all available methods. We may only hope that in the future one will be able to carry out a correct bootstrap procedure, thus reconstructing partial amplitudes simultaneously in the right

half-plane s (in the physical region and its vicinity) and in the left-hand one (in the region of “forces” of t - and u -channel exchanges). But at present, and it is not out of place to emphasise this once more, the description of the left-hand cut cannot pretend to result in the high accuracy calculations, first, because of the arbitrary choice of form factors at the exchange of a particle and, second, owing to multiparticle exchanges. At this point the methods of the partial amplitude description in the left half-plane using both t - and u -channel exchanges and separable vertices are equivalent.

In this section, without going into technicalities which may be found in [3, 4, 10], we give the description of the $NN \rightarrow NN$ reactions in the energy region < 1 GeV [2]. The description of these reactions allows us to reconstruct the NN vertices. At the same time the NN vertices make it possible to describe the deuteron form factors [3] and photodisintegration reactions $\gamma d \rightarrow pn$ [4]. Finally, we present the results for the reaction $NN \rightarrow N\Delta$ [10] that allowed us to conclude about the absence of dibaryon resonances in the mass region 2–3 GeV.

Generally, using two-baryon reactions as an example, we demonstrate the workability of the spectral integration method with separable vertices.

4.9.1 The $pp \rightarrow pp$ and $pn \rightarrow pn$ scattering amplitudes

The fitting procedures performed in [2, 3, 4] differ from each other in some points but have two common features:

- (i) For the description of interaction forces in the NN system, the right and left vertices were introduced with either the same signs (repulsion forces) or opposite signs (attraction forces).
- (ii) The function $f_{n_j}^j(s)$ in (4.110) was chosen in such a way that the loop diagram $B_{iaj}(s)$, Eq. (4.107), has only two types of singularities: a threshold singularity $\sqrt{s' - 4m^2}$ and pole singularities $(s' - s_{jm})^{-1}$ and $(s' - s)^{-1}$. That is, the left-hand side cut in these loops is described by a set of pole terms only. The loop diagram $B_{iaj}(s)$ can be explicitly calculated and its parameters are suitable for the fitting procedure.

Results of the fit obtained in [2] are shown in Fig. 4.12.

A more detailed information on the parameters of the scattering amplitudes may be found in [2].

4.9.1.1 Deuteron form factors

The developed technique was applied to the description of the deuteron electromagnetic form factors, $A(Q^2)$ and $B(Q^2)$ [3]. Based on phase-shift

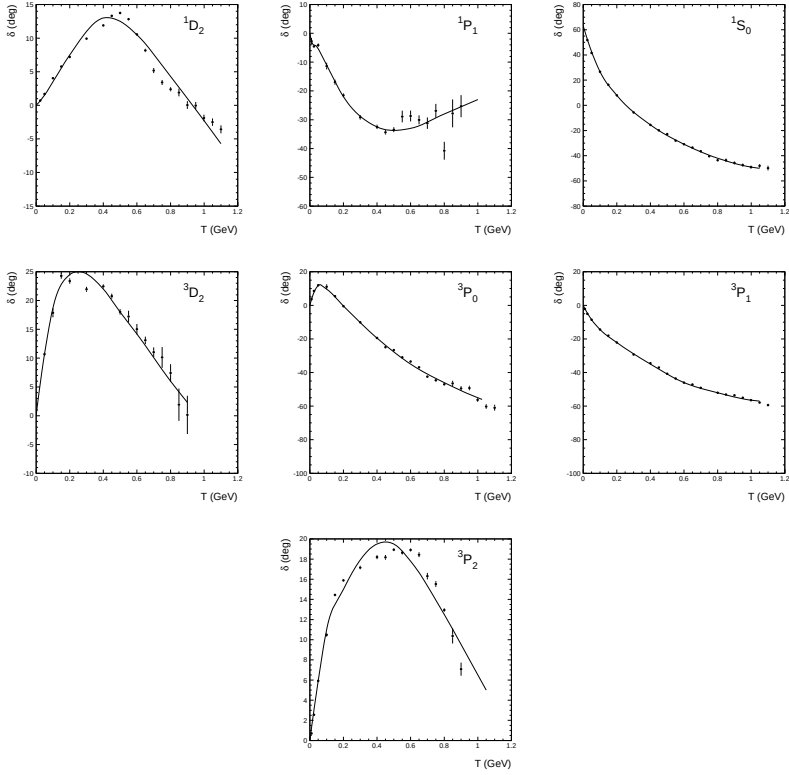


Fig. 4.12 Waves NN -scattering phase shifts at energies $T_{\text{lab}} < 1.0 \text{ GeV}$, and their fit in terms spectral integral technique with separable vertices [2].

data for np scattering at energies $T_{\text{lab}} < 0.8 \text{ GeV}$, vertices (or “wave functions”) for S - and D -wave states were constructed which give correct values for the binding energy, the magnetic moment and the quadrupole moment. These vertices lead to the reasonably good description of the form factors $A(Q^2)$ and $B(Q^2)$ in the regions $Q^2 \leq 1.2 \text{ GeV}^2/c^2$.

In a more detailed form the deuteron–photon interaction amplitude has the following structure:

$$\begin{aligned}
 A_\mu = & -e \left[G_1(-q^2)(P' + P)_\mu g_{\tau\eta} + G_2(-q^2)(q_\eta g_{\mu\tau} - q_\tau g_{\mu\eta}) \right. \\
 & \left. - \frac{G_3(-q^2)}{2M^2} (P' + P)_\mu q_\tau q_\eta \right] \varepsilon_\tau^* \varepsilon_\eta .
 \end{aligned} \tag{4.263}$$

Here P and P' are the incoming and outgoing deuteron momenta, and

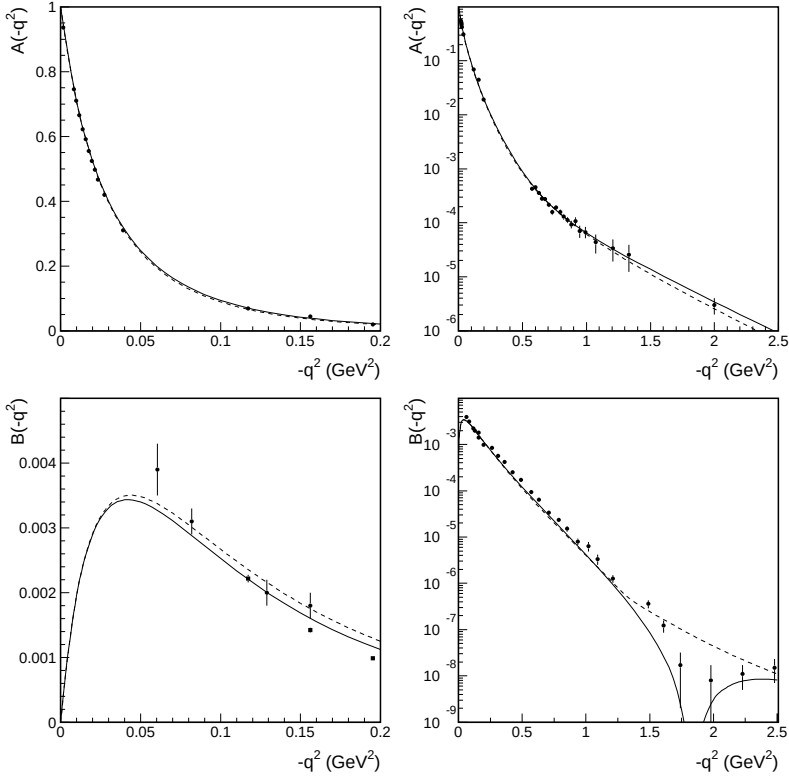


Fig. 4.13 Deuteron form factors $A(-q^2)$ and $B(-q^2)$ versus experimental data [26, 27, 28, 29, 30]. Dashed and solid lines correspond to different fits of scattering amplitudes (and, correspondingly, different deuteron vertices) allowed by error bars in the data.

correspondingly, $-q^2 = Q^2$ is the photon momentum squared, and G_i are the deuteron form factors.

The form factors G_1, G_2 and G_3 are connected with the conventional electric (G_e), magnetic (G_m) and quadrupole (G_Q) form factors as follows:

$$\begin{aligned}
 G_e &= G_1 - \frac{q^2}{6M^2} G_Q, & G_m &= G_2, \\
 G_Q &= G_1 - G_2 + G_3 \left(1 - \frac{q^2}{4M^2} \right).
 \end{aligned}
 \tag{4.264}$$

A comparison of the experimentally measured form factors $A(-q^2)$ and $B(-q^2)$ with experimental data [26, 27, 28, 29, 30] is shown in Fig. 4.13.

We use the following definitions:

$$\begin{aligned}
 A(-q^2) &= G_e^2(-q^2) + 2 \left(\frac{q^2}{6M^2} \right)^2 G_Q^2(-q^2) - \frac{q^2}{6M^2} G_m^2(-q^2), \\
 B(-q^2) &= -\frac{q^2}{6M^2} \left(1 - \frac{q^2}{4M^2} \right) G_m(-q^2).
 \end{aligned} \tag{4.265}$$

In Fig. 4.13 two versions of the form factor fits are presented (dotted and solid curves correspond to versions I and II). They correspond to a freedom in the choice of vertices in the description of data on NN scattering: one can see that experimental error bars are not small, in particular, at $T_{lab} \sim 0.7 - 1.0$ GeV. However, the versions I and II provide close results for $A(-q^2)$ and $B(-q^2)$ at small $|q^2|$, and they differ essentially only at $|q^2| \sim 1$ GeV².

The binding energy, the magnetic moment and the quadrupole moment and the D -wave probability are for cases I and II:

	Case I	Case II	Experiment	
μ_D	$1.719 \mu_B$	$1.709 \mu_B$	$1.715 \mu_B$	
Q_D	25.4 e/fm^2	25.0 e/fm^2	25.5 e/fm^2	(4.266)
ε	2.222 MeV	2.222 MeV	2.222 MeV	
D -wave probability	4%	5%		

We conclude:

In the framework of the dispersion integration over the composite particle mass we have carried out the analysis of the nucleon-nucleon scattering amplitude. The structure of the NN scattering partial amplitude operators was considered. Using the phase shift analysis data the vertex functions of the $^1S_0$, $^3P_0$, $^1P_1$, $^3P_1$, $^3P_2$, $^1D_2$, $^3D_2$ states were reconstructed neglecting the contribution of inelastic channels. Of course, this approximation is valid only at such energies where inelastic corrections are small. This vertex function can be used for the investigation of the deuteron photodisintegration as well as for other processes, few-nucleon system involved.

We see several ways for the development and improvement of this calculation. It would be interesting to compare our approach with some dynamical models which are used for the description of nucleon-nucleon interactions (for example, one-meson exchange models). Here the N function is obtained using dispersion integration in the physical region ($s > 4m^2$) and the analytical continuation into the region of the left-hand cut should be made. This procedure can be carried out after extracting the contribution of the meson exchange from the N function. The other field of activity is

the calculation of the contribution of channels in the $n - p$ scattering amplitude, in particular, in the channels with the production of Δ resonance. The results presented here can be taken as a basis for the multi-channel calculations. A correct calculation of these processes is important for the description of photo- and electrodisintegration of the deuteron at the energies near the Δ threshold.

4.9.1.2 Deuteron disintegration

The determined deuteron vertices (or deuteron form factors) allow us to calculate the deuteron disintegration reaction $\gamma d \rightarrow pn$. However, one should keep in mind that the pn system created in the final state may be in the isotopic states $I = 0$ and $I = 1$. So the rescattering processes may occur in these two states. Besides, one should take into account that the process $\gamma d \rightarrow np$ in the state $I = 1$ can go in two stages: $\gamma d \rightarrow N\Delta \rightarrow np$ (remind that the mass of Δ is close to the nucleon mass, $m_\Delta = 1240$ MeV). Thus, for the calculation of the reaction $\gamma d \rightarrow np$ we must consider the process $N\Delta \rightarrow np$ too. The reaction $\gamma d \rightarrow np$ at comparatively large E_γ (of the order of 300–400 MeV) may be also affected by the processes $\gamma d \rightarrow NN^*(1440) \rightarrow np$ and $\gamma d \rightarrow NN\pi \rightarrow np$.

The analysis of the reaction $\gamma d \rightarrow np$ in terms of the spectral integration technique has been carried out in [4]. The np rescatterings were accounted for in the waves $^1S_0$, $^3S_0$, $^3D_1$, $^3P_0$, $^3P_1$, $^3P_2$, $^1D_2$, $^3D_2$ and $^3F_3$. The transitions $N\Delta \rightarrow np$ were also calculated for the waves with $I = 1$.

The description of data is shown in Figs. 4.14, 4.15. It turned out that the process $\gamma d \rightarrow pn \rightarrow np$ is important for the waves $^1S_0$, $^3P_0$, $^3P_1$ at $E_\gamma = 50 - 100$ MeV, while $\gamma d \rightarrow N\Delta \rightarrow np$ dominates for the waves $^3P_2$, $^1D_2$, $^3F_3$ at $E_\gamma > 300$ MeV.

As a whole, the description of the reaction $\gamma d \rightarrow pn \rightarrow np$ is rather satisfactory at $E_\gamma \leq 100$ MeV. At larger E_γ the description fails; this probably reflects the more complicated character of the process at higher energies. Indeed, the inelastic processes such as $\gamma d \rightarrow NN^*(1440) \rightarrow np$ and $\gamma d \rightarrow NN\pi \rightarrow np$ play a considerable role requiring detailed investigations.

4.9.1.3 Reaction $NN \rightarrow N\Delta$ at energies $T_N \leq 1.5$ GeV

The investigation of the reaction $NN \rightarrow N\Delta$ within the spectral integration technique is interesting from two points of view:

(i) This reaction is important in the analysis of processes such as deuteron

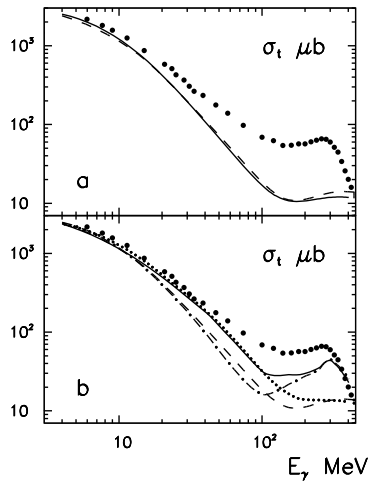


Fig. 4.14 Total cross sections versus data [31, 32, 33, 34, 35, 36, 37, 38]. a) Contribution of the impulse approximation diagram (dashed line) and that with final state rescatterings taken into account (solid line). b) Contributions after accounting for the inelastic intermediate states: in the waves $^1D_2, ^3P_2, ^3F_3$ (dot-dashed line), in the waves $^1S_0, ^3P_0, ^3P_1$ (small-dot line), total cross section with all corrections taken into account is shown by solid line.

photoproduction (see the preceding section).

(ii) It is also essential for the solution of the dibaryon resonance problem: dibaryon resonances in the $N\Delta$ channel may reveal themselves clearly while in the NN channel they are not seen.

In this reaction one should distinguish between two regions, namely, the energy region close to the $N\Delta$ threshold and that above the $N\Delta$ threshold.

At energies near the threshold, $E_{cm} = \sqrt{s} \sim 2200$ MeV, the processes leading to anomalous singularities near the physical region are important, see Section 4.3.2. Apart for the reactions discussed here, see Figs. 4.4 and 4.6, there is a set of diagrams with the chain of transitions of the type $N\Delta \rightarrow N\pi\pi \rightarrow N\Delta \rightarrow N\pi\pi \rightarrow N\Delta$, which also contain singular terms. But, as it is shown for the simplest cases in Figs. 4.4 and 4.6, these singularities are near the threshold and they are not important for the discussed problem of dibaryon resonances.

The study of near-threshold diagrams requires detailed calculations of diagrams of Figs. 4.4 and 4.6. The analysis of the reaction $NN \rightarrow N\Delta$ above the threshold may be carried out with the help of the standard spectral integration technique with separable vertices. Such an analysis at en-

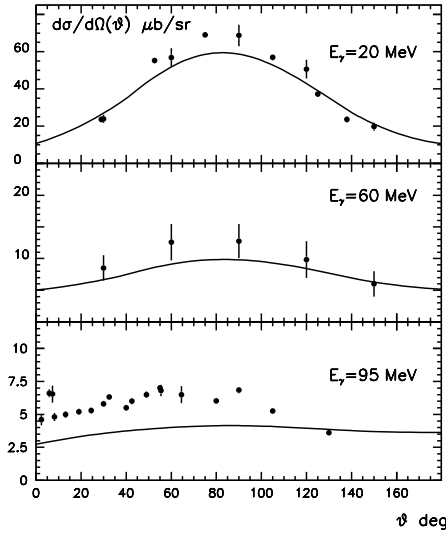


Fig. 4.15 Differential cross sections $d\sigma/d\Omega(\theta)$ at $E_\gamma = 20$ MeV, $E_\gamma = 60$ MeV, and $E_\gamma = 95$ MeV versus data [38, 39, 40, 41, 42].

ergies $\sqrt{s} \sim 2300 - 2700$ MeV has been performed in [10].

The results of the analysis of different waves $NN(^{2S+1}L_J) \rightarrow N\Delta(^{2S'+1}L'_J)$, namely, $NN(^1D_2) \rightarrow N\Delta(^5S_2)$, $NN(^3F_3) \rightarrow N\Delta(^6P_3)$, $NN(^3P_2) \rightarrow N\Delta(^5P_2)$ are shown in Fig. 4.16 and in the following equation (above the line we give T_N values in MeV units):

Wave	Amplitude	492	576	643	729	796
$^1D_2$	$1 - A_{pp \rightarrow pp} ^2$	0.266	0.431	0.539	0.561	0.553
	$ A_{pp \rightarrow d\pi^+} ^2$	0.137	0.192	0.159	0.103	0.066
	$ A_{pp \rightarrow N\Delta} ^2$	0.104	0.288	0.429	0.481	0.510
$^3F_3$	$1 - A_{pp \rightarrow pp} ^2$	0.050	0.141	0.399	0.596	0.614
	$ A_{pp \rightarrow d\pi^+} ^2$	0.010	0.036	0.056	0.052	0.035
	$ A_{pp \rightarrow N\Delta} ^2$	0.008	0.118	0.266	0.429	0.498
$^3P_2$	$1 - A_{pp \rightarrow pp} ^2$	0.028	0.157	0.342	0.472	0.569
	$ A_{pp \rightarrow d\pi^+} ^2$	0.008	0.019	0.031	0.034	0.028
	$ A_{pp \rightarrow N\Delta} ^2$	0.055	0.094	0.347	0.401	0.433

(4.267)

The main results are as follows: the vertices for reaction $NN \rightarrow N\Delta$ were restored, that gives the possibility to investigate amplitudes in

complex- s plane. The analysis of the complex plane has demonstrated the absence of poles, *i.e.* the absence of dibaryon resonances in the studied energy region.

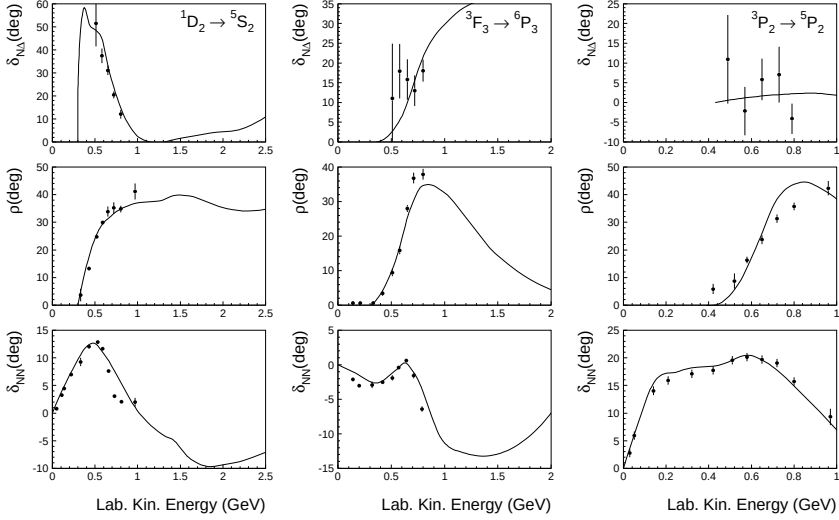


Fig. 4.16 Reaction $NN \rightarrow N\Delta$: description of the partial amplitudes for the transitions $^1D_2 \rightarrow ^5S_2$, $^3F_3 \rightarrow ^6P_3$, $^3P_2 \rightarrow ^5P_2$, (data and the calculated curves from [10], $\eta_{pp} = \cos \rho$).

4.10 Appendix D. $N\Delta$ One-Loop Diagrams

The vertex for the transition of a state with total spin J into a $3/2^+$ particle with the momentum k_1 and a $1/2^+$ particle with the momentum of k_2 has the general form

$$\bar{\psi}_\alpha(k_1) V_{\mu_1 \dots \mu_J}^{(i)\alpha}(k^\perp) u(-k_2), \quad (4.268)$$

where ψ_α is the vector spinor for a spin-3/2 particle, $V_{\mu_1 \dots \mu_J}^{(i)\alpha}$ is the vertex operator and $k^\perp$ is the relative momentum of the final particles orthogonal to their total momentum.

The spin-3/2 and -1/2 particles can form two spin states, $S = 1$ and $S = 2$. Let us start from the $S = 1$ states. Here we have three sets of

operators with $J = L - 1$, $J = L$ and $J = L + 1$:

$$\begin{aligned} V_{\mu_1 \dots \mu_J}^{(1)\alpha}(k^\perp) &= i\gamma_5 X_{\alpha\mu_1 \dots \mu_J}^{(J+1)}(k^\perp) & J = L - 1, \\ V_{\mu_1 \dots \mu_J}^{(2)\alpha}(k^\perp) &= i\gamma_5 X_{\beta_2 \dots \beta_J}^{(J-1)}(k^\perp) O_{\mu_1 \dots \mu_J}^{\alpha\beta_2 \dots \beta_J} & J = L + 1, \\ V_{\mu_1 \dots \mu_J}^{(3)\alpha}(k^\perp) &= \gamma_5 \varepsilon_{\beta_1 \alpha \xi \eta} k_\xi P_\eta X_{\beta_2 \dots \beta_J}^{(J-1)}(k^\perp) O_{\mu_1 \dots \mu_J}^{\beta_1 \dots \beta_J} & J = L, \end{aligned} \quad (4.269)$$

where the projection operator is needed for index symmetrisation. The operators with $S = 1$ and $J = \pm 1$ describe the decay of the particles with quantum numbers $0^-, 1^+, 2^-, 3^+, \dots$ and the operators with $S = 1$ and $J = L$ the particles $(1^-, 2^+, 3^-, 4^+ \dots)$.

In case of $S = 2$, there are five operators with $L - 2 \leq J \leq L + 2$ operators:

$$\begin{aligned} V_{\mu_1 \dots \mu_J}^{(4)\alpha}(k^\perp) &= \gamma_\beta O_{\nu_1 \nu_2}^{\alpha\beta} X_{\nu_3 \dots \nu_J}^{(J-2)}(k^\perp) O_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J} & J = L + 2, \\ V_{\mu_1 \dots \mu_J}^{(5)\alpha}(k^\perp) &= \gamma_\beta X_{\alpha\beta\nu_1 \dots \nu_J}^{(J+2)}(k^\perp) & J = L - 2, \\ V_{\mu_1 \dots \mu_J}^{(6)\alpha}(k^\perp) &= \gamma_\beta O_{\alpha\beta}^{\nu_1 \xi} X_{\xi\nu_2 \dots \nu_J}^{(J)}(k^\perp) O_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J} & J = L, \\ V_{\mu_1 \dots \mu_J}^{(7)\alpha}(k^\perp) &= i\varepsilon_{\nu_1 \beta \tau \eta} k_\tau P_\eta O_{\beta\xi}^{\alpha\chi} \gamma_\chi X_{\xi\nu_2 \dots \nu_J}^{(J)}(k^\perp) O_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J} & J = L - 1, \\ V_{\mu_1 \dots \mu_J}^{(8)\alpha_1}(k^\perp) &= i\varepsilon_{\nu_1 \beta \tau \eta} k_\tau P_\eta O_{\beta\nu_2}^{\alpha\chi} \gamma_\chi X_{\nu_3 \dots \nu_J}^{(J-2)}(k^\perp) O_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J} & J = L + 1. \end{aligned} \quad (4.270)$$

The operators with $s = 2$ and $J = L + 2, L, L - 2$ describe the decay of the particles with quantum numbers $0^+, 1^-, 2^+, 3^-, \dots$ and the operators with $S = 2$ and $J = L \pm 1$ the particles $(1^+, 2^-, 3^-, 4^+ \dots)$.

The calculation of the one-loop diagram for different vertex operators is an important step in the construction of the unitary $N\Delta$ amplitude. Let us define the loop diagram with two vertices $V_{\mu_1 \dots \mu_J}^{(i)}$ and $V_{\nu_1 \dots \nu_J}^{(m)}$ as:

$$\begin{aligned} &W_J^{(im)} O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J} (-1)^J \\ &= \int \frac{d\Omega}{4\pi} Sp \left[V_{\mu_1 \dots \mu_J}^{(i)\alpha}(m_1 + \hat{k}_1) \left(g_{\alpha\beta}^{\perp k_1} - \frac{\gamma_\alpha^{\perp k_1} \gamma_\beta^{\perp k_1}}{3} \right) V_{\nu_1 \dots \nu_J}^{(m)\beta}(m_2 - \hat{k}_2) \right]; \end{aligned} \quad (4.271)$$

here m_1 and m_2 are masses of Δ and the nucleon, respectively.

Using the expression

$$\begin{aligned} &Sp \left[i\gamma_5(m_1 + \hat{k}_1) \left(g_{\alpha\beta}^{\perp k_1} - \frac{\gamma_\alpha^{\perp k_1} \gamma_\beta^{\perp k_1}}{3} \right) i\gamma_5(m_2 - \hat{k}_2) \right] \\ &= -\frac{4}{3} \left(g_{\alpha\beta} - \frac{k_\alpha^\perp k_\beta^\perp}{m_1^2} \right) (s - \delta^2), \end{aligned} \quad (4.272)$$

where $\delta = m_1 - m_2$, the one-loop diagram for the operators with $S = 1$

and $J = L \pm 1$ is given by

$$\begin{aligned} W_J^{(11)} &= \frac{4}{3}(s - \delta^2) \frac{\alpha_J}{J+1} \left(1 + \frac{|\vec{k}|^2(J+1)}{m_1^2(2J+1)}\right) |\vec{k}|^{2J+2}, \\ W_J^{(22)} &= \frac{4}{3}(s - \delta^2) \frac{\alpha_{J-1}}{2J-1} \left(1 + \frac{|\vec{k}|^2 J}{m_1^2(2J+1)}\right) |\vec{k}|^{2J-2}, \\ W_J^{(33)} &= \frac{4}{3}(s - \delta^2) s \alpha_{J-1} \frac{J+1}{4J^2-1} |\vec{k}|^{2J}. \end{aligned} \quad (4.273)$$

The transition loop diagrams between ${}^3L_J$ ($J = L-1$) and ${}^3L_J$ ($J = L+1$) states do not vanish due to the term proportional to $k_\alpha^\perp k_b^\perp$ in (4.272):

$$W_J^{(12)} = -\frac{4}{3}(s - \delta^2) \frac{\alpha_{J-1}}{2J+1} \frac{|\vec{k}|^{2J+2}}{m_1^2}. \quad (4.274)$$

One can introduce the pure spin operators also in a way that the transition loop diagram equals zero. Then Eqs.(4.269)–(4.269) can be rewritten as:

$$\tilde{V}_{\mu_1 \dots \mu_J}^{(i)\beta} = \Gamma_{\alpha\beta}^{3/2} V_{\mu_1 \dots \mu_J}^{(i)\beta} \quad (4.275)$$

where

$$\Gamma_{\alpha\beta}^{3/2} = g_{\alpha\beta} + \frac{4s k_\alpha^\perp k_\beta^\perp}{(s + M\delta)(\sqrt{s} + M)(\sqrt{s} + \delta)}. \quad (4.276)$$

The trace of the $N\Delta$ loop diagram with the $i\gamma_5 \Gamma_{\alpha\beta}^{3/2}$ vertex is equal to:

$$\begin{aligned} &\text{Sp} \left[i\gamma_5 (m_1 + \hat{k}_1) \Gamma_{\alpha\alpha'}^{3/2} \left(g_{\alpha'\beta'}^\perp - \frac{\gamma_{\alpha'}^\perp k_1 \gamma_{\beta'}^\perp k_1}{3} \right) \Gamma_{\beta\beta'}^{3/2} i\gamma_5 (m_2 - \hat{k}_2) \right] = \\ &= -\frac{4}{3} g_{\alpha\beta} (s - \delta^2). \end{aligned} \quad (4.277)$$

and the W_J^{12} function with vertices (4.275) vanishes identically.

To calculate loop diagrams with $S = 2$, the following expression is used:

$$\begin{aligned} &O_{\mu_1\mu_2}^{\alpha_1\alpha_2} \int \frac{d\Omega}{4\pi} \text{Sp} \left[\gamma_{\mu_1} (m_1 + \hat{k}_1) \left(g_{\mu_2\nu_1}^\perp - \frac{\gamma_{\mu_2}^\perp k_1 \gamma_{\nu_1}^\perp k_1}{3} \right) \gamma_{\nu_2} (m_2 - \hat{k}_2) \right] O_{\beta_1\beta_2}^{\nu_1\nu_2} \\ &= a_1 O_{\beta_1\beta_2}^{\alpha_1\alpha_2} + a_2 Z_{\alpha_1\alpha_2}^\xi Z_{\beta_1\beta_2}^\xi + a_3 X_{\alpha_1\alpha_2}^{(2)} X_{\beta_1\beta_2}^{(2)}, \end{aligned} \quad (4.278)$$

where

$$\begin{aligned} a_1 &= 2(s - \delta^2), \quad a_2 = \frac{32\delta}{9m_1} - \frac{16}{27m_1^2} (s - (m_1 + m_2)^2), \\ a_3 &= -\frac{64}{27m_1^2}. \end{aligned} \quad (4.279)$$

Then the one-loop diagrams for states with $S = 2$ and $J = L + 2, L, L - 2$ are:

$$\begin{aligned}
W_J^{(44)} &= \frac{\alpha_{J-2}}{2J-3} |\vec{k}|^{2J-4} \left(a_1 + \frac{9(J-1)}{4(2J-1)} (-a_2 |\vec{k}|^2 + a_3 \frac{J}{2J+1} |\vec{k}|^4) \right), \\
W_J^{(55)} &= \alpha_J |\vec{k}|^{2J+4} \left(\frac{(2J+3)a_1}{(J+1)(J+2)} + \frac{9}{4} \left(-\frac{a_2 |\vec{k}|^2}{J+1} + \frac{a_3 |\vec{k}|^4}{2J+1} \right) \right), \\
W_J^{(45)} &= \frac{9}{4} \frac{\alpha_{J-2}}{2J+1} a_3 |\vec{k}|^{2J+4}, \\
W_J^{(46)} &= \frac{3\alpha_{J-2}(J+1)}{8(2J+1)(2J-1)} |\vec{k}|^{2J} \left(\frac{2J+3}{J} a_2 - 2 |\vec{k}|^2 a_3 \right), \\
W_J^{(56)} &= \frac{3\alpha_J}{8(2J+1)} |\vec{k}|^{2J+4} \left(a_2 - 2 |\vec{k}|^2 a_3 \frac{J+1}{2J-1} \right), \\
W_J^{(66)} &= \frac{\alpha_{J-1}}{2J(2J+1)} |\vec{k}|^{2J} \left[\frac{(2J+3)(J+1)a_1}{3J} - \frac{9}{8} |\vec{k}|^2 a_2 \left(\frac{2J+5}{9} \right. \right. \\
&\quad \left. \left. + \frac{2J+1}{J(2J-1)} \right) + \frac{a_3 |\vec{k}|^4 (J+1)^2}{2(2J-1)} \right]; \tag{4.280}
\end{aligned}$$

for states with $S = 2$ and $J = L \pm 1$ we have

$$\begin{aligned}
W_J^{(77)} &= \frac{s\alpha_{J-1}}{2(2J+1)} |\vec{k}|^{2J+2} \left(\frac{a_1(J+1)(2J^2+J-2)}{J^2(2J-1)} - \frac{9}{8} |\vec{k}|^2 a_2 \frac{J+1}{2J-1} \right), \\
W_J^{(88)} &= \frac{s\alpha_{J-2}(J+1)}{2(2J-1)(2J-3)} |\vec{k}|^{2J-2} \left(a_1 - \frac{9}{8} |\vec{k}|^2 a_2 \frac{J-1}{2J+1} \right), \\
W_J^{(78)} &= \frac{s\alpha_{J-2}}{4J^2-1} |\vec{k}|^{2J} \left(\frac{J+1}{J} a_1 + \frac{9}{16} |\vec{k}|^2 a_2 (J+1) \right), \tag{4.281}
\end{aligned}$$

4.11 Appendix 4.E. Analysis of the Reactions

$p\bar{p} \rightarrow \pi\pi, \eta\eta, \eta\eta'$: Search for f_J -Mesons

The partial wave analysis of the reactions $p\bar{p} \rightarrow \pi\pi, \eta\eta, \eta\eta'$ over the region of invariant masses 1900–2400 MeV indicates the existence of four relatively narrow tensor–isoscalar resonances $f_2(1920)$, $f_2(2020)$, $f_2(2240)$, $f_2(2300)$ and the broad state $f_2(2000)$. The decay couplings of the broad resonance $f_2(2000) \rightarrow \pi^0\pi^0, \eta\eta, \eta\eta'$ satisfy relations which correspond to those of the tensor glueball, while the couplings of other tensor states do not, thus verifying the glueball nature of $f_2(2000)$.

In [8, 43] a combined partial wave analysis was performed for the high statistics data in the reactions $p\bar{p} \rightarrow \pi^0\pi^0, \eta\eta, \eta\eta'$ at antiproton momenta 600, 900, 1150, 1200, 1350, 1525, 1640, 1800 and 1940 MeV/c together

with data obtained for polarised target in the reaction $\bar{p}p \rightarrow \pi^+\pi^-$ [44] that resulted in the determination of a number of isoscalar resonances f_J with $J = 0, 2, 4$ (to review, see [45, 46, 47]). To describe the data on $p\bar{p} \rightarrow \pi^0\pi^0, \eta\eta, \eta\eta'$ in the 02^{++} -sector, five states are required [8, 43]:

Resonance	Mass(MeV)	Width(MeV)
$f_2(1920)$	1920 ± 30	230 ± 40
$f_2(2000)$	2010 ± 30	495 ± 35
$f_2(2020)$	2020 ± 30	275 ± 35
$f_2(2240)$	2240 ± 40	245 ± 45
$f_2(2300)$	2300 ± 35	290 ± 50 .

(4.282)

The resonance $f_2(1920)$ was observed earlier in the $\omega\omega$ [48, 49, 50] and $\eta\eta'$ [51, 52] spectra, respectively; see also the compilation [53]. For the broad tensor-isoscalar resonance recent analyses give in the region around 2000 MeV: $M = 1980 \pm 20$ MeV, $\Gamma = 520 \pm 50$ MeV in $pp \rightarrow pp\pi\pi\pi\pi$ [54] and $M = 2050 \pm 30$ MeV, $\Gamma = 570 \pm 70$ MeV in $\pi^-p \rightarrow \phi\phi n$ [55]. Following [8, 43, 56], we denote the broad resonance as $f_2(2000)$. The description of the data in the reactions $p\bar{p} \rightarrow \pi^0\pi^0, \eta\eta, \eta\eta'$ is illustrated by Fig. 4.17. In Figs. 4.18 and 4.19, one can see the differential cross sections $p\bar{p} \rightarrow \pi^+\pi^-$. Fig. 4.20 presents the polarisation data. In Fig. 4.21 we show cross sections for $p\bar{p} \rightarrow \pi^0\pi^0, \eta\eta, \eta\eta'$ in the $^3P_2\bar{p}p$ and $^3F_2\bar{p}p$ waves (dashed and dotted curves) and the total ($J = 2$) cross section (solid curve) as well as the Argand-plots for the $^3P_2$ and $^3F_2$ wave amplitudes at invariant masses $M = 1.962, 2.050, 2.100, 2.150, 2.200, 2.260, 2.304, 2.360, 2.410$ GeV.

Direct arguments in favour of the glueball nature of $f_2(2000)$ are provided by inter-relations of the decay coupling constants — such relations are presented in [43]. In [8, 45], the extraction of the decay couplings $f_J \rightarrow \pi\pi, \eta\eta, \eta\eta'$ is not performed — in the paper [43] this gap is filled. The $\bar{p}p \rightarrow \pi^0\pi^0, \eta\eta, \eta\eta'$ amplitudes provide us with the following ratios for the f_2 resonance couplings, $g_{\pi^0\pi^0} : g_{\eta\eta} : g_{\eta\eta'}$:

$$\begin{aligned}
 g_{\pi^0\pi^0}[f_2(1920)] : g_{\eta\eta}[f_2(1920)] : g_{\eta\eta'}[f_2(1920)] &= 1 : 0.56 \pm 0.08 : 0.41 \pm 0.07 \\
 g_{\pi^0\pi^0}[f_2(2000)] : g_{\eta\eta}[f_2(2000)] : g_{\eta\eta'}[f_2(2000)] &= 1 : 0.82 \pm 0.09 : 0.37 \pm 0.22 \\
 g_{\pi^0\pi^0}[f_2(2020)] : g_{\eta\eta}[f_2(2020)] : g_{\eta\eta'}[f_2(2020)] &= 1 : 0.70 \pm 0.08 : 0.54 \pm 0.18 \\
 g_{\pi^0\pi^0}[f_2(2240)] : g_{\eta\eta}[f_2(2240)] : g_{\eta\eta'}[f_2(2240)] &= 1 : 0.66 \pm 0.09 : 0.40 \pm 0.14 \\
 g_{\pi^0\pi^0}[f_2(2300)] : g_{\eta\eta}[f_2(2300)] : g_{\eta\eta'}[f_2(2300)] &= 1 : 0.59 \pm 0.09 : 0.56 \pm 0.17.
 \end{aligned}$$
(4.283)

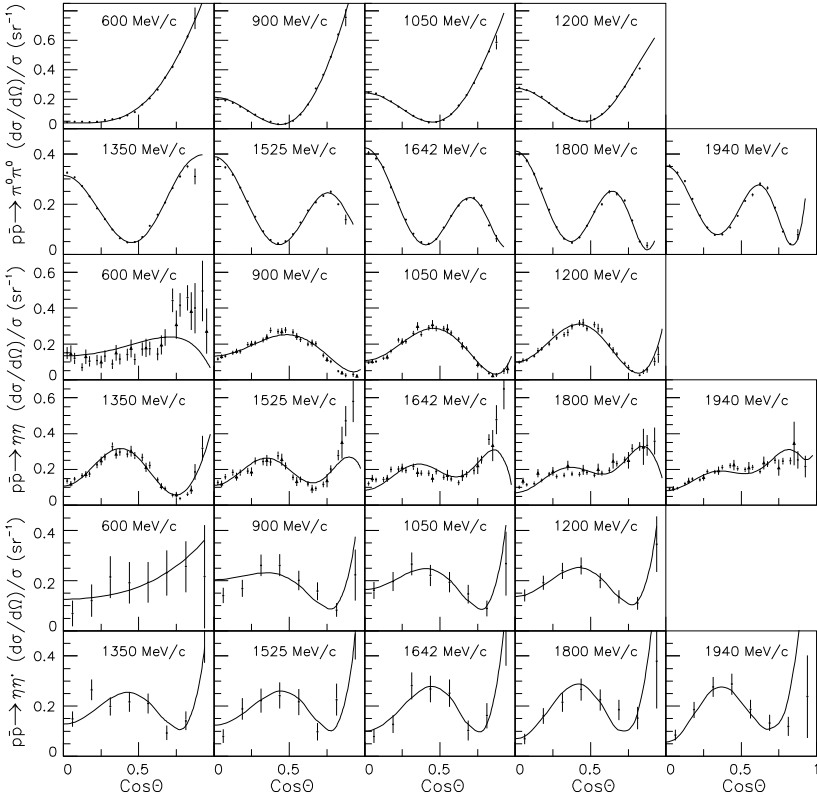


Fig. 4.17 Angle distributions in the reactions $p\bar{p} \rightarrow \pi\pi, \eta\eta, \eta\eta'$ and the fitting to resonances of Eq. (4.282).

These ratios demonstrate that the only broad state $f_2(2000)$ is nearly flavour blind that is a signature of the glueball.

Analyses also gives us the width of $f_2(2000)$ twice as large as other neighbouring states – this is another argument in favour of its glueball nature (remind that glueballs accumulate the widths of the neighbouring $q\bar{q}$ states). In addition, there is no room for $f_2(2000)$ on the (n, M^2) -trajectories [56], and it becomes clear that this resonance is indeed the lowest tensor glueball (this point is discussed in detail in Chapter 2, Section 2.6).

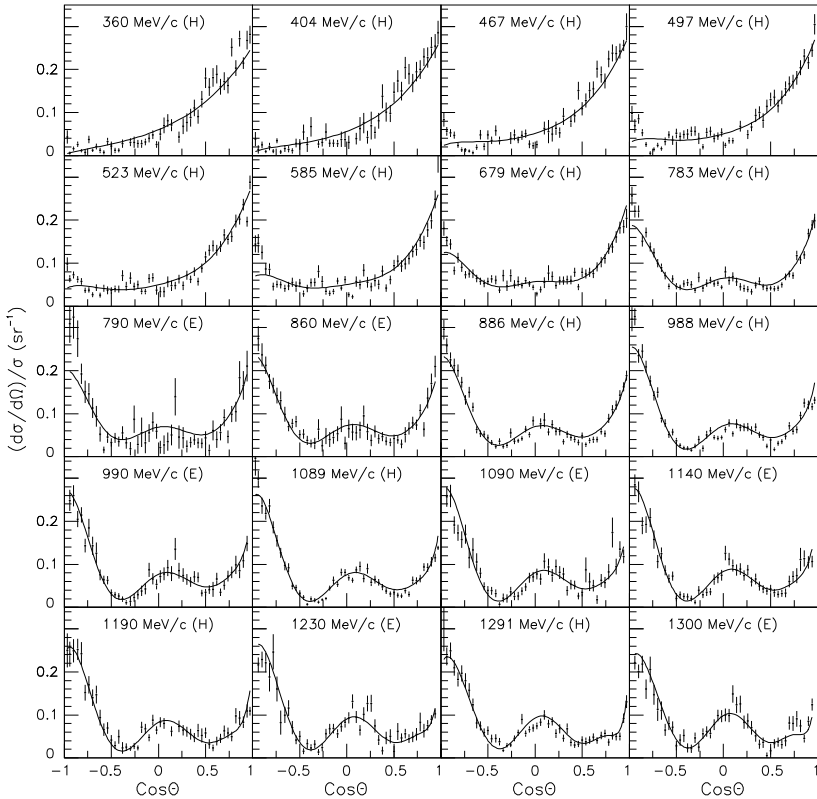


Fig. 4.18 Differential cross sections in the reaction $p\bar{p} \rightarrow \pi^+\pi^-$ at proton momenta 360–1300 MeV and the fitting results to resonances of Eq. (4.282).

4.12 Appendix 4.F. New Thresholds and the Data for $\rho = \text{Im } A / \text{Re } A$ of the UA4 Collaboration at $\sqrt{s} = 546 \text{ GeV}$

The large value of the real part of the forward $p\bar{p}$ scattering amplitude, $\rho = 0.24 \pm 0.04$, measured by the UA4 Collaboration at $\sqrt{s} = 546 \text{ GeV}$ [22], initiated the discussion about the existence of a new threshold at high energies [19, 20, 21, 57, 58].

In this appendix, following [23], we calculate threshold effects for the high energy scattering amplitude taking into account the screening owing to the s -channel unitarity. Different versions of the threshold behaviour

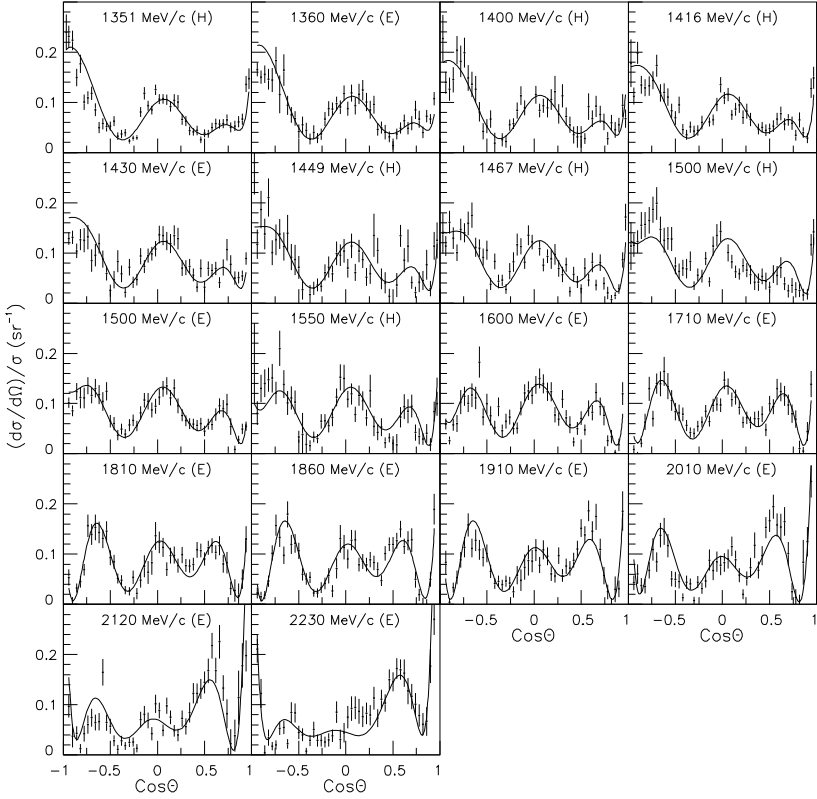


Fig. 4.19 Differential cross sections in the reaction $p\bar{p} \rightarrow \pi^+\pi^-$ at proton momenta 1350–2230 MeV and the fitting results to resonances of Eq. (4.282).

are analysed based on the realistic $p\bar{p}$ scattering amplitude for the energy region $\sqrt{s} = 0.05 - 2.0$ TeV.

Let us specify our calculations. For the scattering amplitude (4.216), the following parametrisation is used:

$$\begin{aligned}\sigma_{tot}(s) &= 2\pi \left(14.9 + 35.0 \frac{1\text{GeV}}{\sqrt{s}} + 2.84 \ln \frac{\sqrt{s}}{25\text{GeV}} \right) \text{GeV}^{-2}, \\ r^2(s) &= \left(4.13 + 9.73 \frac{1\text{GeV}}{\sqrt{s}} + 0.79 \ln \frac{\sqrt{s}}{25\text{GeV}} \right) \text{GeV}^{-2}, \\ \rho(s) &= 0.11 \left(1 - \frac{225 \text{ GeV}^2}{s} \right).\end{aligned}$$

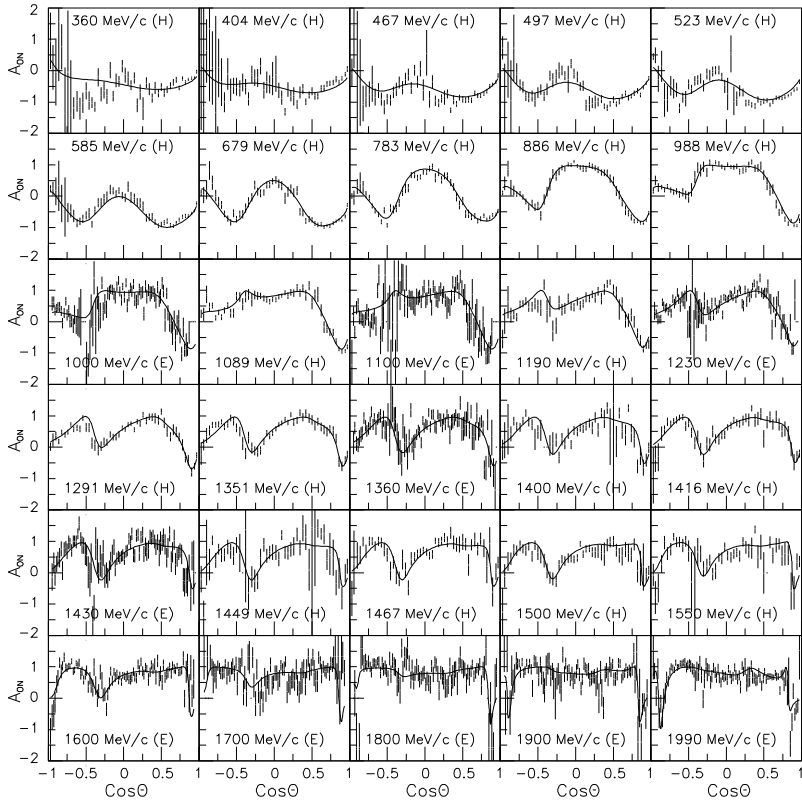


Fig. 4.20 Polarisation in $p\bar{p} \rightarrow \pi^+\pi^-$ and the fitting results to resonances of Eq. (4.282).

At present, the data do not contradict the idea of the maximal growth of hadron total cross sections at superhigh energies. However, this is not the case for the region $\sqrt{s} \sim 0.05 - 2.0$ TeV. At these energies, the growth of the total cross sections is weaker, $\sigma_{tot} \sim \ln s$, while the decrease of ρ is not seen. The parametrisation we use gives a sufficiently good description of the elastic diffractive cross section: we have $\alpha' = 0.20 \text{ GeV}^{-2}$ with the diffractive slope $B = 17 \text{ GeV}^{-2}$ at $\sqrt{s} = 1.8 \text{ TeV}$, in agreement with the data [59].

Because of the saturation of $f(b, s)$ at small b , the amplitude $f(b, s)$ is sensitive only to large b in $\alpha(b, s)$. The large values of b in the energy region not far from the threshold can be caused by the diffractive production of

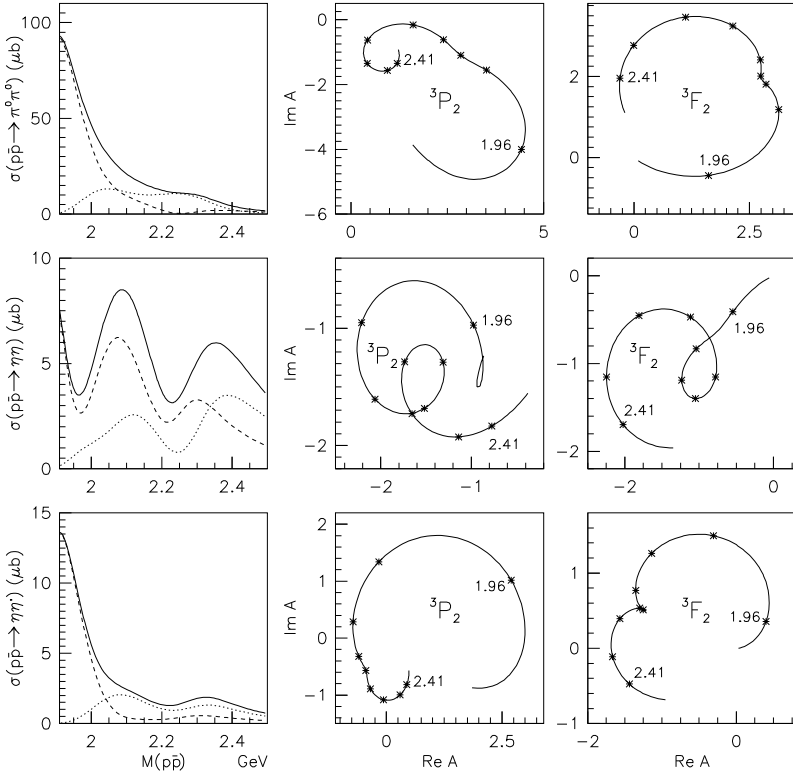


Fig. 4.21 Cross sections and Argand-plots for ${}^3P_2$ and ${}^3F_2$ waves in the reaction $p\bar{p} \rightarrow \pi^0\pi^0, \eta\eta, \eta\eta'$. The upper row refers to $p\bar{p} \rightarrow \pi^0\pi^0$: we demonstrate the cross sections for ${}^3P_2$ and ${}^3F_2$ waves (dashed and dotted lines, correspondingly) and total ($J = 2$) cross section (solid line) as well as Argand-plots for the ${}^3P_2$ and ${}^3F_2$ wave amplitudes at invariant masses $M = 1.962, 2.050, 2.100, 2.150, 2.200, 2.260, 2.304, 2.360, 2.410$ GeV. Figures in the second and third rows refer to the reactions $p\bar{p} \rightarrow \eta\eta$ and $p\bar{p} \rightarrow \eta\eta'$.

new particles. So, we examine the mechanism of diffractive production of heavy particles.

Let $\alpha(b, s)$, being a function of s , have a threshold singularity at $s = s_0$. In the calculations, we use the s -plane threshold singularity of the $(s - s_0)^2 \ln(s - s_0)$ type, which corresponds to a three-particle intermediate state. This singularity should be considered as the strongest one for the diffractive production processes.

We parametrise $\alpha(b, s)$ in the following form:

$$\alpha(b, s) = \sum_{n=1}^3 c_n a_n(s) \exp\left(-\frac{b^2}{4R^2(s)}\right), \quad (4.284)$$

where c_n are constants, the functions $a_n(s)$ have the threshold singularity at $s = s_0$, and the b^2 -dependence is supposed to be exponential.

The threshold bump in ρ depends on R^2 : the larger the value of R^2 , the bigger the bump in ρ . We accept the diffractive mechanism for the new particle production and put $R^2 \sim \frac{1}{3}r^2(s)$. In the region of $\sqrt{s} \sim 0.6 - 1.0$ TeV, it gives us the value which coincides with the slope of diffractive production of “old hadrons” at moderate energies.

Below $x = s/s_0$, the functions $a_n(s)$ are chosen in the form:

$$\begin{aligned} a_1(s) &= \left(1 - \frac{1}{x}\right)^2 \left(\frac{1}{\pi} \ln \left| \frac{x+1}{x-1} \right| + i\right) - \frac{1}{\pi} B_1(x), \\ a_2(s) &= \frac{1}{x^2} \left(1 - \frac{1}{x}\right)^2 \left(-\frac{1}{\pi} \ln |x-1| + i\right) - \frac{1}{\pi} B_2(x), \\ a_3(s) &= \frac{1}{x^4} \left(1 - \frac{1}{x}\right)^2 \left(-\frac{1}{\pi} \ln |x-1| + i\right) - \frac{1}{\pi} B_3(x), \end{aligned} \quad (4.285)$$

and

$$\begin{aligned} B_1(x) &= 2x^{-1} - 4, \\ B_2(x) &= x^{-3} - \frac{3}{2}x^{-2} + \frac{1}{3}x^{-1} + \frac{1}{12}, \\ B_3(x) &= x^{-5} - \frac{3}{2}x^{-4} + \frac{1}{3}x^{-3} + \frac{1}{12}x^{-2} + \frac{1}{30}x^{-1} + \frac{1}{60}. \end{aligned} \quad (4.286)$$

Equations (4.285) are written for $a_n(s)$ at $s > s_0$, while at $s < s_0$ one should omit the imaginary parts of the right-hand sides of Eqs. (4.285). The polynomial terms $B_n(x)$ provide the analyticity of $a_n(s)$ at $s \rightarrow 0$, and the logarithmic ones give the threshold singularity of the type $(s - s_0)^2 \ln(s - s_0)$. The function $a_1(s)$ leads to a nonvanishing new particle production cross section at $s \gg s_0$, whereas the functions $a_2(s)$ and $a_3(s)$ give us the possibility to change the production cross section near threshold.

Figures 4.22 and 4.23 show the results of our calculation of σ_{tot} and ρ for the following sets of (c_1, c_2, c_3) :

$$\begin{aligned} \text{case I} &= (0.27, 0, 27), & \text{case III} &= (0.2, 2, 0) \\ \text{case II} &= (0.22, 5.5, 0), & \text{case IV} &= (0.3, 0, 7.5). \end{aligned} \quad (4.287)$$

We put $\sqrt{s_0} = 500$ GeV and $R^3 = \frac{1}{3}r^2(s)$ for the cases I, II and IV. In case III, we use $R^2 = \frac{1}{2}r^2(s)$.

The versions I–III present examples of the maximal value of ρ near $\sqrt{s} = 550$ GeV being close to 0.22. The calculated total cross sections for these cases are larger than the experimental ones in the region $\sqrt{s} = 0.5\text{--}1.0$ TeV. In the version IV we show the case when the calculated values of σ_{tot} are near the error bars of the experimental data. Here, however, ρ at $\sqrt{s} = 550$ GeV is below the value reported in [22]. (Note that these results differ from the calculation of new threshold effects of Ref. [60] where screening corrections were not taken into account.)

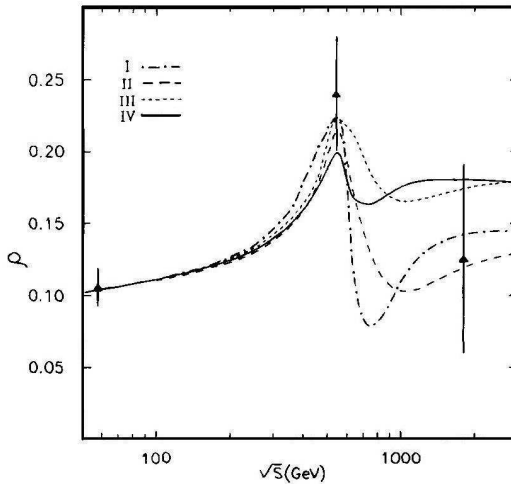


Fig. 4.22 Ratio $\rho = \text{Re } A / \text{Im } A$ for the cases I–IV. The data are from Refs. [59, 61].

Concluding, the analysis [23] demonstrated that the data of UA4 collaboration [22] hardly agree with the hypothesis that the cusp in ρ is the result of the opening of new channels with heavy particles. Later measurements did not confirm the existence of the cusp in ρ .

4.13 Appendix 4.G. Rescattering Effects in Three-Particle States: Triangle Diagram Singularities and the Schmid Theorem

In this appendix for three-particle production reactions, the effects of anomalous singularities caused by resonances in intermediate states are discussed in detail. We consider two types of diagrams: direct resonance

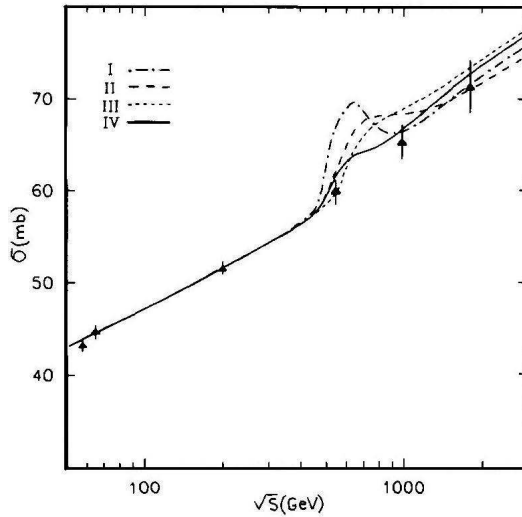


Fig. 4.23 $p\bar{p}$ total cross sections for the cases I-IV. The data are from Refs. [59, 61].

production, Fig. 4.24a (pole singularity $(s_{23} - M_R^2)^{-1}$), and diagrams with rescattering of the produced particles, Fig. 4.24b (anomalous singularity $\ln(s_{12} - s_{tr})$). We present simple and visual rules for the determination of positions of the anomalous triangle-diagram singularity. Then, in terms of the dispersion relation technique, we describe the calculation procedure for these diagrams: the specific feature of calculations of these diagrams is the necessity to take into account the energy dependence of the resonance widths ($M_R^2 = m_R^2 - im_R\Gamma_R(s_{23})$), which contain threshold singularities related to their decays. Finally we reanalyse the Schmid theorem [62] which discusses interference effects of the of diagrams Figs. 4.24a and 4.24b in two-particle spectrum $d\sigma/ds_{12}$. We show that the Schmid theorem (which tells us about the disappearance of the anomalous singularity effects in $d\sigma/ds_{12}$) is not valid when the amplitude of the rescattering particles has several open channels. In this case the irregularities related to anomalous singularities appear not only in the three-particle Dalitz-plot but in the two-particle distributions as well. Examples of the processes $p\bar{p}(at\ rest) \rightarrow three\ mesons$ are discussed.

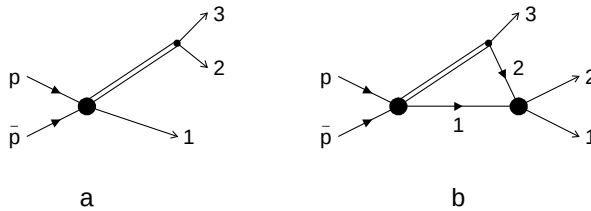


Fig. 4.24 Pole (a) and triangle diagram (b).

4.13.1 Visual rules for the determination of positions of the triangle-diagram singularities

Here, for the sake of simplicity, we consider the case when all final state particles in Figs. 4.24a, b have equal masses: $m_1 = m_2 = m_3 = m$ and resonance width is small $\Gamma_R \rightarrow 0$.

a) Small total energy: $9m^2 < s < (m + M_R)^2$.

Let us start with the case of small total energy when the resonance is not produced yet, $s = (p_1 + p_2 + p_3)^2 < (m + M_R)^2$. To be more illustrative, we consider the resonance with a small width, $\Gamma \ll m$, treating it in the kinematical relations sometimes as a stable particle.

Positions of the resonance and the Dalitz-plot are shown in Fig. 4.25a. The anomalous singularity of the triangle diagram is located on the second sheet of the complex- s_{12} plane at $s_{12} = s_{tr}$ with

$$s_{tr} = 2m^2 + \frac{1}{2}(s - m^2 - M_R^2) - \frac{i}{M_R} \sqrt{(M_R^2 - 4m^2)[(M_R + m)^2 - s][s - (M_R - m)^2]}. \quad (4.288)$$

b) Threshold production of the resonance at $s = (m + M_R)^2$.

This is the energy when an anomalous singularity comes to the physical region from the complex- s_{12} values of the second sheet. Positions of the resonance and the anomalous singularity with respect to the Dalitz plot are shown in Fig. 4.25b. The anomalous singularity of the triangle diagram is located at

$$s_{12} = s_{tr} = 2m^2 + mM_R, \quad (4.289)$$

being slightly shifted on the second sheet of the complex- s_{12} plane (it touches the physical region: remember that we consider here the limit $\Gamma \rightarrow 0$). The anomalous singularity band crosses the resonance band exactly on the border of the Dalitz plot. In the crossing region the pole diagram, Fig. 4.24a, and the triangle one, Fig. 4.24b, do strongly interfere.

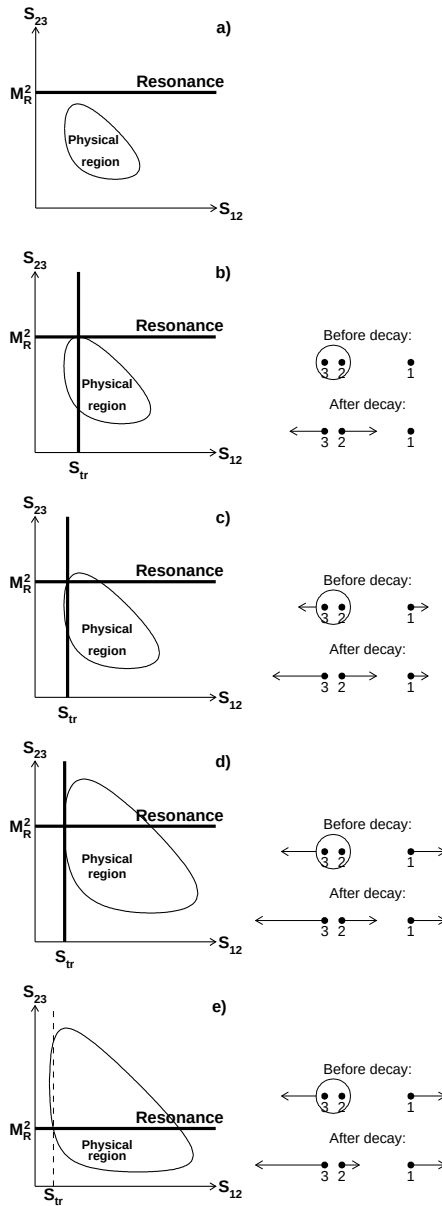


Fig. 4.25 Dalitz plots with pole and triangle diagram singularities and visual rules for the determination of positions of the triangle-diagram singularities.

The position of the anomalous singularity corresponds to the following kinematics. At $s = (m + M_R)^2$ the resonance and particle 1 are produced at rest (see right-hand side of Fig. 4.25b), then the resonance decays: the total invariant energy squared of particles 1 and 2 gives the position of the anomalous singularity, $s_{tr} = (p_1 + p_2)^2$, in this case.

c) Location of the anomalous singularity in the physical region at $(m + M_R)^2 \leq s \leq m^2 + 2M_R^2$.

The anomalous singularity in the limit $\Gamma \rightarrow 0$ is located in the physical region:

$$s_{tr} = 2m^2 + \frac{1}{2}(s - m^2 - M_R^2) - \frac{1}{M_R} \sqrt{(M_R^2 - 4m^2)[s - (M_R + m)^2][s - (M_R - m)^2]}. \quad (4.290)$$

The value s_{tr} corresponds to the crossing of the Dalitz plot border curve with the resonance band, see Fig. 4.25c. Again, in the crossing region the pole diagram, Fig. 4.24a, and the triangle diagram, Fig. 4.24b, strongly interfere.

On the right-hand side of Fig. 4.25c the kinematics which gives s_{tr} is shown: in the c.m. system the resonance and particle 1 are moving in opposite directions. After the resonance decay, the minimal s_{12} corresponds to the case when particle 2 is moving in the same direction as the particle 1. This minimal s_{12} gives us the value s_{tr} :

$$[s_{12}]_{\text{minimal value for real decay}} = s_{tr}. \quad (4.291)$$

d) Maximal total energy when anomalous singularity is located in the immediate region of the physical process: $s = m^2 + 2M_R^2$.

At $s = m^2 + 2M_R^2$ the anomalous singularity is located on the border of the production process, at

$$s_{12} = 4m^2, \quad (4.292)$$

see Fig. 4.25d. The decay kinematics for this case is

$$\mathbf{p}_1 = \mathbf{p}_2. \quad (4.293)$$

e) Location of the anomalous singularity at large total energies, $s > m^2 + 2M_R^2$.

At $s > m^2 + 2M_R^2$ the anomalous singularity goes to the upper part of the second sheet of complex- s_{12} plane, moving away from the physical region. Its position on the upper half-sheet is shown in Fig. 4.25e by the dashed line. The corresponding kinematics is also shown in Fig. 4.25e, in this case we have

$$\mathbf{p}_1 > \mathbf{p}_2. \quad (4.294)$$

4.13.2 Calculation of the triangle diagram in terms of the dispersion relation N/D -method

A convenient way to extract singularities of the amplitudes given by diagrams like Fig. 4.24b is to use the N/D method [63] with some modifications caused by the large invariant mass, s , of the initial system [11].

As an illustrative example, we assume that all particles are scalars and that the interaction occurs in an S -wave with the partial amplitude equal to: $\exp(i\delta_{12}) \sin \delta_{12}$. The amplitude of the triangle diagram of Fig. 4.24b with the subsequent rescattering of particles 1 and 2 is written as:

$$A_{tr}(s_{12}) = \frac{1}{1 - B_{12}(s_{12})} \times \int_{(m_1+m_2)^2}^{\infty} \frac{ds'_{12}}{\pi} \frac{N_{12}(s'_{12})\rho_{12}(s'_{12})}{s'_{12} - s_{12} - i0} \int_{C(s'_{12})} \frac{dz_{23}}{2} R(s_{23}) . \quad (4.295)$$

In equation (4.295) the factor $1/(1 - B_{12}(s_{12}))$ describes the chain of loop diagrams corresponding to the rescattering of particles 1 and 2 (see Chapter 3, Section 3.3.5 for more details). The amplitude of the triangle diagram (the second term in the right-hand side of (4.295)) is given by the spectral integral over s' . This term includes as a factor the integral over the resonance propagator R_{23} averaged over $z_{23} = \cos \theta_{23}$ where θ_{23} is the angle between particles 2 and 3; the factor $N_{12}(s'_{12})$ presents the right-hand side vertex in the triangle diagram, and $\rho_{12}(s'_{12})$ is the invariant phase space for particles 1 and 2 in the intermediate state.

The loop diagram B_{12} is defined by N_{12} only (see Chapter 3, Section 3.3.5), so we have:

$$B_{12}(s_{12}) = \int_{(m_1+m_2)^2}^{\infty} \frac{ds'_{12}}{\pi} \frac{N_{12}(s'_{12})\rho_{12}(s'_{12})}{s'_{12} - s_{12} - i0} , \quad (4.296)$$

$$\frac{N_{12}(s_{12})\rho_{12}(s_{12})}{1 - B_{12}(s_{12})} = \exp(i\delta_{12}) \sin \delta_{12} .$$

In more detail: $R(s_{23})$ is the resonance production amplitude given by Fig. 4.24a,

$$R(s_{23}) = \lambda \frac{1}{M^2 - s_{23} - iM\Gamma(s_{23})} g_{23} , \quad (4.297)$$

where the couplings λ and g_{23} determine the magnitude of the resonance production and its decay into particles 2 and 3. In the general case the width Γ depends on the energy squared s_{23} and has a threshold singularity at

in Eq. (4.295) over s_{23} is carried out in the complex plane (shown by contour III in Fig. 4.26).

The Breit–Wigner pole is located under the cut related to the singular point at $s_{23} = (m_2 + m_3)^2$ (see Fig. 4.26). The final point of the integration (A in Fig. 4.26) is in the proximity of the Breit–Wigner pole at some values of s'_{12} and s : if the final point A touches the Breit–Wigner pole point, a logarithmic singularity is created. In the complex s_{12} -plane the logarithmic singularity is located on the second sheet, which is related to the threshold singular point $s_{12} = (m_1 + m_2)^2$; let us remind that the Breit–Wigner resonance poles are also located on the second sheet. But, contrary to the Breit–Wigner resonance poles, the logarithmic singularity moves with a change of the total energy $\sqrt{s}$ and is near the physical region only at

$$(m_1 + M)^2 < s < m_1^2 + M^2 + \frac{m_1}{m_2} (M^2 + m_2^2 - m_3^2). \quad (4.302)$$

The position of the logarithmic singularity on the second sheet is as follows:

$$\begin{aligned} s_L = & m_1^2 + m_2^2 + \frac{1}{2M_R^2} (s - m_1^2 - M_R^2) (M_R^2 + m_2^2 - m_3^2) \\ & - \frac{1}{2M_R^2} \left\{ [M_R^2 - (\sqrt{s} + m_1)^2] [M_R^2 - (\sqrt{s} - m_1)^2] \right. \\ & \times \left. [M_R^2 - (m_2 + m_3)^2] [M_R^2 - (m_2 - m_3)^2] \right\}^{1/2}, \end{aligned} \quad (4.303)$$

where $M_R^2 = M^2 - iM\Gamma$.

4.13.3 *The Breit–Wigner pole and triangle diagrams: interference effects*

The anomalous triangle singularity is not strong enough: the amplitude diverges as $\ln(s_{12} - s_L)$, so an observation of the corresponding irregularities requires rather precise data. Another problem is related to the fact that in the two-particle spectra of some reactions the leading singular term may be cancelled due to the interference of the triangle diagram with the Breit–Wigner pole contribution. This cancellation has been observed in [62] and is called the Schmid theorem. In this section, following [64], we reanalyse the proof of the Schmid theorem and show the limits of its applicability; several illustrative examples for the reaction $p\bar{p}(\text{at rest}) \rightarrow (\text{three mesons})$ are presented as well.

Specifically, an analogous logarithmic singularity exists in the projection of the resonance term (Fig. 4.24a) on the energy axis of particles 1 and 2.

Denoting this resonance projection as $\alpha \ln(s_{12} - s_L)$, a result of Schmid's theorem is that the addition of the triangle diagram contribution leads only to a shift of the phase of this singular term

$$\alpha \ln(s_{12} - s_L) \rightarrow \exp(2i\delta_{12})\alpha \ln(s_{12} - s_L). \quad (4.304)$$

Here, as previously, the scattering amplitude of particles 1 and 2 is $\exp(i\delta_{12})\sin\delta_{12}$. So the sum of resonance and triangle diagram terms gives us for the projection $d\sigma/ds_{12}$ the factor $|\exp(2i\delta_{12}) \times \alpha \ln(s_{12} - s_L)|^2$ which does not depend on the phase shift δ_{12} . This illustrates the fact that the interference of diagrams presented in Figs. 4.24a and 4.24b is essential in the description of the Dalitz plot for the reaction. It is just the interference which kills the contribution of the diagram of Fig. 4.24b onto the projection of $d\sigma/ds_{12}$.

The formulae given below provide us a simple way of obtaining the Schmid theorem and also illustrate the cases when the theorem is not valid. We calculate the two-particle distribution $d\sigma/ds_{12}$ for the case when the production occurs only by the processes drawn in Figs. 4.24a and 4.24b. To this end we expand the production amplitude over partial waves in the 12-channel. The Breit–Wigner pole term of Eq. (4.297) is a sum over all orbital momenta $R(s_{23}) = \Sigma f_\ell(s_{12})P_\ell(z)$ while the triangle diagram gives a contribution to the $\ell = 0$ state only. Then

$$\frac{d\sigma}{ds_{12}} = N \left(|f_0(s_{12}) + A_{tr}(s_{12})|^2 + \sum_{\ell=1}^{\infty} \frac{|f_\ell(s_{12})|^2}{2\ell+1} \right), \quad (4.305)$$

where N is the kinematical factor depending on the momenta of the produced particle.

The part of A_{tr} which contains the logarithmic singularity can be extracted using a two-step procedure. First, the pole singularity $(s'_{12} - s_{12} - i0)^{-1}$ in Eq. (4.295) is replaced by its residue $2\pi i\delta(s'_{12} - s_{12})$ and, second, the integration contour C is replaced by the contour integration (4.301), *i.e.* by the contour I in Fig. 4.26. The other terms (denoted below as A_{n-s}) are analytical at the point $s_{12} = s_L$. As a result we have

$$\begin{aligned} A_{tr}(s_{12}) &= \frac{1}{1 - B_{12}(s_{12})} \int_{(m_1+m_2)^2}^{\infty} \frac{ds'_{12}}{\pi} N_{12}(s'_{12}) \rho_{12}(s'_{12}) 2i\pi\delta(s'_{12} - s_{12}) \\ &\times \int_{-1}^1 \frac{dz}{2} R(s_{23}) + A_{n-s} = [\exp(2i\delta_{12}) - 1] f_0(s_{12}) + A_{n-s}. \end{aligned} \quad (4.306)$$

So the structure of the singular term A_{tr} is the same as $f_0(s_{12})$ but has an additional factor which depends on the scattering amplitude. The contribution of the partial wave with $\ell = 0$ to the cross section (4.305) is

$$\left| \exp(2i\delta_{12})f_0(s_{12}) + A_{n-s} \right|^2. \quad (4.307)$$

The leading singular term is proportional to $\ln^2(s_{12} - s_L)$ and is contained in $|f_0(s_{12})|^2$. It does not depend on the scattering amplitude of particles 1 and 2 (this is the statement of the Schmid theorem). The next-to-leading terms are proportional to $\ln(s_{12} - s_L)$ and depend on the scattering phase shift δ_{12} .

The equation (4.307) indicates the type of systems for which Schmid's theorem is not valid. These are systems where the scattering amplitude of the outgoing particles ($1 + 2 \rightarrow 1 + 2$) in Fig. 4.24a has several open channels.

Let us discuss certain examples. (i) The Schmid theorem is not valid in the reaction $p\bar{p}(\text{at rest}) \rightarrow 3\pi^0$. Here $p\bar{p}$ annihilates predominantly from the $^1S_0$ state, and pion production happens mainly through the production of the f_0 -resonance. The pion rescattering in this reaction can be considered as a two-channel case which has an S -wave interaction in the isotopic states $I = 0$ and $I = 2$. However, the phase shift in the $I = 2$ state is nearly zero in the region less than 1 GeV, so the rescattering can be neglected in this state. Therefore, if we describe the reaction $p\bar{p} \rightarrow 3\pi^0$ in terms of diagrams of Figs. 4.24a and 4.24b with the production of f_0 -resonances and pion rescattering in the $(I = 0, J = 0)$ -wave state, the amplitude $p\bar{p} \rightarrow 3\pi^0$ is equal to

$$A(p\bar{p} \rightarrow 3\pi^0) = A(s_{12}) + A(s_{13}) + A(s_{23}), \quad (4.308)$$

where

$$A(s_{jk}) = \sum_i \left[R_i(s_{jk}) + \frac{1}{3} T_i(s_{jk}) \right]. \quad (4.309)$$

Here the amplitude R_i describes the production of the $f_0(i)$ -resonance and T_i is given by an equation similar to (4.295), using the $(I = 0, J = 0)$ -wave pion scattering. It means $m_1 = m_2 = m_3 = m_\pi$ and the determination of the N and D functions by the relation

$$\frac{N_{\pi\pi}(s_{jk})\rho_{\pi\pi}(s_{jk})}{1 - B_{\pi\pi}(s_{jk})} = \exp(i\delta_0^0) \sin \delta_0^0.$$

The Schmid theorem would be valid if we were able to replace the factor $\frac{1}{3}$ by 1 in Eq. (4.309). This factor of $\frac{1}{3}$ in Eq. (4.309) is due to the isotopic

Clebsch–Gordan coefficient squared and reflects the two-channel nature of the low-energy $\pi\pi$ scattering, ($I = 0, J = 0$)-wave and ($I = 2, J = 0$)-wave.

(ii) The reactions $p\bar{p}$ (at rest) $\rightarrow \omega + \eta' \rightarrow \gamma + \pi^0 + \eta'$ and $p\bar{p}$ (at rest) $\rightarrow \omega + \phi \rightarrow \gamma + \pi^0 + \phi$ give us another example where the Schmid theorem is invalid. In these reactions the scattering amplitudes $\pi^0\phi$ and $\pi^0\eta'$ are characterised by a large inelasticity, $\eta_{12} < 1$. So, we should replace in (4.306) $\exp(2i\delta_{12}) \rightarrow \eta_{12} \exp(2i\delta_{12})$. It leads to a dip in the spectrum of $\pi^0\phi$ (or $\pi^0\eta'$) compared to the case $\eta_{12} = 1$ when the Schmid theorem is valid.

In conclusion, the singular term is proportional to the scattering amplitude of the outgoing particles so an extraction of it is equivalent to the determination of this amplitude. The anomalous singularity is the subject of an attractive study because it provides a path by which a new method can be developed for the determination of scattering amplitudes of non-stable particles (including resonances such as ω or ϕ). A cancellation of the leading singular term in the two-particle spectra would be an additional problem in the study of the triangle diagram singularities. However, such a cancellation is absent when the amplitude of the rescattering particles has several open channels.

4.14 Appendix 4.H. Excited Nucleon States $N(1440)$ and $N(1710)$ — Position of Singularities in the Complex- M Plane

The nucleon $N(980)$ and its radial excitations – they belong to the same trajectory on (n, M^2) -plane – are a set of states which should be investigated together. The next excited states of the nucleon are the Roper resonance $N(1440)$ and the $N(1710)$ state – these states both evoked a lively discussion.

Invariant characteristics of resonances are their positions in the complex- M plane, therefore it would be interesting to look at them – following to [65], we show the complex- M plane for nucleon states on Fig. 4.27.

Let us pay attention to the fact that

- (i) position of the Roper resonance pole is noticeably lower than its value given in [53], and
- (ii) there is only one pole near the physical region, around $M \sim 1400$ MeV — there is no pole doubling.

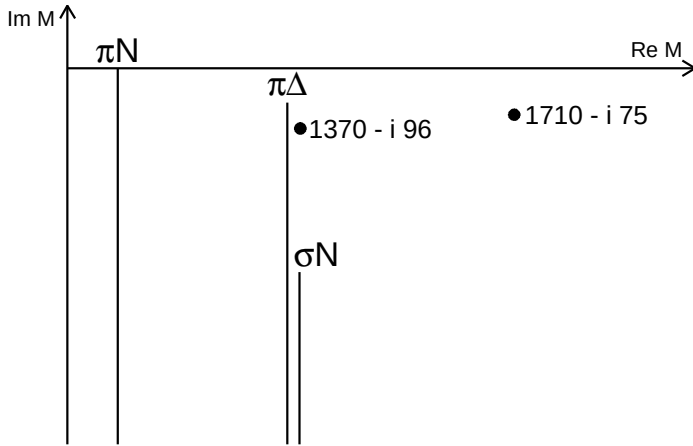


Fig. 4.27 Complex- M plane: position of poles of resonance states $N(1440)$ and $N(1710)$. The cuts related to the threshold singularities πN , ρN and σN are shown by vertical solid lines.

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Chapter 5

Baryons in the πN and γN Collisions

Highly excited baryon states put forward an intriguing question: are they built of three constituents (three quarks) or only of two (quark+di-quark)? To get a definite answer to this fundamental question (whether the highly excited states prefer to be built of only two constituents), refined experimental data for baryon spectra at large masses are needed, being complemented by reliable methods of their interpretation. Because of that we give here an extended presentation of the technique of analysis of baryon resonances, with examples of its application to the existing data.

The structure of the low-lying baryons and baryon resonances is well described in quark models which assume that baryons can be built from three constituent quarks. The spatial and spin-orbital wave functions can be derived using a confinement potential and some residual interactions between constituent quarks. The best known example is the Karl-Isgur model [1], at that time a breakthrough in the understanding of the low-lying baryons. Later refinements differed by the choice of the residual interactions: effective one-gluon exchange, exchanges of Goldstone bosons between the quarks, instanton induced interactions, and so on.

A common feature of these models is the large number of predicted states: the dynamics of three quarks leads to a rich spectrum, much richer than observed experimentally (see discussions in [2] and in Chapter 1, Subsections 1.4.1 and 1.4.2). This problem was called the problem of *missing baryon resonances*. The reason could be that the dynamics of three quark interactions is not understood well enough. The most successful model which partly explains this phenomenon assumed that the two quarks form diquarks ($J = 1^+$ and $J = 0^+$) which reveal themselves in large systems that are highly excited baryons. Then baryons as quark-diquark bound systems are formed in a similar way as quark-antiquark bound states.

Such dynamics reduced dramatically the number of the expected states and matches perfectly well all firmly established states.

Of course, there is also a possibility that the large number of predicted but unobserved states reflects an experimental problem: even diquark models predict more states than experimentally observed up to now. For a long time the main source of information on N^* and Δ^* resonances was derived from pion–nucleon elastic scattering. If a resonance couples weakly to this channel, it could escape identification. Important information is hence expected from experiments studying photoproduction of resonances off nucleons and decaying into multi-particle final states.

The task to extract the positions of poles and residues from multi-particle final states is, however, not a simple one. The main problems can be linked to the large interference effects between different isobars and to contributions from singularities related to multi-body interactions. Meson spectroscopy teaches us that the analysis of reactions with multi-particle final states cannot be done unambiguously without information about reactions with two-body final states. The best way to obtain such an information is to perform a combined analysis of a set of reactions. This issue is even more important in baryon spectroscopy where the polarisation of initial or/and final particles is often not detected. Here, investigating the two-body final states, the combined analysis of the data from different channels plays a vital role. Thus, the development of a method which describes different reactions on the same basis is a key point in the search for new baryon states.

In this chapter the partial wave amplitudes for the production and the decay of baryon resonances are constructed in the framework of the operator expansion method. We present the cross sections for photon and pion induced productions of baryon resonances and their partial decay widths to the two-body and multi-body final states performing calculations in the framework of the relativistically invariant operator expansion method.

The developed method is illustrated by applying it to a combined analysis of photoproduction data on $\gamma p \rightarrow \pi N$, ηN , $K\Lambda$, $K\Sigma$.

5.1 Production and Decay of Baryon States

Here, using the operator expansion method, we construct the partial wave amplitudes for the production and the decay of baryon resonances.

5.1.1 The classification of the baryon states

The baryon states are classified by isospin, total spin and P-parity. The states with isospin $I = 1/2$ are called nucleon states and states with $I = 3/2$ are delta-states. In the literature baryon states are often classified by their decay properties into a nucleon and a pseudoscalar meson: for the sake of simplicity let us consider a πN system. Thus a state called $L_{2I\,2J}$ decays into a nucleon and a pion with the orbital momentum $L = 0, 1, 2, 3, 4, \dots$, it has an isospin I and a total spin J .

A system of a pseudoscalar meson and a nucleon with orbital momentum L can form a baryon state with total spin either equal to $J = L - 1/2$ or to $J = L + 1/2$ and parity $P = (-1)^{L+1}$. The first set of states is called '−' states and the second set '+' states. For each set, the vertex for the decay of a baryon into a pion-nucleon system is formed by the same convolution of the spin operators (Dirac matrices) and orbital momentum operators which will be shown in the next section in detail. In the nucleon sector the '−' states are:

$$I J^P(L_{2I\,2J}) = \frac{1}{2} \frac{1}{2}^+ (P_{11}), \frac{1}{2} \frac{3}{2}^- (D_{13}), \frac{1}{2} \frac{5}{2}^+ (F_{15}), \frac{1}{2} \frac{7}{2}^- (G_{17}), \dots \quad (5.1)$$

and the '+' states:

$$I J^P(L_{2I\,2J}) = \frac{1}{2} \frac{1}{2}^- (S_{11}), \frac{1}{2} \frac{3}{2}^+ (P_{13}), \frac{1}{2} \frac{5}{2}^- (D_{15}), \frac{1}{2} \frac{7}{2}^+ (F_{17}), \dots \quad (5.2)$$

5.1.2 The photon and baryon wave functions

Let us remind the basic properties of the photon and baryon wave functions and introduce notations which are convenient for using in this chapter.

5.1.2.1 The photon projection operator

The sum over the polarisations of the virtual photon which is described by the polarisation vector $\epsilon_\mu^{(\gamma^*)}$ and momentum q ($q^2 \neq 0$) sets up the metric operator:

$$-\sum_{a=1,2,3} \epsilon_\mu^{(\gamma^*)a} \epsilon_\nu^{(\gamma^*)a} = O_\nu^\mu = g_{\mu\nu}^\perp, \quad g_{\mu\nu}^\perp = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}. \quad (5.3)$$

The three independent polarisation vectors are orthogonal to the momentum of the particle,

$$q_\mu \epsilon_\mu^{(\gamma^*)a} = 0 \quad (5.4)$$

and are normalised as

$$\epsilon_{\mu}^{(\gamma^*)a+} \epsilon_{\mu}^{(\gamma^*)b} = -\delta_{ab} . \quad (5.5)$$

A real photon has, however, only two independent polarisations. The invariant expression for the photon projection operator can be constructed only for the photon interacting with another particle. In this case (here we consider the photon–baryon interaction, $\gamma + N \rightarrow \text{baryon state}$) the completeness condition reads:

$$-\sum_{a=1,2} \epsilon_{\mu}^{(\gamma)a} \epsilon_{\nu}^{(\gamma)a+} = g_{\mu\nu} - \frac{P_{\mu} P_{\nu}}{P^2} - \frac{k_{\mu}^{\perp} k_{\nu}^{\perp}}{k_{\perp}^2} = g_{\mu\nu}^{\perp\perp}(P, p_N) . \quad (5.6)$$

In (5.6) the baryon and photon momenta are p_N and q_{γ} (remind that $q_{\gamma}^2 = 0$), the total momentum is denoted as $P = p_N + q_{\gamma}$; we have introduced $k = \frac{1}{2}(p_N - q_{\gamma})$ and $k^{\perp}$:

$$k_{\mu}^{\perp} \equiv k_{\mu}^{\perp P} = \frac{1}{2}(p_N - q_{\gamma})_{\nu} g_{\mu\nu}^{\perp P} = \frac{1}{2}(p_N - q_{\gamma})_{\nu} \left(g_{\mu\nu} - \frac{P_{\mu} P_{\nu}}{P^2} \right) . \quad (5.7)$$

In the c.m. system ($\vec{p}_N + \vec{q}_{\gamma} = 0$ and $P = (\sqrt{s}, 0, 0, 0)$), if the momenta of $\mathbf{p}_N$ and $\mathbf{q}_{\gamma}$ are directed along the z-axis, the metric tensor $g_{\mu\nu}^{\perp\perp}(P, p_N)$ has only two non-zero elements: $g_{xx}^{\perp\perp} = g_{yy}^{\perp\perp} = -1$ (the four-vector components are defined as $p = (p_0, p_x, p_y, p_z)$). For the photon polarisation vector we can use the linear basis: $\epsilon^{(\gamma)x} = (0, 1, 0, 0)$ and $\epsilon^{(\gamma)y} = (0, 0, 1, 0)$, as well as circular one with helicities ± 1 : $\epsilon^{(\gamma)+1} = -(0, 1, +i, 0)/\sqrt{2}$ and $\epsilon^{(\gamma)-1} = (0, 1, -i, 0)/\sqrt{2}$.

The tensor $g_{\mu\nu}^{\perp\perp}(P, p_N)$ acts in the space which is orthogonal to the momenta of both particles, p_N and q_{γ} , and extracts the gauge invariant part of the amplitude: $A = A_{\mu} \epsilon_{\mu}^{(\gamma)} = A_{\nu} g_{\nu\mu}^{\perp\perp}(P, p_N) \epsilon_{\mu}^{(\gamma)}$. Indeed, $A_{\nu} g_{\nu\mu}^{\perp\perp}(P, p_N)$ is gauge invariant: $A_{\nu} g_{\nu\mu}^{\perp\perp}(P, p_N) q_{\gamma\mu} = 0$.

5.1.2.2 Baryon projection operators

In this chapter it is convenient to use the baryon wave functions introduced in Chapter 4 (Subsection 4.1.1), $u_j(p)$ and $\bar{u}_j(p)$, which are normalised as $\bar{u}_j(p) u_{\ell}(p) = \delta_{j\ell}$ and obey the completeness condition $\sum_{j=1,2} u_j(p) \bar{u}_j(p) = (m + \hat{p})/2m$. For a baryon with fixed polarisation one has to substitute:

$$\frac{m + \hat{p}}{2m} \rightarrow \frac{m + \hat{p}}{2m} (1 + \gamma_5 \hat{S}) , \quad (5.8)$$

with the following constraints for the polarisation vector S_{μ} :

$$S^2 = -1, \quad (pS) = 0. \quad (5.9)$$

(i) Projection operators for particles with $J > 1/2$.

The wave function of a particle with spin $J = n + 1/2$, momentum p and mass m is given by a tensor four-spinor $\Psi_{\mu_1 \dots \mu_n}$. It satisfies the constraints

$$(\hat{p} - m)\Psi_{\mu_1 \dots \mu_n} = 0, \quad p_{\mu_i}\Psi_{\mu_1 \dots \mu_n} = 0, \quad \gamma_{\mu_i}\Psi_{\mu_1 \dots \mu_n} = 0, \quad (5.10)$$

and the symmetry properties

$$\begin{aligned} \Psi_{\mu_1 \dots \mu_i \dots \mu_j \dots \mu_n} &= \Psi_{\mu_1 \dots \mu_j \dots \mu_i \dots \mu_n}, \\ g_{\mu_i \mu_j} \Psi_{\mu_1 \dots \mu_i \dots \mu_j \dots \mu_n} &= g_{\mu_i \mu_j}^{\perp p} \Psi_{\mu_1 \dots \mu_i \dots \mu_j \dots \mu_n} = 0. \end{aligned} \quad (5.11)$$

Conditions (5.10), (5.11) define the structure of the denominator of the fermion propagator (the projection operator) which can be written in the following form:

$$F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}(p) = (-1)^n \frac{m + \hat{p}}{2m} R_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}(\perp p). \quad (5.12)$$

The operator $R_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}(\perp p)$ describes the tensor structure of the propagator. It is equal to 1 for a ($J = 1/2$)-particle and is proportional to $g_{\mu\nu}^{\perp p} - \gamma_{\mu}^{\perp} \gamma_{\nu}^{\perp} / 3$ for a particle with spin $J = 3/2$ (remind that $\gamma_{\mu}^{\perp} = g_{\mu\nu}^{\perp p} \gamma_{\nu}$, see Chapter 4, Subsection 4.3.1).

The conditions (5.11) are identical for fermion and boson projection operators and therefore the fermion projection operator can be written as:

$$R_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}(\perp p) = O_{\alpha_1 \dots \alpha_n}^{\mu_1 \dots \mu_n}(\perp p) T_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(\perp p) O_{\nu_1 \dots \nu_n}^{\beta_1 \dots \beta_n}(\perp p). \quad (5.13)$$

The operator $T_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(\perp p)$ can be expressed in a rather simple form since all symmetry and orthogonality conditions are imposed by O -operators. First, the T -operator is constructed of metric tensors only, which act in the space of $\perp p$ and $\gamma^{\perp}$ -matrices. Second, a construction like $\gamma_{\alpha_i}^{\perp} \gamma_{\alpha_j}^{\perp} = \frac{1}{2} g_{\alpha_i \alpha_j}^{\perp} + \sigma_{\alpha_i \alpha_j}^{\perp}$ (remind that here $\sigma_{\alpha_i \alpha_j}^{\perp} = \frac{1}{2} (\gamma_{\alpha_i}^{\perp} \gamma_{\alpha_j}^{\perp} - \gamma_{\alpha_j}^{\perp} \gamma_{\alpha_i}^{\perp})$) gives zero if multiplied by an $O_{\alpha_1 \dots \alpha_n}^{\mu_1 \dots \mu_n}$ -operator: the first term is due to the traceless conditions and the second one to symmetry properties. The only structures which can then be constructed are $g_{\alpha_i \beta_j}^{\perp}$ and $\sigma_{\alpha_i \beta_j}^{\perp}$. Moreover, taking into account the symmetry properties of the O -operators, one can use any pair of indices from sets $\alpha_1 \dots \alpha_n$ and $\beta_1 \dots \beta_n$, for example, $\alpha_i \rightarrow \alpha_1$ and $\beta_j \rightarrow \beta_1$. Then

$$T_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(\perp p) = \frac{n+1}{2n+1} (g_{\alpha_1 \beta_1}^{\perp} - \frac{n}{n+1} \sigma_{\alpha_1 \beta_1}^{\perp}) \prod_{i=2}^n g_{\alpha_i \beta_i}^{\perp}. \quad (5.14)$$

Since $R_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}(\perp p)$ is determined by convolutions of O -operators, see Eq. (5.13), we can replace in (5.14)

$$T_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(\perp p) \rightarrow T_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(p) = \frac{n+1}{2n+1} (g_{\alpha_1 \beta_1} - \frac{n}{n+1} \sigma_{\alpha_1 \beta_1}) \prod_{i=2}^n g_{\alpha_i \beta_i}. \quad (5.15)$$

The coefficients in (5.15) are chosen to satisfy the constraints (5.10) and the convolution condition:

$$F_{\alpha_1 \dots \alpha_n}^{\mu_1 \dots \mu_n}(p) F_{\nu_1 \dots \nu_n}^{\alpha_1 \dots \alpha_n}(p) = (-1)^n F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}(p). \quad (5.16)$$

(ii) Projection operators for a baryon system with $J > 1/2$.

If a γN system produces a baryon system with momentum P , the role of the mass is played by the invariant energy $\sqrt{P^2} = \sqrt{s}$. We write:

$$F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}(P) = (-1)^n \frac{\sqrt{s} + \hat{P}}{2\sqrt{s}} R_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}(\perp P). \quad (5.17)$$

The factor $1/(2\sqrt{s})$ compensates the divergency of the numerator at high energies, and this form is more convenient in fitting mechanisms.

5.1.3 Pion–nucleon and photon–nucleon vertices

To be specific, let us consider the processes $\pi N \rightarrow \text{baryon system} \rightarrow \pi N$ and $\gamma N \rightarrow \text{baryon system} \rightarrow \pi N$ in detail.

5.1.3.1 πN vertices

Let us now construct vertices for the decay of a composite baryon system with momentum P into the πN final state with relative momentum $k = 1/2(p'_N - p'_\pi)$. Here p'_N is the nucleon momentum, p'_π is the momentum of the pion and $P = p'_N + p'_\pi$.

(i) πN vertices for the '+' states.

A particle with spin $J^P = 1/2^-$ belongs to the '+' set and decays into the πN channel in an S -wave. Indeed, the parity of the system in an S -wave is equal to the production of the nucleon and pion parities, the P -wave would change the parity while D -wave does not form a $1/2$ state. The orbital angular momentum operator for the S -wave is a scalar, *e.g.* a unit operator. Then the transition amplitude (up to the energy dependent part) can be written as:

$$A = \bar{u}(p'_N) u(P). \quad (5.18)$$

Here $u(P)$ is a bispinor of the composite particle and $\bar{u}(p'_N)$ is the bispinor of the nucleon.

The next '+' state has quantum numbers $3/2^+$ and decays into πN with an orbital angular momentum $L = 1$. It means that the decay vertex must be a vector constructed of $k_\mu^\perp$ and gamma matrices. However, it is sufficient to take only $k_\mu^\perp$: first, due to the orthogonality of γ_μ to the

polarisation vector of the $3/2^+$ particle and, second, due to the fact that the projection operator (the numerator of the fermion propagator) will automatically provide the correct structure. Continuing this procedure for the higher spin states, we obtain for the decay of the '+' baryons:

$$\begin{aligned} G_{B(J^P) \rightarrow \pi N}^{(+)}(P, p'_N) &= \bar{u}(p'_N) N_{\mu_1 \dots \mu_n}^{(+)}(k^\perp) \Psi_{\mu_1 \dots \mu_n}(P) \\ &= \bar{u}(p'_N) X_{\mu_1 \dots \mu_n}^{(n)}(k^\perp) \Psi_{\mu_1 \dots \mu_n}(P) . \end{aligned} \quad (5.19)$$

Here $n = J - 1/2$ and the operator $N_{\mu_1 \dots \mu_n}^{(+)}(k^\perp)$ is called the vertex operator for the '+' state set.

(ii) πN vertices for the '-' states.

Let us construct now the vertices for the decay of '-' states into a πN system. The state with $1/2^+$ decays into πN with the orbital momentum $L = 1$ to satisfy the parity conservation. The $1/2^+$ state is described by a bispinor, and it is a scalar in the vector space. Such a scalar should be constructed from $k_\mu^\perp$ and gamma matrices. However, the simple convolution of the relative momentum and the γ -matrix, $\hat{k}^\perp = k_\mu^\perp \gamma_\mu$ corresponds to the $1/2^-$ state:

$$\begin{aligned} \bar{u}(p'_N) \hat{k}^\perp u(P) &= \bar{u}(p'_N) (\hat{p}'_N - a(s) \hat{P}) u(P) \\ &= \bar{u}(p'_N) u(P) (m_N - a(s) \sqrt{s}) , \end{aligned} \quad (5.20)$$

where $a(s) = (P p'_N)/P^2 = (s + m_N^2 - m_\pi^2)/(2s)$. This is due to the fact that the γ -matrix has also changed the parity of the system. A restoration of the parity can be done by adding a $i\gamma_5$ matrix. Then the basic operator for the decay of a $1/2^+$ state into a nucleon and a pseudoscalar meson has the form:

$$i\gamma_5 \hat{k}^\perp . \quad (5.21)$$

After simple calculations one obtains a standard expression for the nucleon-pion vertex:

$$\begin{aligned} \bar{u}(P) i\gamma_5 \hat{k}^\perp u(p'_N) &= \bar{u}(P) i\gamma_5 (\hat{p}'_N - a(s) \hat{P}) u(p'_N) \\ &= \bar{u}(P) i\gamma_5 u(p'_N) (m_N + a(s) \sqrt{s}) . \end{aligned} \quad (5.22)$$

The factor $\hat{k}^\perp$ introduces only an energy dependence and does not provide any angular dependence for a cross section. Nevertheless, we would like to keep this factor to have a clear correspondence to the LS classification. In the calculations below we denote the scalar factor $(m_N + a(s) \sqrt{s})$ as follows: $\chi_i = m_i + a(s) \sqrt{s} \rightarrow$ (in c.m.s) $m_i + k_{i0}$.

Generally, one can introduce also another scalar expression using γ matrices, $k^\perp$ and the antisymmetrical tensor $\varepsilon_{ijkl} \gamma_i \gamma_j k_k^\perp P_l$. However, making

use of the property of the γ matrices $i\gamma^5\gamma_i\gamma_j\gamma_k = \varepsilon_{ijkl}\gamma_l$, one can show that this operator leads to the same angular dependence as (5.21).

The general form for the decay of systems with $J = L - 1/2$ into πN can be written as:

$$\begin{aligned} G_{B(J^P) \rightarrow \pi N}^{(-)}(P, p'_N) &= \bar{u}(p'_N) N_{\mu_1 \dots \mu_n}^{(-)}(k'^\perp) \Psi_{\mu_1 \dots \mu_n}(P) \\ &= \bar{u}(p'_N) X_{\mu_1 \dots \mu_n \alpha}^{(n+1)}(k'^\perp) i\gamma_5 \gamma_\alpha \Psi_{\mu_1 \dots \mu_n}(P). \end{aligned} \quad (5.23)$$

5.1.3.2 πN scattering

The angular dependent part of the $\pi N \rightarrow \text{resonance} \rightarrow \pi N$ transition amplitude is constructed as a convolution of the vertex functions describing the production and decay of the resonance with the intermediate state propagator and nucleon bispinors:

$$\bar{u}_f \tilde{N}_{\mu_1 \dots \mu_L}^\pm F_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L}(P) N_{\nu_1 \dots \nu_L}^\pm u_i. \quad (5.24)$$

Here $\tilde{N}^\pm$ is the left-hand vertex function (with two particles joining into one resonance) which is different from the decay vertex function $N^\pm$ by the ordering of γ -matrices. Let us define q and k as the relative momenta before and after the interaction and p'_N and p_N as the corresponding nucleon momenta; the amplitude for πN scattering via '+' states can be written in the form

$$A = \bar{u}(p'_N) X_{\mu_1 \dots \mu_L}(k^\perp) F_{\nu_1 \dots \nu_L}^{\mu_1 \dots \mu_L}(P) X_{\nu_1 \dots \nu_L}(q^\perp) u(p_N) BW_L^+(s), \quad (5.25)$$

where $BW_L^+(s)$ describes the energy dependence of the intermediate state propagator given, *e.g.*, by a Breit-Wigner amplitude, a K -matrix or an N/D expression.

Using the properties of the Legendre polynomials and formulae for the convolutions of X -operators with one free index (see Appendix 5.A), we obtain:

$$\begin{aligned} A &= (\sqrt{-k_\perp^2} \sqrt{-q_\perp^2})^L \bar{u}(p'_N) \frac{\sqrt{s + \hat{P}}}{2\sqrt{s}} \frac{\alpha(L)}{2L+1} BW_L^+(s) \\ &\times \left[(L+1)P_L(z) - \frac{\sigma_{\mu\nu} k_\mu q_\nu}{\sqrt{k_\perp^2} \sqrt{q_\perp^2}} P'_L(z) \right] u(p_N), \end{aligned} \quad (5.26)$$

where z is defined as follows:

$$z = \frac{-(k^\perp q^\perp)}{\sqrt{-k_\perp^2} \sqrt{-q_\perp^2}} = (\text{in c.m.s.}) \cos(\vec{k} \vec{q}). \quad (5.27)$$

For a resonance belonging to a '–' state set the amplitude for the transition $\pi N \rightarrow R \rightarrow \pi N$ can be written in the form

$$A = \bar{u}(p'_N) X_{\alpha\mu_1 \dots \mu_{L-1}}^{(L)}(k) \gamma_\alpha^\perp i \gamma_5 F_{\nu_1 \dots \nu_{L-1}}^{\mu_1 \dots \mu_{L-1}}(P) i \gamma_5 \gamma_\xi^\perp X_{\xi\nu_1 \dots \nu_{L-1}}^{(L)}(q) u(p_N) \times BW_L^-(s). \quad (5.28)$$

Taking into account that

$$\begin{aligned} \hat{k}^\perp \hat{q}^\perp &= (k^\perp q^\perp) + \sigma_{\mu\nu}^\perp k_\mu^\perp q_\nu^\perp = \sqrt{-k_\perp^2} \sqrt{-q_\perp^2} \left(z + \frac{\sigma_{\mu\nu}^\perp k_\mu^\perp q_\nu^\perp}{\sqrt{-k_\perp^2} \sqrt{-q_\perp^2}} \right), \\ \hat{k}^\perp \frac{\sigma_{\mu\nu}^\perp k_\mu^\perp q_\nu^\perp}{\sqrt{-k_\perp^2} \sqrt{-q_\perp^2}} \hat{q}^\perp &= \sqrt{-k_\perp^2} \sqrt{-q_\perp^2} \left(1 - z^2 - z \frac{\sigma_{\mu\nu}^\perp k_\mu^\perp q_\nu^\perp}{\sqrt{-k_\perp^2} \sqrt{-q_\perp^2}} \right), \end{aligned} \quad (5.29)$$

we get:

$$\begin{aligned} A &= \bar{u}_i(p'_N) \frac{\sqrt{s} + \hat{P}}{2\sqrt{s}} \frac{\alpha(L)}{L} \left[LP_L(z) + \frac{\sigma_{\mu\nu}^\perp k_\mu^\perp q_\nu^\perp}{\sqrt{k_\perp^2} \sqrt{q_\perp^2}} P'_L(z) \right] u_f(p_N) \\ &\times (\sqrt{-k_\perp^2} \sqrt{-q_\perp^2})^L BW_L^-(s). \end{aligned} \quad (5.30)$$

The total $\pi N \rightarrow \pi N$ transition amplitude is equal to the sum over all possible intermediate quantum numbers. Then

$$\begin{aligned} A &= (\sqrt{-k_\perp^2} \sqrt{-q_\perp^2})^L \bar{u}_i(p'_N) \frac{\sqrt{s} + \hat{P}}{2\sqrt{s}} \left[f_1 + \frac{\sigma_{\mu\nu}^\perp k_\mu^\perp q_\nu^\perp}{\sqrt{-k_\perp^2} \sqrt{-q_\perp^2}} f_2 \right] u_f(p_N), \\ f_1 &= \sum_L \left[\frac{\alpha(L)}{2L+1} (L+1) BW_L^+(s) + \frac{\alpha(L)}{L} L BW_L^-(s) \right] P_L(z), \\ f_2 &= \sum_L \left[\frac{\alpha(L)}{2L+1} BW_L^+(s) - \frac{\alpha(L)}{L} BW_L^-(s) \right] P'_L(z). \end{aligned} \quad (5.31)$$

In the c.m.s. of the resonance where $P = (\sqrt{s}, \vec{0})$ the amplitude (5.31) can be rewritten in the form:

$$\begin{aligned} A_{\pi N \rightarrow \pi N} &= \varphi^* [G(s, z) + H(s, z) i(\vec{\sigma} \vec{n})] \varphi', \\ G(s, z) &= \sum_L [(L+1) F_L^+(s) - L F_L^-(s)] P_L(z), \\ H(s, z) &= \sum_L [F_L^+(s) + F_L^-(s)] P'_L(z), \end{aligned} \quad (5.32)$$

where φ^* and φ' are non-relativistic spinors for initial and final nucleons,

$$\vec{n}_j = -\varepsilon_{\mu\nu j} \frac{k_\mu q_\nu}{|\vec{k}| |\vec{q}|}, \quad \vec{n}^2 = (1 - z^2). \quad (5.33)$$

$F_L^\pm$ are functions which depend only on the energy:

$$\begin{aligned} F_L^+ &= (|\vec{k}||\vec{q}|)^L \sqrt{\chi_i \chi_f} \frac{\alpha(L)}{2L+1} BW_L^+(s) , \\ F_L^- &= (|\vec{k}||\vec{q}|)^L \sqrt{\chi_i \chi_f} \frac{\alpha(L)}{L} BW_L^-(s) , \\ \chi_i &= m_i + a(s)\sqrt{s} = m_i + k_{i0} , \end{aligned} \quad (5.34)$$

where k_{i0} is given in the c.m. system.

Let us consider some simple examples. The $1/2^-$ state belongs to the '+' set of states with $L = 0$. Then

$$A_{\pi N \rightarrow \pi N} = \varphi^* F_0^+(s) \varphi' , \quad (5.35)$$

and the cross section, which is proportional to the amplitude squared, has a uniform angular distribution. For a $1/2^+$ state ($L = 1$) the amplitude has a complicated z -dependence:

$$A_{\pi N \rightarrow \pi N} = \varphi^* (z + i(\vec{\sigma}\vec{n})) F_1^-(s) \varphi' . \quad (5.36)$$

However, the cross section defined by the production of the $1/2^+$ partial wave has a flat angular distribution:

$$\sigma \sim |A|^2 = (z^2 + 1 - z^2) |F_1^-(s)|^2 = |F_1^-(s)|^2 . \quad (5.37)$$

The z dependence of the $1/2^+$ amplitude reveals itself in a polarisation experiment or in the case of interferences with other partial waves. For example, in the case of mixing the $1/2^-$ and $1/2^+$ partial waves the amplitude has two components:

$$A_{\pi N \rightarrow \pi N} = \varphi^* \left[F_0^+(s) + (z + i(\vec{\sigma}\vec{n})) F_1^-(s) \right] \varphi' . \quad (5.38)$$

In this case the differential cross section is depending linearly on z with a slope defined by the ratio of the real part of the product of amplitudes and the sum of amplitudes squared:

$$\sigma \sim |A|^2 = |F_0^+(s)|^2 + |F_1^-(s)|^2 + 2z \operatorname{Re} \left[F_0^+(s) F_1^-(s) \right] . \quad (5.39)$$

5.1.4 Photon–nucleon vertices

One should distinguish between decays with the emission of a virtual photon and that of a real one. We present vertices for both cases.

5.1.4.1 Operators for a photon–nucleon system

A vector particle (*e.g.* a virtual photon γ^*) has spin 1 and therefore the γ^*N system can form two S -wave states with total spins $1/2$ and $3/2$. These states are usually called spin states. In a combination of these two spin states with the orbital momentum L , six sets of states can be formed; three ‘+’ states:

$$\begin{aligned}
 J = L + \frac{1}{2}, \quad S = \frac{1}{2}, \quad P = (-1)^{L+1}, \quad L = 0, 1, \dots, \quad J^P &= \frac{1}{2}^-, \frac{3}{2}^+, \frac{5}{2}^- \dots \\
 J = L - \frac{3}{2}, \quad S = \frac{3}{2}, \quad P = (-1)^{L+1}, \quad L = 2, 3, \dots, \quad J^P &= \frac{1}{2}^-, \frac{3}{2}^+, \frac{5}{2}^- \dots \\
 J = L + \frac{1}{2}, \quad S = \frac{3}{2}, \quad P = (-1)^{L+1}, \quad L = 1, 2, \dots, \quad J^P &= \frac{3}{2}^+, \frac{5}{2}^- \dots
 \end{aligned} \tag{5.40}$$

and three ‘-’ states:

$$\begin{aligned}
 J = L - \frac{1}{2}, \quad S = \frac{1}{2}, \quad P = (-1)^{L+1}, \quad L = 1, 2, \dots, \quad J^P &= \frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+ \dots \\
 J = L - \frac{1}{2}, \quad S = \frac{3}{2}, \quad P = (-1)^{L+1}, \quad L = 1, 2, \dots, \quad J^P &= \frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+ \dots \\
 J = L + \frac{3}{2}, \quad S = \frac{3}{2}, \quad P = (-1)^{L+1}, \quad L = 0, 1, \dots, \quad J^P &= \frac{3}{2}^-, \frac{5}{2}^+ \dots
 \end{aligned} \tag{5.41}$$

States which have different L and S but the same J^P can mix.

5.1.4.2 Operators for γ^*N states with $J^P = \frac{1}{2}^-, \frac{3}{2}^+, \frac{5}{2}^-, \dots$

Let us start with operators for ‘+’ states. The lowest $1/2^-$ γ^*N system can either be formed by the spin state $S = 1/2$ and $L = 0$ or by the spin state $S = 3/2$ and $L = 2$. For the S -wave system the orbital angular momentum operator is a unit operator and the index of the photon polarisation vector can be convoluted with a γ matrix only. The γ matrix, however, changes the parity of the system. To compensate this unwanted change, an additional $i\gamma_5$ matrix has to be introduced. Therefore the operator describing the transition of the state with spin $1/2^-$ into a γ and $1/2^+$ S -wave fermion is

$$\bar{u}(P)\gamma_\mu i\gamma_5 u(p_N)\epsilon_\mu. \tag{5.42}$$

Here $\bar{u}(P)$ is the bispinor describing a baryon resonance with momentum P , $u(p_N)$ is the bispinor for the final fermion with momentum p_N and ϵ_μ is the polarisation vector of the vector particle. In combination with the

orbital angular momentum operators $X_{\mu_1 \dots \mu_n}^{(n)}$, the operator (5.42) defines the first set of the operators for states (5.40):

$$G_{\gamma^* N \rightarrow B(JP)}^{(1+)}(P, q) = \bar{\Psi}_{\alpha_1 \dots \alpha_n} \gamma_\mu i \gamma_5 X_{\alpha_1 \dots \alpha_n}^{(n)}(k^\perp) u(p_N) \epsilon_\mu. \quad (5.43)$$

As before, $\Psi_{\alpha_1 \dots \alpha_L}$ is a tensor bispinor wave function for the system with $J = n + 1/2$, and $k^\perp$ is the relative momentum of the $\gamma^* N$ system orthogonal to the total momentum of the system. For these partial waves $L = n$.

The decay of a $1/2^- \gamma^* N$ system in the D -wave must be described by the D -wave orbital angular momentum operator. The only non-zero convolution is defined as:

$$\bar{u}(P) \gamma_\nu i \gamma_5 X_{\mu\nu}^{(2)}(k^\perp) u(p_N) \epsilon_\mu. \quad (5.44)$$

Here again, the γ_5 matrix is introduced to provide a correct P-parity. One can easily write the second set of operators (5.40) with $J = L - 3/2$:

$$G_{\gamma^* N \rightarrow B(JP)}^{(2+)}(P, q) = \bar{\Psi}_{\alpha_1 \dots \alpha_n} \gamma_\nu i \gamma_5 X_{\mu\nu\alpha_1 \dots \alpha_n}^{(n+2)}(k^\perp) u(p_N) \epsilon_\mu. \quad (5.45)$$

The third set of operators starts from the total momentum $J = 3/2$. The basic operator describes the P-wave decay of a $3/2^+$ system into a baryon and a vector particle. It has the form

$$\bar{\Psi}_\mu \gamma_\nu i \gamma_5 X_\nu^{(1)}(k^\perp) u(p_N) \epsilon_\mu. \quad (5.46)$$

The operators for a baryon with $J = L + 1/2$ can be written as

$$G_{\gamma^* N \rightarrow B(JP)}^{(3+)}(P, q) = \bar{\Psi}_{\mu\alpha_1 \dots \alpha_{n-1}} \gamma_\nu i \gamma_5 X_{\nu\alpha_1 \dots \alpha_{n-1}}^{(n)}(k^\perp) u(p_N) \epsilon_\mu. \quad (5.47)$$

Owing to gauge invariance, in the case of the photoproduction the operators (5.45) are reduced to those given in (5.43), the gauge invariance requires $\epsilon_\mu k_{1\mu} = \epsilon_\mu k_{2\mu} = \epsilon_\mu k_\perp^\perp = 0$.

Using the recurrent expression for the X -operators (Appendix 4.A of Chapter 4), we obtain

$$\begin{aligned} & \bar{\Psi}_{\alpha_1 \dots \alpha_n} \gamma_\nu i \gamma_5 X_{\mu\nu\alpha_1 \dots \alpha_n}^{(n+2)}(k^\perp) u(p_N) \epsilon_\mu \\ &= \frac{-k_\perp^2 \alpha(n)}{(2n-1)\alpha(n-2)} \bar{\Psi}_{\alpha_1 \dots \alpha_n} \gamma_\mu i \gamma_5 X_{\alpha_1 \dots \alpha_n}^{(n)}(k^\perp) u(p_N) \epsilon_\mu. \end{aligned} \quad (5.48)$$

Hence, in the case of real photons both sets of operators (5.43) and (5.45) produce the same angular dependence.

It is convenient to write the decay amplitudes as a convolution of the spinor wave functions and the vertex functions $V_{\alpha_1 \dots \alpha_L}^{(i+)\mu}$ $i = 1, 2, 3$. Then Eqs. (5.43), (5.45), (5.47) can be rewritten as

$$\begin{aligned} G_{\gamma^* N \rightarrow B(JP)}^{(i+)}(P, q) &= \bar{\Psi}_{\alpha_1 \dots \alpha_n} V_{\alpha_1 \dots \alpha_n}^{(i+)\mu}(k^\perp) u(p_N) \epsilon_\mu, \\ V_{\alpha_1 \dots \alpha_n}^{(1+)\mu}(k^\perp) &= \gamma_\mu i \gamma_5 X_{\alpha_1 \dots \alpha_n}^{(n)}(k^\perp), \\ V_{\alpha_1 \dots \alpha_n}^{(2+)\mu}(k^\perp) &= \gamma_\nu i \gamma_5 X_{\mu\nu\alpha_1 \dots \alpha_n}^{(n+2)}(k^\perp), \\ V_{\alpha_1 \dots \alpha_n}^{(3+)\mu}(k^\perp) &= \gamma_\nu i \gamma_5 X_{\nu\alpha_1 \dots \alpha_{n-1}}^{(n)}(k^\perp) g_{\mu\alpha_n}^\perp. \end{aligned} \quad (5.49)$$

5.1.4.3 Operators for $1/2^+$, $3/2^-$, $5/2^+$, ... states

A $1/2^+$ particle decays into a fermion with $J^P = 1/2^+$ and 1^- particle in a relative P-wave only. The operator for spin $1/2$ of the γ^*N system can be constructed in the same way as the corresponding operator for the $'++'$ -states. The P-wave orbital angular momentum operator must be convoluted with a γ -matrix as well as with a γ_5 operator to provide the correct parity. The transition amplitude can be written as

$$\bar{u}(P)i\gamma_5\gamma_\xi i\gamma_5\gamma_\mu X_\xi^{(1)}u(p_N)\epsilon_\mu = \bar{u}(P)\gamma_\xi\gamma_\mu X_\xi^{(1)}u(p_N)\epsilon_\mu . \quad (5.50)$$

and the vertex for the system with $S = 1/2$ and $J = L + 1/2$ has the form:

$$G_{\gamma^*N \rightarrow B(J^P)}^{(1-)}(P, q) = \bar{\Psi}_{\alpha_1 \dots \alpha_n} \gamma_\xi \gamma_\mu X_{\xi \alpha_1 \dots \alpha_n}^{(n+1)}(k^\perp) u(p_N) \epsilon_\mu . \quad (5.51)$$

with $n = J - 1/2$.

For the $'-'$ states, the operators with $S = 3/2$ and $J = L - 1/2$ have the same orbital angular momentum as the $S = 1/2$ operator. However, here the polarisation vector convolutes with the index of the orbital angular momentum operator. Then

$$G_{\gamma^*N \rightarrow B(J^P)}^{(2-)}(P, q) = \bar{\Psi}_{\alpha_1 \dots \alpha_n} X_{\mu \alpha_1 \dots \alpha_n}^{(n+1)}(k^\perp) u(p_N) \epsilon_\mu . \quad (5.52)$$

The third set of operators starts with the total spin $3/2$. The basic operator describes the decay of the $3/2^-$ system into the nucleon and a photon in a relative S -wave. Thus

$$\bar{\Psi}_\mu u(p_N) \epsilon_\mu , \quad (5.53)$$

and we obtain the set

$$G_{\gamma^*N \rightarrow B(J^P)}^{(3-)}(P, q) = \bar{\Psi}_{\alpha_1 \dots \alpha_n} X_{\alpha_2 \dots \alpha_n}^{(n-1)}(k^\perp) g_{\alpha_1 \mu}^\perp u(p_N) \epsilon_\mu . \quad (5.54)$$

Remember that for these states $J = L + 3/2$.

For real photons the operator (5.52) vanishes for $J = 1/2^+$, and for higher states these operators provide the same angular dependence as the (5.54) operators.

For the sake of convenience we introduce the vertex functions $V_{\alpha_1 \dots \alpha_n}^{(i-)\mu}$ $i = 1, 2, 3$ as it was done in the case of $'++'$ states

$$\begin{aligned} G_{\gamma^*N \rightarrow B(J^P)}^{(i-)}(P, q) &= \bar{\Psi}_{\alpha_1 \dots \alpha_n} V_{\mu \alpha_1 \dots \alpha_n}^{(i-)\mu}(k^\perp) u(p_N) \epsilon_\mu , \\ V_{\alpha_1 \dots \alpha_n}^{(1-)\mu}(k^\perp) &= \gamma_\xi \gamma_\mu X_{\xi \alpha_1 \dots \alpha_n}^{(n+1)}(k^\perp) , \\ V_{\alpha_1 \dots \alpha_n}^{(2-)\mu}(k^\perp) &= X_{\mu \alpha_1 \dots \alpha_n}^{(n+1)}(k^\perp) , \\ V_{\alpha_1 \dots \alpha_n}^{(3-)\mu}(k^\perp) &= X_{\alpha_2 \dots \alpha_n}^{(n-1)}(k^\perp) g_{\alpha_1 \mu}^\perp . \end{aligned} \quad (5.55)$$

As for the $'++'$ states, in the case of real photons only two sets of operators are independent.

5.1.4.4 Vertices for γN states

Vertices for real photon–nucleon states are determined by formulae analogous to (5.49) and (5.55), substituting:

$$\epsilon_{\mu}^{(\gamma^*)} \rightarrow \epsilon_{\mu}^{(\gamma)}, \quad G_{\gamma^* N \rightarrow B(J^P)}^{(i)}(P, q) \rightarrow G_{\gamma N \rightarrow B(J^P)}^{(i)}(P, q_{\gamma}). \quad (5.56)$$

Recall that here $q_{\gamma}^2 = 0$ and $\epsilon_{\mu}^{(\gamma)} P_{\mu} = 0$, $\epsilon_{\mu}^{(\gamma)} q_{\gamma\mu} = 0$ (for details, see Section 5.1.1). This procedure reduces the number of independent vertices (and spin operators, respectively).

5.2 Single Meson Photoproduction

The amplitude for the photoproduction of a single pseudoscalar meson is well known and can be found in the literature. In the centre-of-mass frame of the reaction the general structure of the amplitude can be derived from the gauge invariance and parity conservation. Thus

$$A = \varphi^* J_{\mu} \epsilon_{\mu} \varphi',$$

$$J_{\mu} = i\mathcal{F}_1 \sigma_{\mu} + \mathcal{F}_2 (\vec{\sigma} \vec{q}) \frac{\epsilon_{\mu ij} \sigma_i k_j}{|\vec{k}| |\vec{q}|} + i\mathcal{F}_3 \frac{(\vec{\sigma} \vec{k})}{|\vec{k}| |\vec{q}|} q_{\mu} + i\mathcal{F}_4 \frac{(\vec{\sigma} \vec{q})}{|\vec{q}|^2} q_{\mu}, \quad (5.57)$$

where $\vec{q}$ is the momentum of the nucleon in the πN channel and $\vec{k}$ is the momentum of the nucleon in the γN channel calculated in the c.m.s. of the reaction, and σ_i are Pauli matrices. Remember that in the c.m.s. the momentum of the nucleon p_N is equal to the relative momentum between the nucleon and the second particle.

The functions $\mathcal{F}_i$ have the following angular dependence:

$$\begin{aligned} \mathcal{F}_1(z) &= \sum_{L=0}^{\infty} [LM_L^+ + E_L^+] P'_{L+1}(z) + [(L+1)M_L^- + E_L^-] P'_{L-1}(z), \\ \mathcal{F}_2(z) &= \sum_{L=1}^{\infty} [(L+1)M_L^+ + LM_L^-] P'_L(z), \\ \mathcal{F}_3(z) &= \sum_{L=1}^{\infty} [E_L^+ - M_L^+] P''_{L+1}(z) + [E_L^- + M_L^-] P''_{L-1}(z), \\ \mathcal{F}_4(z) &= \sum_{L=2}^{\infty} [M_L^+ - E_L^+ - M_L^- - E_L^-] P''_L(z). \end{aligned} \quad (5.58)$$

Here L corresponds to the orbital angular momentum in the πN system, $P_L(z)$ are Legendre polynomials $z = (\vec{k} \vec{q}) / (|\vec{k}| |\vec{q}|)$ and $E_L^{\pm}$ and $M_L^{\pm}$ are

electric and magnetic multipoles describing transitions to states with $J = L \pm 1/2$. There are no contributions from M_0^+ , E_0^- and E_1^- for spin $1/2$ resonances. In what follows we will construct the $\gamma N \rightarrow \pi N$ transition amplitudes using the operators defined in the previous sections and show that in the centre-of-mass frame these amplitudes satisfy the equations (5.57), (5.58).

5.2.1 Photoproduction amplitudes for $1/2^-, 3/2^+, 5/2^-, \dots$ states

The angular dependence of the single-meson production amplitude via an intermediate resonance has the general form

$$\bar{u}(q_1) \tilde{N}_{\alpha_1 \dots \alpha_n}^{\pm}(q^\perp) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(P) V_{\beta_1 \dots \beta_n}^{(\pm)\mu}(k^\perp) u(p_N) \epsilon_\mu. \quad (5.59)$$

Here q_1 and p_N are the momenta of the nucleon in the πN and γN channel and $q^\perp$ and $k^\perp$ are the components of the relative momenta which are orthogonal to the total momentum of the resonance.

If states with $J = L + 1/2$ are produced from a γN partial wave with spin $1/2$, one has the following expression for the amplitude:

$$A^+(1/2) = \bar{u}(q_1) X_{\alpha_1 \dots \alpha_L}^{(L)}(q^\perp) F_{\beta_1 \dots \beta_L}^{\alpha_1 \dots \alpha_L}(P) \gamma_\mu i \gamma_5 X_{\beta_1 \dots \beta_L}^{(L)}(k^\perp) u(p_N) \epsilon_\mu \times BW(s), \quad (5.60)$$

where $BW(s)$ represents the dynamical part of the amplitude.

Calculating this amplitude in the centre-of-mass frame of the reaction, we obtain the following correspondence between the spin operators and multipoles (for details see Appendix 5.C):

$$E_L^{+(\frac{1}{2})} = (-1)^L \sqrt{\chi_i \chi_f} \frac{\alpha(L)}{2L+1} \frac{(|\vec{k}| |\vec{q}|)^L}{L+1} BW(s), \quad M_L^{+(\frac{1}{2})} = E_L^{+(\frac{1}{2})}. \quad (5.61)$$

Here and below the $E_L^{+(\frac{1}{2})}$ and $M_L^{+(\frac{1}{2})}$ multipoles correspond to the decomposition of spin $1/2$ amplitudes. In the case of photoproduction, only two γN operators are independent for every resonance with spin $3/2$ and higher (for $J = 1/2$ states there is only one independent operator). For the set of $J = L + 1/2$ states the second operator has the amplitude structure:

$$A^+(3/2) = \bar{u}(q_1) X_{\alpha_1 \dots \alpha_L}^{(L)}(q^\perp) F_{\mu\beta_2 \dots \beta_L}^{\alpha_1 \dots \alpha_L}(P) \gamma_\xi i \gamma_5 X_{\xi\beta_2 \dots \beta_L}^{(L)}(k^\perp) u(p_N) \epsilon_\mu \times BW(s). \quad (5.62)$$

Using expressions given in Appendix 5.B one obtains the multipole decomposition

$$E_L^{+(\frac{3}{2})} = (-1)^L \sqrt{\chi_i \chi_f} \frac{\alpha(L)}{2L+1} \frac{(|\vec{k}| |\vec{q}|)^L}{L+1} BW(s), \quad M_L^{+(\frac{3}{2})} = -\frac{E_L^{+(\frac{3}{2})}}{L}. \quad (5.63)$$

The $E_L^{+(\frac{3}{2})}$ and $M_L^{+(\frac{3}{2})}$ multipoles correspond to the decomposition of spin-3/2 amplitudes.

5.2.2 Photoproduction amplitudes for $1/2^+$, $3/2^-$, $5/2^+$, ... states

The $\gamma N \rightarrow \pi N$ amplitude for states with $J = L - 1/2$ in the πN channel has the structure

$$A^-(1/2) = \bar{u}(q_1) \gamma_\xi i \gamma_5 X_{\xi\alpha_1 \dots \alpha_{L-1}}^{(L)}(q^\perp) F_{\beta_1 \dots \beta_{L-1}}^{\alpha_1 \dots \alpha_{L-1}}(P) \gamma_\xi \gamma_\mu X_{\xi\beta_1 \dots \beta_{L-1}}^{(L)}(k^\perp) \times u(p_N) \epsilon_\mu BW(s). \quad (5.64)$$

For the amplitude (5.64) we find the following connection to the multipoles:

$$E_L^{-(\frac{1}{2})} = (-1)^L \sqrt{\chi_i \chi_f} |\vec{k}| |\vec{q}|^L \frac{\alpha(L)}{L^2} BW(s), \quad M_L^{-(\frac{1}{2})} = -E_L^{-(\frac{1}{2})}. \quad (5.65)$$

Amplitudes including spin 3/2 operators have the structure

$$A^-(3/2) = \bar{u}(q_1) \gamma_\xi i \gamma_5 X_{\xi\alpha_1 \dots \alpha_{L-1}}^{(L)}(q^\perp) F_{\mu\beta_2 \dots \beta_{L-1}}^{\alpha_1 \dots \alpha_{L-1}}(P) X_{\beta_2 \dots \beta_{L-1}}^{(L-2)}(k^\perp) u(p_N) \epsilon_\mu \times BW(s), \quad (5.66)$$

and, correspondingly,

$$E_L^{-(\frac{3}{2})} = (-1)^L \sqrt{\chi_i \chi_f} |\vec{k}|^{L-2} |\vec{q}|^L \frac{\alpha(L-2)}{(L-1)L} BW(s), \quad M_L^{-(\frac{3}{2})} = 0. \quad (5.67)$$

5.2.3 Relations between the amplitudes in the spin-orbit and helicity representation

The helicity transition amplitudes are combinations of the spin-1/2 and spin-3/2 amplitudes $A^\pm(1/2)$, $A^\pm(3/2)$. For '+' multipoles the relations between the helicity amplitudes and multipoles are

$$\begin{aligned} \tilde{A}^{1/2} &= -\frac{1}{2}(LM_L^+ + (L+2)E_L^+), \\ \tilde{A}^{3/2} &= \frac{1}{2}\sqrt{L(L+2)}(E_L^+ - M_L^+). \end{aligned} \quad (5.68)$$

For the '-' sector the relations are

$$\begin{aligned} \tilde{A}^{1/2} &= \frac{1}{2}((L+1)M_L^- - (L-1)E_L^-), \\ \tilde{A}^{3/2} &= -\frac{1}{2}\sqrt{(L-1)(L+1)}(E_L^- + M_L^-). \end{aligned} \quad (5.69)$$

The energy dependence of the helicity transition amplitudes $\tilde{A}^{1/2}$ and $\tilde{A}^{3/2}$ is a model-dependent subject which will be discussed in the next section. Note that these amplitudes differ from the helicity vertex functions $A^{1/2}$, $A^{3/2}$ given in PDG by a constant factor:

$$(A^{1/2}, A^{3/2}) = C(\tilde{A}^{1/2}, \tilde{A}^{3/2}). \quad (5.70)$$

The ratio of the transition amplitudes $\tilde{A}^{1/2}$, $\tilde{A}^{3/2}$ (which is equal to the ratio of the helicity vertex functions in the case of the Breit–Wigner parametrisation) depends on the γN interaction only, and it should be the same in all photoproduction reactions.

For ‘+’ states we obtain the following decomposition of the spin-1/2 amplitude (5.61):

$$\tilde{A}^{1/2} = -(L+1)E_L^{+(\frac{1}{2})}, \quad \tilde{A}^{3/2} = 0. \quad (5.71)$$

Obviously, a spin-1/2 state cannot have a helicity 3/2 projection. For the spin-3/2 state one gets

$$\tilde{A}^{1/2} = -\frac{L+1}{2}E_L^{+(\frac{3}{2})}, \quad \tilde{A}^{3/2} = \frac{1}{2}\sqrt{\frac{L+2}{L}}(L+1)E_L^{+(\frac{3}{2})}. \quad (5.72)$$

The ratio of the helicity amplitudes can be calculated directly if the ratio of the spin amplitudes is known. The $BW(s)$ is in both amplitudes an energy-dependent part of the amplitude which depends on the model used in the analysis. If we extract explicitly the initial coupling constants $g_{1/2}^+(L)$ and $g_{3/2}^+(L)$ for the spins 1/2 and 3/2 (here L is the orbital momentum in the πN system which is equal to the orbital momentum in the γN system for 1 and 3 operator sets), then the expression for the total amplitude for ‘+’ states has the form

$$A_{tot}^{L+} = [g_{1/2}^+(L) A^+(1/2) + g_{3/2}^+(L) A^+(3/2)]. \quad (5.73)$$

In this case the multipole amplitudes can be rewritten as follows:

$$\begin{aligned} E_L^{+(\frac{1}{2})} &= (-1)^L \sqrt{\chi_i \chi_f} \frac{\alpha(L)}{2L+1} \frac{(|\vec{k}||\vec{q}|)^L}{L+1} g_{1/2}^+(L) BW(s), \\ E_L^{+(\frac{3}{2})} &= (-1)^L \sqrt{\chi_i \chi_f} \frac{\alpha(L)}{2L+1} \frac{(|\vec{k}||\vec{q}|)^L}{L+1} g_{3/2}^+(L) BW(s), \\ E_L^+ &= E_L^{+(\frac{1}{2})} + E_L^{+(\frac{3}{2})}. \end{aligned} \quad (5.74)$$

Using (5.71) and (5.72), one can calculate the ratio between the helicity amplitudes for ‘+’ states:

$$\begin{aligned} \frac{\tilde{A}^{3/2}}{\tilde{A}^{1/2}} &= \frac{A^{3/2}}{A^{1/2}} = -\frac{\frac{1}{2}\sqrt{\frac{L+2}{L}}(L+1)E_L^{+(\frac{3}{2})}}{\frac{L+1}{2}E_L^{+(\frac{3}{2})} + (L+1)E_L^{+(\frac{1}{2})}} = -\sqrt{\frac{L+2}{L}} \frac{1}{1+2R^+}, \\ R^+ &= \frac{g_{1/2}^+(L)}{g_{3/2}^+(L)}. \end{aligned} \quad (5.75)$$

This ratio does not depend on the final state of the photoproduction process, it is valid for any photoproduction reaction and should be compared with PDG values.

In the case of the $'-'$ states we get for the spin-1/2 amplitude:

$$\tilde{A}^{1/2} = -LE_L^{-(\frac{1}{2})}, \quad \tilde{A}^{3/2} = 0, \quad (5.76)$$

and for the spin 3/2 amplitudes

$$\tilde{A}^{1/2} = -\frac{L-1}{2}E_L^{-(\frac{3}{2})}, \quad \tilde{A}^{3/2} = -\frac{1}{2}\sqrt{(L-1)(L+1)}E_L^{-(\frac{3}{2})}. \quad (5.77)$$

For $'-'$ states the γp vertex has the same orbital momentum as the πN vertex (L) for spin-1/2 amplitudes, and $L-2$ for spin-3/2 amplitudes:

$$A_{tot}^{L-} = \left[g_{1/2}^-(L)A^-(1/2) + g_{3/2}^-(L-2)A^+(3/2) \right] \quad (5.78)$$

The multipole amplitudes can be rewritten as follows:

$$\begin{aligned} E_L^{-(\frac{1}{2})} &= (-1)^L \sqrt{\chi_i \chi_f} |\vec{k}|^L |\vec{q}|^L \frac{\alpha(L)}{L^2} g_{1/2}^-(L) BW(s), \\ E_L^{-(\frac{3}{2})} &= (-1)^L \sqrt{\chi_i \chi_f} |\vec{k}|^{L-2} |\vec{q}|^L \frac{\alpha(L-2)}{(L-1)L} g_{3/2}^-(L-2) BW(s), \\ E_L^- &= E_L^{-(\frac{1}{2})} + E_L^{-(\frac{3}{2})}. \end{aligned} \quad (5.79)$$

For the ratio of helicity amplitudes one obtains:

$$\frac{\tilde{A}^{3/2}}{\tilde{A}^{1/2}} = \frac{A^{3/2}}{A^{1/2}} = \frac{\frac{1}{2}\sqrt{(L-1)(L+1)}E_L^{-(\frac{3}{2})}}{\frac{L-1}{2}E_L^{-(\frac{3}{2})} + LE_L^{-(\frac{1}{2})}} = \sqrt{\frac{L+1}{L-1}} \frac{1}{1+2R^-}, \quad (5.80)$$

where

$$R^- = \frac{(2L-1)(2L-3)}{L(L-1)} |\vec{k}|^2 \frac{g_{1/2}^-(L)}{g_{3/2}^-(L-2)}. \quad (5.81)$$

This ratio calculated in the resonance mass should be compared with PDG values.

5.3 The Decay of Baryons into a Pseudoscalar Particle and a 3/2 State

The system of a $3/2^+$ particle and a pseudoscalar particle 0^- can form a state with $J^P = 3/2^-$ in the S -wave. This means that for large orbital momenta this system can form $J = L-3/2, L-1/2, L+1/2$ and $L+3/2$ states. Two of these states belong to the $'+'$ set of states and two others to the $'-'$ set.

5.3.1 Operators for '+' states

Let us start from the lowest member of the '+' set of states. A $1/2^-$ particle decays into a $J^P = 3/2^+$ -particle (e.g. Δ) and a pseudoscalar meson (e.g. a pion) in D -wave. Only one index of the orbital angular momentum operator can be convoluted with the γ -matrix owing to the tracelessness and the symmetry properties. Therefore, the second index should be convoluted with the vector index of the $3/2^+$ state wave function. Again, to compensate the change of parity due to the γ -matrix one has to introduce an additional γ_5 -matrix. Thus the amplitude describing the transition of a state with spin $1/2^-$ into a $\Delta\pi$ system can be written as

$$\bar{u}(P) i\gamma_5 \gamma_\nu X_{\mu\nu}^{(2)} \Psi_\mu^\Delta, \quad (5.82)$$

where $\bar{u}(P)$ is a spinor describing an initial state and Ψ_μ^Δ is a vector spinor for the final spin-3/2 fermion. It is easy to derive the whole set of operators which describe the decay of states with $J = n + 1/2 = L - 3/2$ into a pseudoscalar meson and $3/2^+$ state:

$$\bar{\Psi}_{\alpha_1 \dots \alpha_n} i\gamma_5 \gamma_\nu X_{\mu\nu\alpha_1 \dots \alpha_n}^{(n+2)} \Psi_\mu^\Delta. \quad (5.83)$$

The second set of operators with the total spin equal to the orbital momentum $J = L$ starts with the total spin 3/2. The basic operator describes the decay of the $3/2^+$ system into another $3/2^+$ state and a pion in a P-wave. Here the index of the orbital momentum operator convolutes with the γ -matrix and the vector index of the initial state with the vector index of the final particle. Hence,

$$\bar{\Psi}_\alpha i\gamma_5 \gamma_\nu X_\nu^{(1)} g_{\alpha\mu}^\perp \Psi_\mu^\Delta. \quad (5.84)$$

From this expression one can easily deduce the second set of operators:

$$\bar{\Psi}_{\alpha_1 \dots \alpha_n} i\gamma_5 \gamma_\nu X_{\nu\alpha_2 \dots \alpha_n}^{(n)} g_{\alpha_1\mu}^\perp \Psi_\mu^\Delta, \quad L = 1, 2, \dots \quad (5.85)$$

Thus the vertex functions for '+' states are

$$\begin{aligned} \bar{\Psi}_{\alpha_1 \dots \alpha_n} N_{\alpha_1 \dots \alpha_n}^{(i+)\mu} \Psi_\mu^\Delta, \quad & N_{\alpha_1 \dots \alpha_n}^{(1+)\mu} = i\gamma_5 \gamma_\nu X_{\mu\nu\alpha_1 \dots \alpha_n}^{(n+2)}, \\ & N_{\alpha_1 \dots \alpha_n}^{(2+)\mu} = i\gamma_5 \gamma_\nu X_{\nu\alpha_2 \dots \alpha_n}^{(n)} g_{\alpha_1\mu}^\perp. \end{aligned} \quad (5.86)$$

5.3.2 Operators for $1/2^+$, $3/2^-$, $5/2^+$, ... states

We consider here the decay of the '-' states into a $3/2^+$ particle and a pseudoscalar meson. A $1/2^+$ particle may decay into a $J^P = 3/2^+$ baryon and a 0^- meson in P-wave. In this case the P-wave orbital angular momentum

operator must be converted with the vector spinor Ψ_μ^Δ . The γ_5 operator is not needed to provide a correct parity for the state. Then the amplitude is

$$\bar{u}(P)X_\mu^{(1)}\Psi_\mu^\Delta. \quad (5.87)$$

The decay of higher states will occur with a higher orbital momentum and the tensor indices of the polarisation vector should be convoluted with indices of the orbital momentum. This set of operators has a spin $S = 3/2$, the total spin is $J = L - 1/2$ and we can write in a general form:

$$\bar{\Psi}_{\alpha_1 \dots \alpha_n} X_{\mu\alpha_1 \dots \alpha_n}^{(n+1)} \Psi_\mu^\Delta, \quad n = 1, 2, \dots \quad (5.88)$$

The second set of operators starts from the total spin $J = 3/2$. The basic operator describes the decay of the $3/2^-$ system into a $3/2^+$ particle and a pion in S -wave. Consequently,

$$\bar{\Psi}_\mu \Psi_\mu^\Delta, \quad (5.89)$$

and we obtain for this set

$$\bar{\Psi}_{\alpha_1 \dots \alpha_n} X_{\alpha_2 \dots \alpha_n}^{(n-1)} g_{\alpha_1 \mu}^\perp \Psi_\mu^\Delta, \quad n = 1, 2, \dots \quad (5.90)$$

The vertex functions for '–' states are given by:

$$\begin{aligned} \bar{\Psi}_{\alpha_1 \dots \alpha_n} N_{\alpha_1 \dots \alpha_n}^{(i-)\mu} \Psi_\mu^\Delta, \quad & N_{\alpha_1 \dots \alpha_n}^{(1-)\mu} = X_{\mu\alpha_1 \dots \alpha_n}^{(n+1)}, \\ & N_{\alpha_1 \dots \alpha_n}^{(2-)\mu} = X_{\alpha_2 \dots \alpha_n}^{(n-1)} g_{\alpha_1 \mu}^\perp. \end{aligned} \quad (5.91)$$

5.3.3 Operators for the decays $J^+ \rightarrow 0^- + 3/2^+$, $J^+ \rightarrow 0^+ + 3/2^-$, $J^- \rightarrow 0^+ + 3/2^+$ and $J^- \rightarrow 0^- + 3/2^-$

The operators given in the previous sections provide a full set of operators for the decay of a baryon into a meson with spin 0 and a fermion with spin $3/2$. Indeed, for the construction of operators only the total spin of the system plays a role. Thus the operators for $J^+ \rightarrow 0^- + 3/2^+$ decays and those for $J^+ \rightarrow 0^+ + 3/2^-$, $J^- \rightarrow 0^+ + 3/2^+$ and $J^- \rightarrow 0^- + 3/2^-$ decays have the same form.

5.4 Double Pion Photoproduction Amplitudes

The operators introduced in the previous sections provide a direct way to construct amplitudes in the case of many particle photo- and pion production. In this section we will show an example for the construction of the double pion photoproduction amplitudes.

The reactions as shown in Fig. 5.1 are taken into account where the decay into the final state proceeds via the production of an intermediate baryon or meson resonance. The general form of the angular dependent

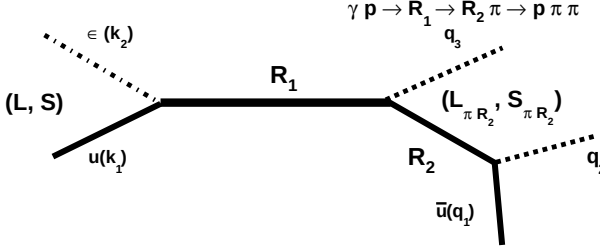


Fig. 5.1 Photoproduction of two mesons due to the cascade of a resonance.

part of the amplitude for such a process is

$$\begin{aligned} & \bar{u}(q_1) \tilde{N}_{\alpha_1 \dots \alpha_n}(R_2 \rightarrow \mu N) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(q_1 + q_2) \tilde{N}_{\gamma_1 \dots \gamma_m}^{(j) \beta_1 \dots \beta_n}(R_1 \rightarrow \mu R_2) \\ & \times F_{\xi_1 \dots \xi_m}^{\gamma_1 \dots \gamma_m}(P) V_{\xi_1 \dots \xi_m}^{(i) \mu}(R_1 \rightarrow \gamma N) u(p_N) \epsilon_\mu, \end{aligned} \quad (5.92)$$

where $P = q_1 + q_2 + q_3 = p_N + p_\pi$. The resonance R_1 with spin $J = m + 1/2$ is produced in the γN interaction, it propagates and then decays into a meson (μ) and a baryon resonance R_2 with spin $J = n + 1/2$. Then the resonance R_2 propagates and decays into the final meson and a nucleon.

In the following the full vertex functions used for the construction of amplitudes are given. One should remember that the $\tilde{N}$ functions are different from N -functions by the order of the γ -matrices. For $R \rightarrow 0^- + 1/2^+$ transitions

$$\tilde{N}_{\mu_1 \dots \mu_n}^+ = X_{\mu_1 \dots \mu_n}^{(n)}, \quad \tilde{N}_{\mu_1 \dots \mu_n}^- = i \gamma_\nu \gamma_5 X_{\nu \mu_1 \dots \mu_n}^{(n+1)} \quad (5.93)$$

holds, while we have

$$\begin{aligned} \tilde{N}_{\alpha_1 \dots \alpha_n}^{(1+) \mu} &= i \gamma_\nu \gamma_5 X_{\mu \nu \alpha_1 \dots \alpha_n}^{(n+2)}, & \tilde{N}_{\alpha_1 \dots \alpha_n}^{(1-) \mu} &= X_{\mu \alpha_1 \dots \alpha_n}^{(n+1)}, \\ \tilde{N}_{\alpha_1 \dots \alpha_n}^{(2+) \mu} &= i \gamma_\nu \gamma_5 X_{\nu \alpha_2 \dots \alpha_n}^{(n)} g_{\alpha_1 \mu}^\perp, & \tilde{N}_{\alpha_1 \dots \alpha_n}^{(2-) \mu} &= X_{\alpha_2 \dots \alpha_n}^{(n-1)} g_{\alpha_1 \mu}^\perp, \end{aligned} \quad (5.94)$$

for $R \rightarrow 0^- + 3/2^+$ transitions, and

$$\begin{aligned} V_{\alpha_1 \dots \alpha_n}^{(1+) \mu} &= \gamma_\mu i \gamma_5 X_{\alpha_1 \dots \alpha_n}^{(n)}, & V_{\alpha_1 \dots \alpha_n}^{(1-) \mu} &= \gamma_\xi \gamma_\mu X_{\xi \alpha_1 \dots \alpha_n}^{(n+1)}, \\ V_{\alpha_1 \dots \alpha_n}^{(2+) \mu} &= \gamma_\nu i \gamma_5 X_{\mu \nu \alpha_1 \dots \alpha_n}^{(n+2)}, & V_{\alpha_1 \dots \alpha_n}^{(2-) \mu} &= X_{\mu \alpha_1 \dots \alpha_n}^{(n+1)}, \\ V_{\alpha_1 \dots \alpha_n}^{(3+) \mu} &= \gamma_\nu i \gamma_5 X_{\nu \alpha_1 \dots \alpha_n}^{(n+1)} g_{\mu \alpha_n}^\perp, & V_{\alpha_1 \dots \alpha_n}^{(3-) \mu} &= X_{\alpha_2 \dots \alpha_n}^{(n-1)} g_{\alpha_1 \mu}^\perp, \end{aligned} \quad (5.95)$$

for $R \rightarrow 1^- + 1/2^+$ transitions.

5.4.1 Amplitudes for baryons states decaying into a $1/2$ state and a pion

In this section explicit expressions for the angular dependent part of the amplitudes are given for the case of a baryon produced in a γ^*N collision. The baryon decays into a pseudoscalar particle and another (intermediate) baryon with spin $1/2$ (decaying in turn into a meson and a nucleon), Fig. 5.1.

The $1/2^-$, $3/2^+$, $5/2^- \dots$ states.

The amplitude for a '+' state (R_1) produced in a γ^*N collision in a partial wave decaying into a 0^- -meson and an intermediate $1/2^+$ -baryon (R_2) has the form

$$\begin{aligned}
 A^{(i)} &= \bar{u}(q_1) \tilde{N}^-(q_{12}^\perp) \frac{\hat{q}_1 + \hat{q}_2 + \sqrt{s_{12}}}{2\sqrt{s_{12}}} \tilde{N}_{\alpha_1 \dots \alpha_n}^+(q_1^\perp) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(P) V_{\beta_1 \dots \beta_n}^{(i+)\mu}(k^\perp) \\
 &\times u(p_N) \epsilon_\mu \\
 &= \bar{u}(q_1) i \hat{q}_{12}^\perp \gamma_5 \frac{\hat{q}_1 + \hat{q}_2 + \sqrt{s_{12}}}{2\sqrt{s_{12}}} X_{\alpha_1 \dots \alpha_L}^{(L)}(q_1^\perp) \frac{\sqrt{s} + \hat{P}}{2\sqrt{s}} R_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n} V_{\beta_1 \dots \beta_n}^{(i+)\mu}(k^\perp) \\
 &\times u(p_N) \epsilon_\mu,
 \end{aligned} \tag{5.96}$$

where the p_N and q_1 are the momenta of the nucleon in the initial and final state, $k^\perp = 1/2(p_N - p_\pi)^\perp$ and $q_1^\perp = 1/2(q_1 + q_2 - q_3)^\perp$ are their components orthogonal to the total momentum of the first resonance R_1 . Further, $s_{12} = (q_1 + q_2)^2$ and the factors $1/(2\sqrt{s})$ and $1/(2\sqrt{s_{12}})$ are introduced to suppress the divergence of the numerator of the fermion propagators at large energies. The relative momentum $q_{12}^\perp \equiv q_{12}^{\perp(q_1+q_2)}$ is defined as $q_{12\mu}^\perp = (q_1 - q_2)_\nu [g_{\mu\nu} - (q_1 + q_2)_\mu (q_1 + q_2)_\nu / (q_1 + q_2)^2] / 2$.

The vertex functions (5.93)–(5.94) are presented for the case when the nucleon wave function is placed in the right-hand side of the amplitude. Therefore the order of the γ -matrices needs to be changed for the meson–nucleon vertices in Eq. (5.92).

If the baryon R_2 has spin $1/2^-$, one has to construct the vertex for the decay of '+' states into a 0^- and a $1/2^-$ particle. Such operators coincide, however, with the operators for the decay of '-' states into a $0^- + 1/2^+$ system. Therefore,

$$\begin{aligned}
A^{(i)} &= \bar{u}(q_1) \tilde{N}^+(q_{12}^\perp) \frac{\hat{q}_1 + \hat{q}_2 + \sqrt{s_{12}}}{2\sqrt{s_{12}}} \tilde{N}_{\alpha_1 \dots \alpha_n}^-(q_1^\perp) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(P) V_{\beta_1 \dots \beta_n}^{(i+)\mu}(k^\perp) \\
&\times u(p_N) \epsilon_\mu \\
&= \bar{u}(q_1) \frac{\hat{q}_1 + \hat{q}_2 + \sqrt{s_{12}}}{2\sqrt{s_{12}}} i\gamma_\nu \gamma_5 X_{\nu\alpha_1 \dots \alpha_n}^{(n+1)}(q_1^\perp) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(n) V_{\beta_1 \dots \beta_n}^{(i+)\mu}(k^\perp) \\
&\times u(p_N) \epsilon_\mu.
\end{aligned} \tag{5.97}$$

In case of the photoproduction with real photons, the $V_{\beta_1 \dots \beta_n}^{(2+)\mu}$ vertex is reduced to $V_{\beta_1 \dots \beta_n}^{(1+)\mu}$ and can be omitted.

The $1/2^+$, $3/2^-$, $5/2^+ \dots$ states.

If a ' $-$ ' state is produced in a $\gamma^* N$ interaction and then decays into a pseudoscalar pion and a $1/2^+$ baryon, the amplitude has the structure

$$\begin{aligned}
A^{(i)} &= \bar{u}(q_1) \tilde{N}^-(q_{12}^\perp) \frac{\hat{q}_1 + \hat{q}_2 + \sqrt{s_{12}}}{2\sqrt{s_{12}}} \tilde{N}_{\alpha_1 \dots \alpha_n}^-(q_1^\perp) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(P) V_{\beta_1 \dots \beta_n}^{(i-)\mu}(k^\perp) \\
&\times u(p_N) \epsilon_\mu \\
&= \bar{u}(q_1) i\hat{q}_{12}^\perp \gamma_5 \frac{\hat{q}_1 + \hat{q}_2 + \sqrt{s_{12}}}{2\sqrt{s_{12}}} i\gamma_\nu \gamma_5 X_{\nu\alpha_1 \dots \alpha_n}^{(n+1)}(q_1^\perp) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(P) V_{\beta_1 \dots \beta_n}^{(i-)\mu}(k^\perp) \\
&\times u(p_N) \epsilon_\mu.
\end{aligned} \tag{5.98}$$

If the intermediate baryon has spin $1/2^-$, then

$$\begin{aligned}
A^{(i)} &= \bar{u}(q_1) \tilde{N}^+(q_{12}^\perp) \frac{\hat{q}_1 + \hat{q}_2 + \sqrt{s_{12}}}{2\sqrt{s_{12}}} \tilde{N}_{\alpha_1 \dots \alpha_n}^+(q_1^\perp) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(P) V_{\beta_1 \dots \beta_n}^{(i-)\mu}(k^\perp) \\
&\times u(p_N) \epsilon_\mu \\
&= \bar{u}(q_1) \frac{\hat{q}_1 + \hat{q}_2 + \sqrt{s_{12}}}{2\sqrt{s_{12}}} X_{\alpha_1 \dots \alpha_{n+1}}^{(n+1)}(q_1^\perp) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(P) V_{\beta_1 \dots \beta_n}^{(i-)\mu}(k^\perp) \\
&\times u(p_N) \epsilon_\mu.
\end{aligned} \tag{5.99}$$

For photoproduction with real photons only amplitudes with $V^{(1-)}$ and $V^{(3-)}$ vertex functions should be taken into account.

5.4.2 Photoproduction amplitudes for baryon states decaying into a $3/2$ state and a pseudoscalar meson

Experimentally important is the photoproduction of resonances decaying into $\Delta(1232)\pi$ followed by a $\Delta(1232)$ decay into a nucleon and a pion.

The ' $+$ ' states produced in a $\gamma^* N$ collision can decay into a pseudoscalar meson and intermediate baryon with spin $3/2^+$ in two partial waves. The

amplitude depends on indices (ij) where index (i) is related, as before, to the partial wave in the γN channel while index (j) is related to the partial wave in the decay of the resonance into the spin-0 meson and the $3/2$ resonance R_2 :

$$A^{(ij)} = \bar{u}(q_1) \tilde{N}_\delta^+(q_{12}^\perp) F_\nu^\delta(q_1 + q_2) \tilde{N}_{\alpha_1 \dots \alpha_n}^{(j+)\nu}(q_1^\perp) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(P) V_{\beta_1 \dots \beta_n}^{(i+)\mu}(k^\perp) \times u(p_N) \epsilon_\mu. \quad (5.100)$$

If the intermediate baryon R_2 has $J^P = 3/2^-$, the structure of the amplitude structure is

$$A^{(ij)} = \bar{u}(q_1) \tilde{N}_\delta^-(q_{12}^\perp) F_\nu^\delta(q_1 + q_2) \tilde{N}_{\alpha_1 \dots \alpha_n}^{(j-)\nu}(q_1^\perp) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(P) V_{\beta_1 \dots \beta_n}^{(i-)\mu}(k^\perp) \times u(k_1) \epsilon_\mu. \quad (5.101)$$

The amplitudes for $'-$ states decaying into a 0^- -meson and $3/2^+$ -baryon are equal to

$$A^{(ij)} = \bar{u}(q_1) \tilde{N}_\delta^+(q_{12}^\perp) F_\nu^\delta(q_1 + q_2) \tilde{N}_{\alpha_1 \dots \alpha_n}^{(j-)\nu}(q_1^\perp) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(P) V_{\beta_1 \dots \beta_n}^{(i-)\mu}(k^\perp) \times u(k_1) \epsilon_\mu, \quad (5.102)$$

while for the intermediate baryon R_2 with quantum numbers $3/2^-$ we have:

$$A^{(ij)} = \bar{u}(q_1) \tilde{N}_\delta^-(q_{12}^\perp) F_\nu^\delta(q_1 + q_2) \tilde{N}_{\alpha_1 \dots \alpha_n}^{(j+)\nu}(q_1^\perp) F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}(P) V_{\beta_1 \dots \beta_n}^{(i+)\mu}(k^\perp) \times u(k_1) \epsilon_\mu. \quad (5.103)$$

5.5 πN and γN Partial Widths of Baryon Resonances

Here we consider two-particle partial widths of baryon resonances.

5.5.1 πN partial widths of baryon resonances

The operators, which describe the vertices for transition of a baryon into the πN states ($'+$ and $'-$), are introduced in Section 5.1.3: $N_{\mu_1 \dots \mu_n}^+(k^\perp) u(p_N) = X_{\mu_1 \dots \mu_n}^{(n)}(k^\perp) u(p_N)$ and $N_{\mu_1 \dots \mu_n}^-(k^\perp) u(p_N) = i\gamma_5 \gamma_\nu X_{\nu \mu_1 \dots \mu_n}^{(n+1)}(k^\perp) u(p_N)$, where, as usually, $n = J - 1/2$.

The width for the case of πN scattering has the form

$$F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}(P) M \Gamma_{\pi N}^\pm = F_{\xi_1 \dots \xi_n}^{\mu_1 \dots \mu_n}(P) \int \frac{d\Omega}{4\pi} N_{\xi_1 \dots \xi_n}^\pm \frac{\hat{p}_N + m_N}{2m_N} N_{\beta_1 \dots \beta_n}^\pm \times \rho(s, m_\pi, m_N) g^2(s) F_{\nu_1 \dots \nu_n}^{\beta_1 \dots \beta_n}(P). \quad (5.104)$$

Recall that the operator $F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}(P)$ was introduced in Section 5.1.2 and the phase space factor was determined in a standard way $\rho(s, m_\pi, m_N) = \int d\Phi_2(P; p_N, k_\pi) = \sqrt{[s - (m_N + m_\pi)^2][s - (m_N - m_\pi)^2]} / (16\pi s)$.

The momentum of the nucleon can be decomposed in the total momentum P and momentum $k^\perp$ as follows: $p_{N\mu} = P_\mu(s + m_N^2 - m_\pi^2)/(2s) + k_\mu^\perp$ and $k_\mu^\perp = (p_N - k_\pi)_\mu/2 - P_\mu(m_N^2 - m_\pi^2)/(2s)$. For ‘ $\pm$ ’ states the calculation can be easily performed (see also Appendix 5.B), and we have

$$M\Gamma_{\pi N}^+ = \frac{\alpha_n}{2n+1} |\vec{k}|^{2n} \frac{m_N + p_{N0}}{2m_N} \rho(s, m_\pi, m_N) g^2(s), \quad (5.105)$$

$$M\Gamma_{\pi N}^- = \frac{\alpha_{n+1}}{n+1} |\vec{k}|^{2n+2} \frac{m_N + p_{N0}}{2m_N} \rho(s, m_\pi, m_N) g^2(s),$$

where p_{N0} , $k_{\pi 0}$ and $\vec{p}_N = -\vec{k}$ are the components of nucleon and pion momenta in the c.m. system: $k_{\pi 0} = (s - m_N^2 + m_\pi^2)/(2\sqrt{s})$, $|\vec{k}| = \sqrt{k_{\pi 0}^2 - m_\pi^2}$ and $p_{N0} = (s + m_N^2 - m_\pi^2)/(2\sqrt{s})$.

5.5.2 The γN widths and helicity amplitudes

The decay of the baryon state with $J = n + 1/2$ into γN is described by the amplitude

$$\bar{\Psi}_{\alpha_1 \dots \alpha_n} V_{\alpha_1 \dots \alpha_n}^{(i\pm)\mu}(k^\perp) u(p_N) \epsilon_\mu,$$

where p_N is the momentum of the nucleon and $k^\perp$ is the component of the relative momentum between the nucleon and the photon which is orthogonal to the total momentum of the system $P = p_N + k_\gamma$ with $s = P^2$. Therefore, here $k_\mu^\perp = g_{\mu\nu}^\perp(p_N - k_\gamma)_\nu/2$ with $g_{\mu\nu}^\perp = g_{\mu\nu} - (P_\mu P_\nu)/s$ and $|\vec{k}|^2 = -k_\perp^2 = (s - m_N^2)^2/(4s)$.

5.5.2.1 The ‘ $+$ ’ states

For the ‘ $+$ ’ states, three vertices are constructed of the spin and orbital momentum operators $V_{\alpha_1 \dots \alpha_n}^{(1+)\mu}(k^\perp)$, $V_{\alpha_1 \dots \alpha_n}^{(2+)\mu}(k^\perp)$, $V_{\alpha_1 \dots \alpha_n}^{(3+)\mu}(k^\perp)$, they are presented in (5.49).

In the case of photoproduction, the second vertex is reduced to the third one and only two amplitudes (one for $J = 1/2$) are independent. The width factor $W^{(i,j+)}$ for the transition between vertices is expressed as follows:

$$F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} W_{i,j}^+ = F_{\alpha_1 \dots \alpha_n}^{\mu_1 \dots \mu_n} \int \frac{d\Omega}{4\pi} V_{\alpha_1 \dots \alpha_n}^{(i+)\mu}(k^\perp) \frac{m_N + \hat{p}_N}{2m_N} V_{\beta_1 \dots \beta_n}^{(j+)\nu}(k^\perp) g_{\mu\nu}^{\perp\perp} \\ \times \rho(s, m_N, m_\gamma) F_{\nu_1 \dots \nu_n}^{\beta_1 \dots \beta_n}, \quad (5.106)$$

where the standard metric tensor $g_{\mu\nu}^{\perp\perp} = g_{\mu\nu} - (P_\mu P_\nu)/P^2 - (k_\mu^\perp k_\nu^\perp)/k_\perp^2$ and the phase space factor $\rho(s, m_N, m_\gamma = 0) = (s - m_N^2)/(16\pi s)$ are used.

The width factors $W^{(i,j+)}$ for the first and third vertices are equal to:

$$\begin{aligned} W_{1,1}^+ &= \frac{2\alpha_n}{2n+1} |\vec{k}|^{2n} \frac{m_N + p_{N0}}{2m_N} \rho(s), \\ W_{3,3}^+ &= \frac{\alpha_n(n+1)}{(2n+1)n} |\vec{k}|^{2n} \frac{m_N + p_{N0}}{2m_N} \rho(s), \\ W_{1,3}^+ &= \frac{\alpha_n}{2n+1} |\vec{k}|^{2n} \frac{m_N + p_{N0}}{2m_N} \rho(s), \end{aligned} \quad (5.107)$$

where we use notations $p_{N0} = (s + m_N^2)/(2\sqrt{s})$ and $\alpha_n = \prod_{l=1}^n (2l-1)/l$.

If a state with total spin $J = n + 1/2$ decays into γN having intrinsic spins $1/2$ and $3/2$ with couplings g_1 and g_3 , the corresponding decay amplitude can be written as follows:

$$A_{\alpha_1 \dots \alpha_n}^{\mu(+)} = V_{\alpha_1 \dots \alpha_n}^{(1+)\mu} g_1(s) + V_{\alpha_1 \dots \alpha_n}^{(3+)\mu} g_3(s). \quad (5.108)$$

If so, the γN width is equal to

$$M\Gamma_{\gamma N}^+ = W_{1,1}^+ g_1^2(s) + 2W_{1,3}^+ g_1(s)g_3(s) + W_{3,3}^+ g_3^2(s). \quad (5.109)$$

The helicity $1/2$ amplitude has an operator proportional to the spin $1/2$ operator $V_{\alpha_1 \dots \alpha_n}^{(1+)\mu}$. The helicity- $3/2$ operator can be constructed as a linear combination of the spin- $3/2$ and spin- $1/2$ operators orthogonal to the $V_{\alpha_1 \dots \alpha_n}^{(1+)\mu}$:

$$\begin{aligned} A_{\alpha_1 \dots \alpha_n}^{\mu(+)} &= A_{\mu; \alpha_1 \dots \alpha_n}^{h=3/2} - A_{\mu; \alpha_1 \dots \alpha_n}^{h=1/2}, \\ A_{\mu; \alpha_1 \dots \alpha_n}^{h=1/2} &= -V_{\alpha_1 \dots \alpha_n}^{(1+)\mu} \left(g_1(s) + \frac{1}{2} g_3(s) \right), \\ A_{\mu; \alpha_1 \dots \alpha_n}^{h=3/2} &= \left(V_{\alpha_1 \dots \alpha_n}^{(3+)\mu} - \frac{1}{2} V_{\alpha_1 \dots \alpha_n}^{(1+)\mu} \right) g_3(s), \end{aligned} \quad (5.110)$$

where the sign '-' for the helicity $1/2$ amplitude was introduced in accordance with the standard multipole definition. The width defined by the helicity amplitudes can be calculated using (5.107):

$$\begin{aligned} M\Gamma^{\frac{1}{2}} &= \rho(s) W_{1,1}^+ \left(g_1(s) + \frac{1}{2} g_3(s) \right)^2, \\ M\Gamma^{\frac{3}{2}} &= \rho(s) \left(W_{3,3}^+ - \frac{1}{2} W_{1,3}^+ \right) g_3^2(s). \end{aligned} \quad (5.111)$$

Taking into account the standard definition of the γN width via helicity amplitudes,

$$M\Gamma_{tot} = M\Gamma^{\frac{3}{2}} + M\Gamma^{\frac{1}{2}} = \frac{|\vec{k}|^2}{\pi} \frac{2m_N}{2J+1} \left(|A_n^{\frac{1}{2}}|^2 + |A_n^{\frac{3}{2}}|^2 \right), \quad (5.112)$$

we obtain

$$|A_n^{\frac{1}{2}}|^2 = \frac{\alpha_n(n+1)}{2n+1} \rho(s, m_N, 0) \pi |\vec{k}|^{2n-2} \frac{m_N + k_{N0}}{m_N^2} \left(g_1(s) + \frac{1}{2} g_3(s) \right)^2, \\ |A_n^{\frac{3}{2}}|^2 = \alpha_n \rho(s, m_N, 0) \pi |\vec{k}|^{2n-2} \frac{m_N + k_{N0}}{m_N^2} \frac{(n+2)(n+1)}{4n(2n+1)} g_3^2(s). \quad (5.113)$$

In the case of resonance production, the vertex functions are usually normalised with certain form factors, *e.g.* the Blatt–Weisskopf form factors (the explicit form can be found in [4] or in Appendix 5.B). These form factors depend on the orbital momentum and the radius r and regularize the behaviour of the amplitude at high energies. For the '+' states the orbital momentum for both spin-1/2 and spin-3/2 operators are equal to $L = J - 1/2 = n$. Then, rewriting

$$g_1(s) = \frac{g_{1/2}}{F(n, |\vec{k}|^2, r)}, \quad g_3(s) = \frac{g_{3/2}}{F(n, |\vec{k}|^2, r)}, \quad (5.114)$$

and using Eq. (5.113), the ratio of helicity amplitudes given in [4] is reproduced.

5.5.2.2 The '-' states

For the decay of a '-' state with total spin J into γN , the vertex functions $V_{\alpha_1 \dots \alpha_n}^{(1-)\mu}(k^\perp)$, $V_{\alpha_1 \dots \alpha_n}^{(2-)\mu}(k^\perp)$, $V_{\alpha_1 \dots \alpha_n}^{(3-)\mu}(k^\perp)$ are given in (5.55). These vertices are constructed of the spin and orbital momentum operators with $(S = 1/2, L = n + 1)$, $(S = 3/2, L = n + 1)$ and $(S = 3/2 \text{ and } L = n - 1)$. As in the case of '+' states, for real photons the second vertex provides us the same angular distribution as the third vertex. For the first and third vertices, the width factors $W_{i,j}^-$ are equal to

$$W_{1,1}^- = \frac{2\alpha_{n+1}}{n+1} |\vec{k}|^{2n+2} \frac{m_N + p_{N0}}{2m_N} \rho(s, m_N, 0), \\ W_{3,3}^- = \frac{\alpha_{n-1}(n+1)}{(2n+1)(2n-1)} |\vec{k}|^{2n-2} \frac{m_N + p_{N0}}{2m_N} \rho(s, m_N, 0), \\ W_{1,3}^- = \frac{\alpha_{n-1}}{n+1} |\vec{k}|^{2n} \frac{m_N + p_{N0}}{2m_N} \rho(s, m_N, 0). \quad (5.115)$$

The decay amplitude is defined by the sum of two vertices as follows:

$$A_{\alpha_1 \dots \alpha_n}^{\mu(-)} = V_{\alpha_1 \dots \alpha_n}^{(1-)\mu} g_1(s) + V_{\alpha_1 \dots \alpha_n}^{(3-)\mu} g_3(s), \quad (5.116)$$

and the γN width of the state is calculated as a sum over possible transitions:

$$M\Gamma_{\gamma N}^- = W_{1,1}^- g_1^2(s) + 2W_{1,3}^- g_1(s)g_3(s) + W_{3,3}^- g_3^2(s). \quad (5.117)$$

The helicity-1/2 amplitude an the operator proportional to the spin-1/2 operator $V_{\alpha_1 \dots \alpha_n}^{(1+)\mu}$. The helicity-3/2 operator can be constructed as a linear combination of the spin-3/2 and spin-1/2 operators orthogonal to the $V_{\alpha_1 \dots \alpha_n}^{(1+)\mu}$:

$$\begin{aligned} A_{\alpha_1 \dots \alpha_n}^{\mu(-)} &= A_{\mu; \alpha_1 \dots \alpha_n}^{h=1/2} - A_{\mu; \alpha_1 \dots \alpha_n}^{h=3/2} , \\ A_{\mu; \alpha_1 \dots \alpha_n}^{h=1/2} &= V_{\alpha_1 \dots \alpha_n}^{(1-)\mu} (g_1(s) - R g_3(s)) , \\ A_{\mu; \alpha_1 \dots \alpha_n}^{h=3/2} &= - \left(V_{\alpha_1 \dots \alpha_n}^{(3-)\mu} + R V_{\alpha_1 \dots \alpha_n}^{(1-)\mu} \right) g_3(s) , \end{aligned} \quad (5.118)$$

where the factor R is given by

$$R = - \frac{1}{2|\vec{k}|^2} \frac{\alpha_{n-1}}{\alpha_{n+1}} = - \frac{1}{2|\vec{k}|^2} \frac{n(n+1)}{(2n-1)(2n+1)} . \quad (5.119)$$

Here, again, the signs for the helicity-1/2 amplitudes are taken to correspond to the multipole definition. The widths defined by the helicity amplitudes are equal to

$$\begin{aligned} M\Gamma^{\frac{1}{2}} &= \rho(s, m_N, 0) W_{1,1}^- \left(g_1(s) - R g_3(s) \right)^2 , \\ M\Gamma^{\frac{3}{2}} &= \rho(s, m_N, 0) \left(W_{3,3}^- + R W_{1,3}^- \right) g_3^2(s) , \end{aligned} \quad (5.120)$$

and, therefore,

$$\begin{aligned} |A_n^{\frac{1}{2}}|^2 &= \alpha_{n+1} \rho(s, m_N, 0) \frac{\pi(m_N + p_{N0})}{m_N^2} |\vec{k}|^{2n} \left(g_1(s) - R g_3(s) \right)^2 , \\ |A_n^{\frac{3}{2}}|^2 &= \alpha_{n-1} \frac{(n+1)(n+2)}{4(4n^2-1)} \rho(s, m_N, 0) \frac{\pi(m_N + p_{N0})}{m_N^2} |\vec{k}|^{2n-4} g_3^2(s) . \end{aligned} \quad (5.121)$$

The vertices with couplings $g_1(s)$ and $g_3(s)$ are formed by different orbital momenta. For the state with total spin J ($n = J - 1/2$), the orbital momentum is equal to $L = n + 1$ for the first decay ($S = 1/2$) and $L = n - 1$ for the second one ($S = 3/2$). Using the Blatt–Weisskopf form factors for the normalisation (see Appendix 5.B), we obtain

$$g_1(s) = \frac{g_{1/2}}{F(n+1, |\vec{k}|^2, r)}, \quad g_3(s) = \frac{g_{3/2}}{F(n-1, |\vec{k}|^2, r)} . \quad (5.122)$$

5.5.3 Three-body partial widths of the baryon resonances

The total width of the state is calculated by averaging over polarisations of the resonance and summing over polarisations of the final state particles. Then, for the three-particle final state, the amplitude squared depends on three invariants: s_{12} , s_{13} and s_{23} where $s_{ij} = (q_i + q_j)^2$. They are related to

the total momentum squared $s + m_1^2 + m_2^2 + m_3^2 = s_{12} + s_{13} + s_{23}$. Therefore, we can write:

$$\int d\Phi_3(P; q_1, q_2, q_3) = \int_{\text{Dalitz plot}} \frac{1}{32s(2\pi)^3} ds_{12} ds_{23}, \quad (5.123)$$

with the following integration limits for s_{12} and s_{23} :

$$(m_1 + m_2)^2 \leq s_{12} \leq (\sqrt{s} - m_3)^2, \quad s_{23}^{(-)} \leq s_{23} \leq s_{23}^{(+)}, \quad (5.124)$$

with

$$s_{23}^{(\pm)} = (E_2 + E_3)^2 - (\sqrt{E_2^2 - m_2^2} \pm \sqrt{E_3^2 - m_3^2})^2, \\ E_2 = \frac{s_{12} - m_1^2 + m_2^2}{2\sqrt{s_{12}}}, \quad E_3 = \frac{s - s_{12} - m_3^2}{2\sqrt{s_{12}}}.$$

The three-body phase space can also be written as a product of the two two-body phase spaces:

$$d\Phi_3(P; q_1, q_2, q_3) = d\Phi_2(q_1 + q_2; q_1, q_2) d\Phi_2(P; q_1 + q_2, q_3) \frac{ds_{12}}{\pi}. \quad (5.125)$$

This expression is very useful for the study of the cascade decays when a resonance accompanied by a spectator particle decays into two particles.

Let us write the explicit form of the expression $Q \otimes Q$ for the width of baryon with spin J ($n = J - 1/2$), which decays into a nucleon with momentum $q_3 \equiv q_N$ and a meson resonance, R_j , which decays subsequently into two pseudoscalar mesons with momenta q_1 and q_2 . The decay of the intermediate resonance with spin j into two pseudoscalar mesons (P_1 and P_2) is described by the orbital momentum operator $X^{(j)}$, so we write:

$$Q_{\mu_1 \dots \mu_n} \otimes Q_{\nu_1 \dots \nu_n} = \tilde{P}_{\mu_1 \dots \mu_n}^{\alpha_1 \dots \alpha_j} \frac{m_N + \hat{q}_N}{2m_N} \frac{f_{\beta_1 \dots \beta_j}^{\alpha_1 \dots \alpha_j} g_{R_j \rightarrow P_1 P_2}(s_{12})}{M_{R_j}^2 - s_{12} - iM_{R_j} \Gamma_{tot}^{R_j}} \\ \times X_{\beta_1 \dots \beta_j}^{(j)}(q_{12}^\perp) X_{\xi_1 \dots \xi_j}^{(j)}(q_{12}^\perp) \frac{g_{R_j \rightarrow P_1 P_2}(s_{12}) f_{\eta_1 \dots \eta_j}^{\xi_1 \dots \xi_j}}{M_{R_j}^2 - s_{12} + iM_{R_j} \Gamma_{tot}^{R_j}} P_{\eta_1 \dots \eta_n}^{\nu_1 \dots \nu_j}. \quad (5.126)$$

Here the operator $P_{\eta_1 \dots \eta_n}^{\nu_1 \dots \nu_j}$ describes the decay of the initial state into the resonance R_j and the spectator nucleon, while the operator $\tilde{P}_{\mu_1 \dots \mu_n}^{\alpha_1 \dots \alpha_j}$ differs from the first one by the permutation of γ -matrices. We denote the propagator of the intermediate state resonance as $f_{\beta_1 \dots \beta_j}^{\alpha_1 \dots \alpha_j}$; $g_{R_j \rightarrow P_1 P_2}(s_{12})$ is the coupling of the intermediate resonance to the final state mesons. As usually, $q_{12\mu}^\perp = g_{\mu\nu}^{\perp(q_1+q_2)}(q_1 - q_2)_\nu/2$ and $g_{\mu\nu}^{\perp(q_1+q_2)} = g_{\mu\nu} - (q_1 + q_2)_\mu(q_1 + q_2)_\nu/(q_1 + q_2)^2$.

Using the method presented in Appendix 5.B we obtain the final expression for the width of the initial resonance R :

$$F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n} M\Gamma = F_{\mu_1 \dots \mu_n}^{\alpha_1 \dots \alpha_n} \int \frac{ds_{12}}{\pi} d\Phi(P, q_1 + q_2, q_N) g_{R \rightarrow NR_j}^2(s) \\ \times \tilde{P}_{\mu_1 \dots \mu_n}^{\xi_1 \dots \xi_j} \frac{f_{\eta_1 \dots \eta_j}^{\xi_1 \dots \xi_j} M_{R_j} \Gamma_{P_1 P_2}^{R_j}}{(M_{R_j}^2 - s_{12})^2 + (M_{R_j} \Gamma_{tot}^{R_j})^2} P_{\nu_1 \dots \nu_n}^{\eta_1 \dots \eta_j} F_{\beta_1 \dots \beta_n}^{\nu_1 \dots \nu_n}. \quad (5.127)$$

In the limit of zero width of the intermediate state we have

$$\left[\int \frac{ds_{12}}{\pi} \frac{M_{R_j} \Gamma_{tot}^{R_j}}{(M_{R_j}^2 - s_{12})^2 + (M_{R_j} \Gamma_{tot}^{R_j})^2} \right]_{\Gamma_{tot}^{R_j} \rightarrow 0} \rightarrow \int ds_{12} \delta(M_{R_j}^2 - s_{12}) \quad (5.128)$$

and equation (5.127) is reduced to the two-body equation multiplied by the branching ratio of the decay of the intermediate state, $\text{Br}_{P_1 P_2} = \Gamma_{P_1 P_2}^R / \Gamma_{tot}^R$.

Let us note that, provided a resonance has many decay modes (or the mode can be in different kinematical channels), the decay amplitude can be written as a vector with components corresponding to these decay modes. In this case, equation (5.127) gives us only diagonal transition elements. To obtain non-diagonal elements between different kinematical channels it is necessary to consider the general case: *state 'in' → intermediate state particles → state 'out'*.

5.5.4 Miniconclusion

In this section explicit expressions for cross sections and resonance partial widths are given for a large number of the pion induced and photoproduction reactions with two or three particles in the final state.

Partial widths of the baryon resonances into channels $f_0 N$, vector meson- N , πP_{11} , πS_{11} , $\pi \Delta(3/2^+)$, $\pi 3/2^-$ can be found in [4, 5, 26, 37, 38].

5.6 Photoproduction of Baryons Decaying into $N\pi$ and $N\eta$

To be illustrative, a combined analysis [37, 38] of the photoproduction data on $\gamma p \rightarrow \pi N$, ηN , $K\Lambda$, $K\Sigma$, based on the method presented above, is shown in this section. Three baryon resonances have a substantial coupling to ηN , the well-known $N(1535)S_{11}$, $N(1720)P_{13}$, and $N(2070)D_{15}$. The data with open strangeness reveal the presence of further resonances, $N(1840)P_{11}$, $N(1890)P_{13}$ and provide proof for the existence of $N(1875)D_{13}$ and $N(2170)D_{13}$.

5.6.1 *The experimental situation — an overview*

The properties of baryon resonances are currently under intense investigations. Photoproduction experiments are carried out at several facilities like ELSA (Bonn), GRAAL (Grenoble), JLab (Newport News, VA), MAMI (Mainz), and SPring-8 (Hyogo). The aim is to identify the resonance spectrum, to determine spins, parities, and decay branching ratios and thus to provide constraints for models.

The information from photoproduction experiments is complementary to experiments with hadronic beams, and it gives access to additional characteristics like helicity amplitudes. The data obtained with polarised photons can be very sensitive to resonances which contributed weakly to the total cross section. A clear example of such an effect is the observation of the $N(1520)D_{13}$ resonance in ηN photoproduction. It contributes very little to the unpolarised cross section but its interference with the $N(1535)S_{11}$ produces a strong effect in the beam asymmetry. Photoproduction can also provide a very strong selection tool: combination of a circularly polarised photon beam and a longitudinally polarised target selects states with helicity $1/2$ or $3/2$ depending on whether the target polarisation is parallel or antiparallel to the photon helicity.

Baryon resonances with large widths overlap, making difficult the study of individual states, in particular, of those excited weakly. We can overcome this problem partly by looking at specific decay channels. For example, the η meson has an isospin $I = 0$ and, consequently, the $N\eta$ final state can be reached only via the formation of N^* resonances. Then even a small coupling of a resonance to $N\eta$ identifies it as an N^* state. A key point in the identification of new baryon resonances is the combined analysis of data on photo and pion induced reactions, with different final states. The resonance is characterised by the position of pole singularity and pole residues. So, the resonance must have the same mass, total width, and gamma–nucleon coupling in all the considered reactions. This imposes strong constraints for parameters of the analysed amplitudes.

In the analysis described below the primary goal is to get information about the pole singularities of the photoproduction amplitude. For this purpose, a representation of the amplitude as a sum of s -channel resonances together with some t - and u -exchange diagrams is an appropriate approach. Strongly overlapping resonances are parametrised by the K -matrix representation. In many cases, for non-overlapping resonances, it is sufficient to use a relativistic Breit–Wigner parametrisation.

5.6.1.1 Parametrisations of amplitudes

The η photoproduction cross section is dominated by $N(1535)S_{11}$. It overlaps with $N(1650)S_{11}$, so the two S_{11} resonances are described by a five-channel K -matrix (πN , ηN , $K\Lambda$, $K\Sigma$ and $\Delta\pi$), with two poles. The photoproduction amplitude can be written in the P -vector approach, since the γN couplings are weak and do not contribute to rescattering. The amplitude is then given by the standard formula $A_a = \hat{P}_b(\hat{I} - i\hat{\rho}\hat{K})_{ba}^{-1}$. The phase space is a diagonal matrix: $\rho_{ab} = \delta_{ab}\rho_a$ with $a, b = \pi N, \eta N, K\Lambda, K\Sigma$. Two-body phase volumes are defined as $\rho_a(s) = 2k_a/\sqrt{s}$, and the $\Delta\pi$ phase volume is defined according to the prescription of Section 5.5.3 and Appendix 5.B. The $\hat{P}$ -vector and the matrix $\hat{K}$ are parametrised in the following way:

$$K_{ab} = \sum_{\alpha} \frac{g_a^{(\alpha)} g_b^{(\alpha)}}{M_{\alpha}^2 - s} + f_{ab}, \quad P_b = \sum_{\alpha} \frac{g_{\gamma N}^{(\alpha)} g_b^{(\alpha)}}{M_{\alpha}^2 - s} + \tilde{f}_b, \quad (5.129)$$

where M_{α} , $g_a^{(\alpha)}$ and $g_{\gamma N}^{(\alpha)}$ are the masses and couplings of bare states, while f_{ab} and $\tilde{f}_b$ are constant terms.

Other resonances are parametrised as the Breit–Wigner terms:

$$A_a = \frac{g_{\gamma N} \tilde{g}_a(s)}{M^2 - s - i M \tilde{\Gamma}_{tot}(s)}. \quad (5.130)$$

States with masses above 2000 MeV were parametrised with a constant width to fit exactly to the pole position. For resonances below 2000 MeV, $\tilde{\Gamma}_{tot}(s)$ was parametrised by

$$\tilde{\Gamma}_{tot}(s) = \sum_a \Gamma_a \frac{\rho_a(s) k_a^{2L}(s) F^2(L, k_a^2(M^2), r)}{\rho_a(M^2) k_a^{2L}(M^2) F^2(L, k_a^2(s), r)}. \quad (5.131)$$

Here L is the orbital momentum and k is the relative momentum for the decay into the final channel, $F(L, k^2, r)$ are Blatt–Weisskopf form factors, taken with a radius $r = 0.8$ fm (see Appendix 5.B and [5]). The $g_{\gamma N}$ is the production coupling and $\tilde{g}_a$ are couplings of the resonance decay into meson nucleon channels.

At high energies, there are clear peaks in the forward direction of photo-produced mesons. The forward peaks are connected with meson exchanges in the t -channel. These contributions are parametrised as reggeised π , ρ , ω , K , and K^* exchanges.

For ρ and ω exchanges we use the trajectory $\alpha_{\rho/\omega}(t) = 0.50 + 0.85t$. The pion trajectory is given by $\alpha(t)_{\pi} = -0.014 + 0.72t$, the K^* and K trajectories are represented by $\alpha_{K^*}(t) = 0.32 + 0.85t$ and $\alpha_K(t) = -0.25 + 0.85t$, respectively. The full expression for the t -channel amplitudes can be found in [5].

The u -channel exchanges were parametrised as N , Λ , or Σ exchanges.

5.6.2 Fits to the data

The list of the fitted reactions is given in Table 5.1. These data comprise CB-ELSA π^0 and η photoproduction data [7, 12], the Mainz-TAPS data [13] on η photoproduction, beam asymmetry measurements of π^0 and η [9, 14], target and recoil asymmetry measurements for π^0 photoproduction and data on $\gamma p \rightarrow n\pi^+$ [11].

A more comprehensive set of data exists for the hyperon-kaon final state due to the natural possibility for measurements of the final hyperon polarisation. Here there are data on the differential cross section for $K^+\Lambda$, $K^+\Sigma$, and $K^0\Sigma^+$ photoproduction from SAPHIR [19] and CLAS [20], beam asymmetry data for $K^+\Lambda$, $K^+\Sigma$ from LEPS [22] and the first double asymmetry data measured by CLAS [17].

The analysis includes also data on photon induced $\pi^0\pi^0$ production [25, 26] and $\pi^0\eta$ [27] and the recent BNL data on $\pi^-p \rightarrow n\pi^0\pi^0$ [28] fitted in an event-based likelihood method.

The fit uses 14 N^* resonances coupled to $N\pi$, $N\eta$, $K\Lambda$, and $K\Sigma$ and 7 Δ resonances coupled to $N\pi$ and $K\Sigma$. Most resonances are described first by relativistic Breit-Wigner amplitudes and then in the framework of the K -matrix approach. The background is described by reggeised t -channel π , ρ , ω , K and K^* exchanges and by baryon exchanges in the s - and u -channels.

Table 5.1 Single meson photoproduction data used in the partial wave analysis (N is the number of points).

Observable	N	Ref.
$\sigma(\gamma p \rightarrow p\pi^0)$	1106	[7]
$\sigma(\gamma p \rightarrow p\pi^0)$	861	[8]
$\Sigma(\gamma p \rightarrow p\pi^0)$	469	[8]
$\Sigma(\gamma p \rightarrow p\pi^0)$	593	[9]
$P(\gamma p \rightarrow p\pi^0)$	594	[10]
$T(\gamma p \rightarrow p\pi^0)$	380	[10]
$\sigma(\gamma p \rightarrow n\pi^+)$	1583	[11]
$\sigma(\gamma p \rightarrow p\eta)$	6677	[12]
$\sigma(\gamma p \rightarrow p\eta)$	100	[13]
$\Sigma(\gamma p \rightarrow p\eta)$	51	[14]
$\Sigma(\gamma p \rightarrow p\eta)$	100	[15]
$P_{11}(\pi N \rightarrow N\pi)$	110	[16]
$P_{13}(\pi N \rightarrow N\pi)$	134	[16]
$S_{11}(\pi N \rightarrow N\pi)$	126	[16]
$D_{33}(\pi N \rightarrow N\pi)$	108	[16]

Observable	N	Ref.
$C_x(\gamma p \rightarrow \Lambda K^+)$	160	[17]
$C_z(\gamma p \rightarrow \Lambda K^+)$	160	[17]
$\sigma(\gamma p \rightarrow \Lambda K^+)$	1377	[18]
$\sigma(\gamma p \rightarrow \Lambda K^+)$	720	[19]
$P(\gamma p \rightarrow \Lambda K^+)$	202	[20]
$P(\gamma p \rightarrow \Lambda K^+)$	66	[21]
$\Sigma(\gamma p \rightarrow \Lambda K^+)$	66	[21]
$\Sigma(\gamma p \rightarrow \Lambda K^+)$	45	[22]
$C_x(\gamma p \rightarrow \Sigma^0 K^+)$	94	[17]
$C_z(\gamma p \rightarrow \Sigma^0 K^+)$	94	[17]
$\sigma(\gamma p \rightarrow \Sigma^0 K^+)$	1280	[18]
$\sigma(\gamma p \rightarrow \Sigma^0 K^+)$	660	[19]
$P(\gamma p \rightarrow \Sigma^0 K^+)$	95	[20]
$\Sigma(\gamma p \rightarrow \Sigma^0 K^+)$	42	[21]
$\Sigma(\gamma p \rightarrow \Sigma^0 K^+)$	45	[22]
$\sigma(\gamma p \rightarrow \Sigma^+ K^0)$	48	[20]
$\sigma(\gamma p \rightarrow \Sigma^+ K^0)$	120	[23]
$\sigma(\gamma p \rightarrow \Sigma^+ K^0)$	72	[24]

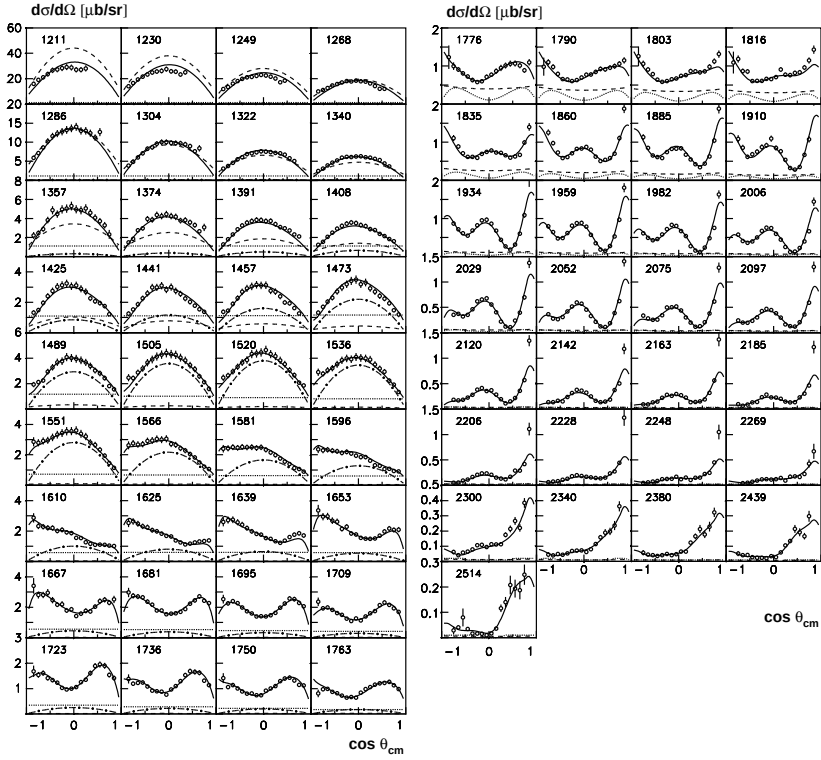


Fig. 5.2 Differential cross section for $\gamma p \rightarrow p\pi^0$ from CB-ELSA and PWA results (solid line). The left panel shows also the following contributions: $\Delta(1232)P_{33}$ together with a non-resonance background (dashed line), the $N(1535)S_{11}$ and $N(1650)S_{11}$ (dotted line) and $N(1520)D_{13}$ (dash-dotted line). In the right panel, the contributions of $\Delta(1700)D_{33}$ (dashed line) and $N(1680)F_{15}$ (dotted line) are shown.

The differential cross sections for the CB-ELSA $\gamma p \rightarrow p\pi^0$ data are shown in Fig. 5.2. The main fit is represented as a solid line. The contribution of $\Delta(1232)$ (given on the left panel as a dashed line) dominates the low-energy region, for small photon energies it even exceeds the experimental cross section, thus underlining the importance of interference effects. Non-resonance background amplitudes, given by a pole at $s \simeq -1 \text{ GeV}^2$ and by an u channel exchange diagram, are needed to describe the shape of the $\Delta(1232)$; the poles at negative s represent effectively the left-hand cuts.

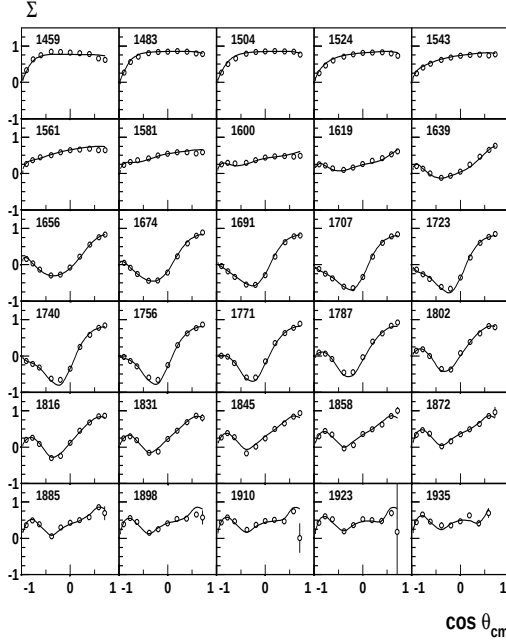


Fig. 5.3 Photon beam asymmetry Σ for $\gamma p \rightarrow p\pi^0$ from GRAAL [9] and PWA result (solid line).

Two S_{11} resonances at 1535 and at 1650 MeV are described by the K -matrix. Their contribution is depicted by a dotted line. The S_{11} contribution is flat with respect to $\cos \Theta_{\text{cm}}$. The contribution of the $D_{13}(1520)$ is shown as a dash-dotted line in Fig. 5.2 (left panel). It is strong in the 1400–1600 MeV mass region. At higher energies (Fig. 5.2, right panel) the most significant contributions come from $\Delta(1700)D_{33}$ (dashed line) and from $N(1680)F_{15}$ (dotted line).

The values of the cross sections can be determined by the summation of the differential cross sections (dots with error bars) with the extrapolation for bins with no data.

In the total cross section for π^0 photoproduction in Fig. 5.4 (left panel), clear peaks are observed for the first, second, and third resonance region. With some good will, the fourth resonance region can be identified as a broad enhancement at about 1900 MeV.

Recent data from GRAAL [29] on the differential cross section and on the photon beam asymmetry Σ for $\gamma p \rightarrow p\pi^0$ were included into the fit. The data on this reaction can be described reasonably well with only well

known resonances in the fit. The number of resonances needed for the description of other channels improved the fit only marginally. One of the examples is the $D_{15}(2070)$ observed in the ηp final state (see Fig. 5.4 (right panel)). If the couplings of this resonance to πp and ηp channels are equal to each other, its cross section in the $\gamma p \rightarrow \pi^0 p$ reactions should be less than $1 \mu\text{b}$, thus contributing a little to this reaction.

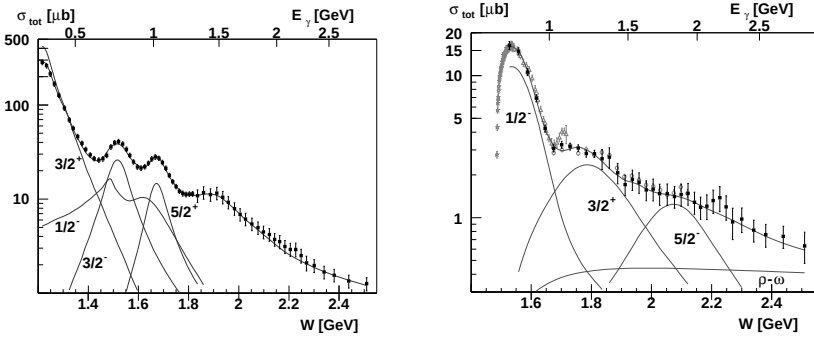


Fig. 5.4 Total cross sections (logarithmic scale) for the reactions $\gamma p \rightarrow \pi \pi^0$ (left panel) and $\gamma p \rightarrow p \eta$ (right panel) obtained by integration of angular distributions of the CB-ELSA data, with extrapolation into forward and backward regions using our PWA result. The solid line represents the result of the PWA.

5.6.2.1 Fit to the $p\eta$ channel

Differential cross sections for $\gamma p \rightarrow p\eta$ in the threshold region were measured by the TAPS Collaboration. Data and fitting results are shown in Fig. 5.5. In the threshold region the dominant contribution comes from the $N(1535)S_{11}$ which gives a flat angular distribution. This resonance overlaps strongly with $N(1650)S_{11}$, and the two-pole K -matrix parametrisation is used in the fit.

The CB-ELSA differential cross section is given in Fig. 5.6. The contribution of the two S_{11} resonances (dashed line, below 2 GeV) dominates the region of η production up to 1650 MeV. Further, the most significant contributions stem from the production of $N(1720)P_{13}$ (dotted line, below 2 GeV), of $N(2070)D_{15}$ (dashed line, above 2 GeV) and ρ/ω exchanges (dotted line, above 2 GeV).

Data on the photon beam asymmetry Σ for $\gamma p \rightarrow p\eta$, measured by GRAAL [29] are shown in Fig. 5.7. This data provide essential information on baryon resonances even if their $p\gamma$ and $p\eta$ couplings are weak. In addition, the beam asymmetry data are necessary to determine the ratio of

helicity amplitudes.

The masses and widths of the observed states are presented in Tables 5.2–5.4 as well as helicity couplings and branchings to different final states.

A large number of fits (explorative fits plus more than 1000 documented fits) were performed to validate the solution. In these fits the number of resonances, their spin and parity, their parametrisation, and the relative weight of the different data sets were changed. The errors are estimated from a sequence of fits in which one variable, *e.g.* a width of one resonance, was set to a fixed value. All other variables were allowed to adjust freely; the χ^2 changes were monitored as a function of this variable. The errors given in Tables 5.2–5.4 correspond to χ^2 changes of 9, hence to three standard deviations. However, the 3σ interval corresponds better to the systematic changes observed when changing the fit hypothesis.

The resonance properties are compared to PDG values [8]. Most resonance parameters converge in the fits to values compatible with previous findings within a 2σ interval of the combined error.

Three new resonances are necessary to describe the data, $N(1875)D_{13}$, $N(2070)D_{15}$ and $N(2200)$ with uncertain spin and parity. For the last one the best fit is achieved for P_{13} quantum numbers.

Finally, a comment is needed on known resonances which were not observed in this analysis, such as $N(1990)F_{17}$, $\Delta(2420)H_{311}$, and $N(2190)G_{17}$. It looks like the resonances with high spin have quite small

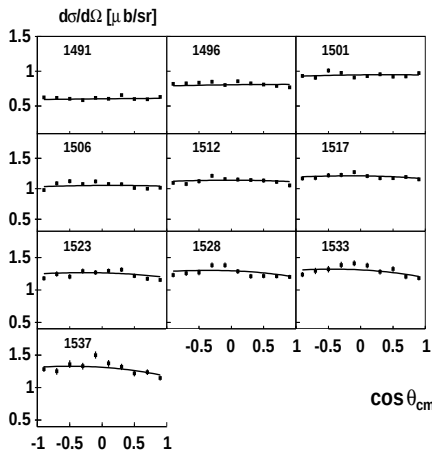


Fig. 5.5 Differential cross section for $\gamma p \rightarrow p\eta$ from Mainz-TAPS data [13] and PWA result (solid line).

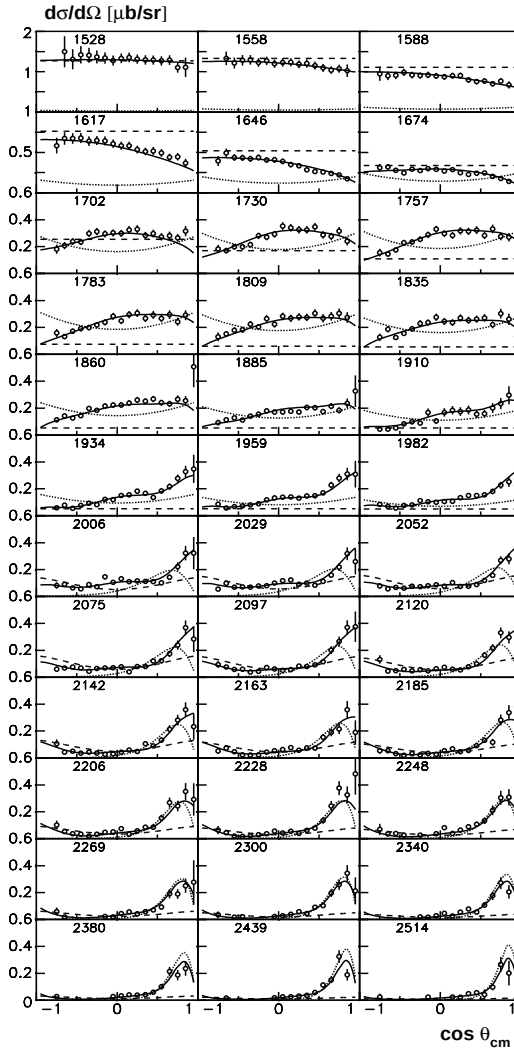


Fig. 5.6 Differential cross section for $\gamma p \rightarrow p\eta$ from CB-ELSA and PWA result (solid line). In the mass range below 2 GeV the contribution of the two S_{11} resonances is shown as a dashed line and that of $N(1720)P_{13}$ as a dotted line. Above 2 GeV the contributions of $N(2070)D_{15}$ (dashed line) and ρ/ω exchange (dotted line) are shown.

γp couplings and are not produced in the photoproduction reactions.

$N(2070)D_{15}$ is the most significant new resonance. Omitting it changes the χ^2 substantially for the η photoproduction and notably for the π^0 pho-

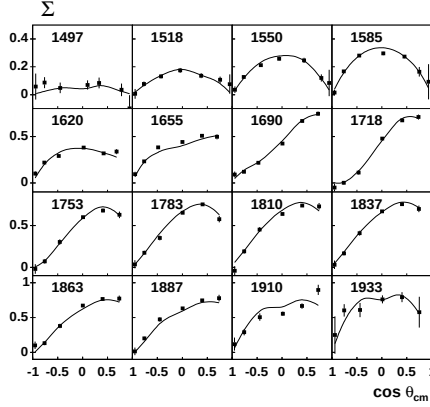


Fig. 5.7 Photon beam asymmetry Σ for $\gamma p \rightarrow p\eta$ from GRAAL [14] and PWA result (solid line).

toproduction. Replacing the J^P assignment from $5/2^-$ to $1/2^\pm$, ..., $9/2^\pm$, the χ^2_{tot} deteriorates by more than 750. The deterioration of the fits is visible in the comparison of data and fit. One of the closest description of η photoproduction was obtained, making the fit with a $7/2^-$ state. The beam asymmetry also clearly favours the $5/2^-$ state. The π^0 photoproduction cross sections measured by CB-ELSA are visually not too sensitive to distinguish between $5/2^-$ and $7/2^-$ quantum numbers. However, there is a clear difference between the two descriptions in the very backward region. The latest GRAAL results on the $p\pi^0$ differential cross section, which were obtained after discovery of the $N(2070)D_{15}$ [7], confirms $5/2^-$ as favoured quantum numbers.

The $N(2200)$ resonance is less significant for the description of data. Omitting $N(2200)$ from the analysis changes the χ^2 for the CB-ELSA data on η photoproduction by 56, and by 20 for the π^0 -photoproduction data. Other quantum numbers than the preferred P_{13} lead to marginally larger χ^2 values.

The following scenario can be suggested for the measured states, it is depicted in Fig. 5.8.

The three largest contributions to the η photoproduction cross section stem from $N(1535)S_{11}$, $N(1720)P_{13}$, and $N(2070)D_{15}$ — we tentatively assign $(J = 1/2; L = 1, S = 1/2)$ quantum numbers to the first state. The $N(1720)P_{13}$ and $N(1680)F_{15}$ form a spin doublet, it argues that the dominant quantum numbers of $N(1720)P_{13}$ are $(J = 3/2; L = 2, S = 1/2)$.

$L=3$					
$S=3/2$		ND ₁₃	N(2200)D ₁₅	N(2190)G ₁₇	N(2250)G ₁₉
$S=1/2$			N(2060)D ₁₅	NG ₁₇	
$L=2$					
$S=3/2$		N(2100)P ₁₁	N(1900)P ₁₃	N(2000)F ₁₅	N(1990)F ₁₇
$S=1/2$			N(1720)P ₁₃	N(1680)F ₁₅	
$L=1$					
$S=3/2$		N(1650)S ₁₁	N(1700)D ₁₃	N(1675)D ₁₅	
$S=1/2$		N(1535)S ₁₁	N(1520)D ₁₃		
		↑	↑	↑	↑
		$J=1/2$	$J=3/2$	$J=5/2$	$J=7/2$
					$J=9/2$

Fig. 5.8 N^*L_{2I2J} states with quantum numbers which can be assigned to orbital angular momentum excitations with $L = 1, 2, 3$ and quark spin $S = 1/2$ or $S = 3/2$ (mixing between states of the same parity and total angular momentum is possible). Resonances with strong coupling to the $N\eta$ channel are marked in grey.

Thus it is tempting to assign $(J = 5/2; L = 3, S = 1/2)$ to the $N(2070)D_{15}$. The three baryon resonances with strong contributions to the $p\eta$ channel thus all have spin $S = 1/2$, and orbital and spin angular momenta are antiparallel, $J = L - 1/2$.

The large $N(1535)S_{11} \rightarrow N\eta$ coupling has been a topic of a controversial discussion. In the quark model, this coupling arises naturally from a mixing of the two $(J = 1/2; L = 1, S = 1/2)$ and $(J = 1/2; L = 1, S = 3/2)$ harmonic-oscillator states [31]. It was assumed in [32] that this resonance originates from coupled-channel meson–baryon chiral dynamics, because $N(1535)S_{11}$ is very close to the $K\Lambda$ and $K\Sigma$ thresholds. Alternatively, the strong $N(1535)S_{11} \rightarrow N\eta$ coupling can be explained as a delicate interplay between confining and fine structure interactions [33].

A consistent picture of states depicted in Fig. 5.8 should explain the similarity of $N\eta$ couplings: the three resonances with large $N\eta$ partial decay widths are those for which $N\eta$ decays are allowed with decay orbital angular momenta $\vec{\ell}_{decay} = 0, 1, 2$, being antiparallel to $\vec{J}$.

5.7 Hyperon Photoproduction $\gamma p \rightarrow \Lambda K^+$ and $\gamma p \rightarrow \Sigma K^+$

The new CLAS data on hyperon photoproduction [17] show a remarkably large spin transfer probability. In the reactions $\gamma p \rightarrow \Lambda K^+$ and $\gamma p \rightarrow \Sigma K^+$ using a circularly polarised photon beam, the polarisations of the Λ and Σ hyperons were monitored by measurements of their decay angular distributions. For photons with helicity $h_\gamma = 1$, the magnitude of the Λ polarisation

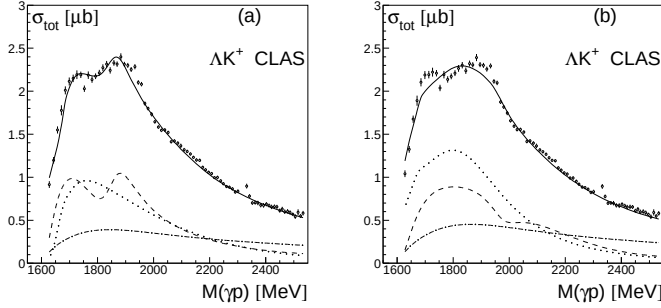


Fig. 5.9 The total cross section for $\gamma p \rightarrow \Lambda K^+$ [18] for solution 1 (a) and solution 2 (b). The solid curves are the results of the PWA fits, dashed lines are the P_{13} contribution, dotted lines are the S_{11} contribution and dash-dotted lines are the contribution from K^* exchange.

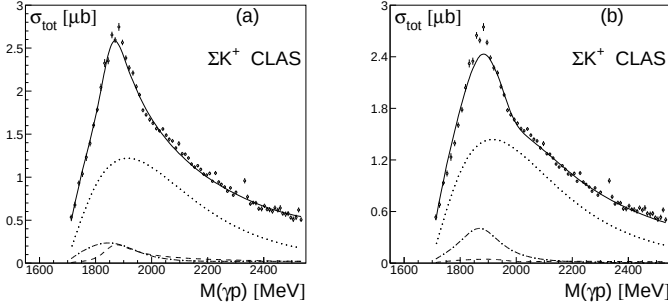


Fig. 5.10 The total cross section for $\gamma p \rightarrow \Sigma K$ [18] for solution 1 (a) and solution 2 (b). The solid curves are the results of the PWA fits, dashed lines are the P_{13} contribution, dash-dotted lines are the P_{31} contribution and dotted lines are the contribution from K exchange.

vector was found to be close to unity, 1.01 ± 0.02 , when integrated over all production angles and all centre-of-mass energies W . For Σ photoproduction, the polarisation was determined to be 0.82 ± 0.03 (again integrated over all energies and angles), still a remarkably large value. The polarisation was determined from the expression $\sqrt{C_x^2 + C_z^2 + P^2}$, where C_z is the projection of the hyperon spin onto the photon beam axis, P the spin projection on the normal-to-the-reaction plane, and C_x the spin projection in the centre-of-mass frame onto the third axis. The measurement of polarisation effects for both Λ and Σ hyperons is particularly useful. The ud pair in the Λ is antisymmetric in both spin and flavour; the ud quark carries no spin, and the Λ polarisation vector is given by the direction of the spin of

the strange quark. In the Σ hyperon, the ud quark is in a spin-1 state and points to the direction of the Σ spin, while the spin of the strange quark is opposite to it.

Independently of the question whether the polarisation phenomena require an interpretation on the quark or on the hadron level, the large polarisation seems to contradict an isobar picture of the process in which intermediate N^* 's and Δ^* 's play a dominant role. It is therefore important to see if the data are compatible with such an isobar interpretation or not.

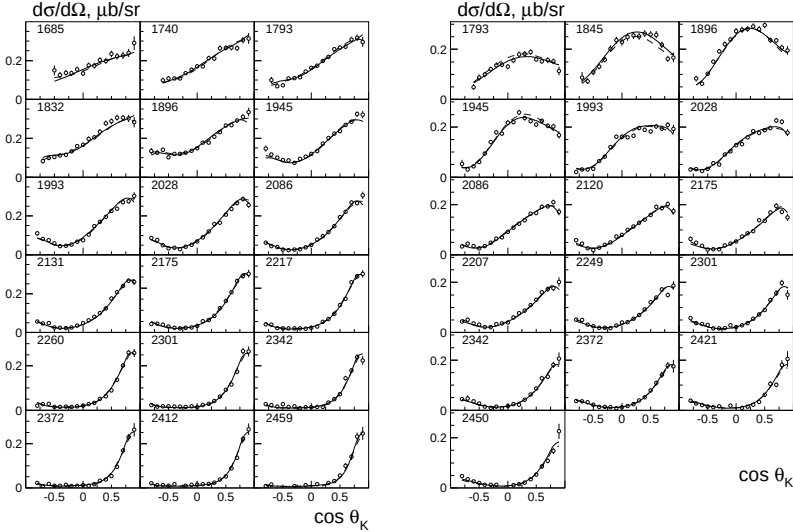
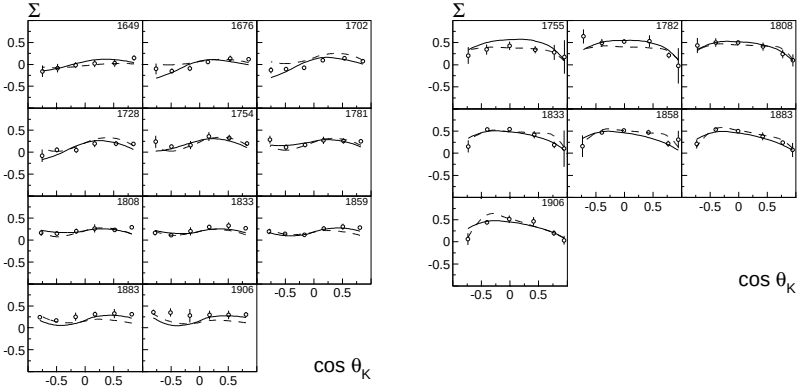


Fig. 5.11 Differential cross sections for $\gamma p \rightarrow \Lambda K^+$ (left panel) and $\gamma p \rightarrow \Sigma K$ (right panel) [18]. Only energy bins where C_x and C_z were measured are shown. The solution 1 is shown as a solid line and solution 2 (hardly visible since overlapping) as a dashed line (the total energy is given in MeV).

The data used in PWA analysis [38] comprise differential cross sections for $\gamma p \rightarrow \Lambda K^+$, $\gamma p \rightarrow \Sigma K$, and $\gamma p \rightarrow \Sigma^+ K_S^0$ including their recoil polarisation, the photon beam asymmetry, and recent spin transfer measurements.

Two new resonances are added to describe the full set of hyperon production data: $N(1840)P_{11}$ and $N(1900)P_{13}$. The $N(1840)P_{11}$ state was needed to describe the $\gamma p \rightarrow K^0 \Sigma$ data. These data show a relatively narrow peak in the region 1870 MeV which can be described either by this state alone or by contributions from P_{11} and P_{13} states. The new data on double polarisation measurements showed that both states are needed. Before, the evidence for $N(1900)P_{13}$ resonance had been weak.



However, the data do not provide a unique solution: in the partial wave analysis [38] two reasonably good descriptions of data were found. In the first one the $N(1900)P_{13}$ resonance provides a dominant contribution for the $K\sigma$ cross sections; in the second one the dominant contribution comes from the $N(1840)P_{11}$ state.

The total cross section seems to be better described by solution 1 (see Figs. 5.9 and 5.10) but the quality of the description of angular distributions is very similar for both solutions (see Fig. 5.11).

The total and differential cross sections for $\gamma p \rightarrow \Sigma K$ are presented in Fig. 5.10 and in the right panel of Fig. 5.11.

The GRAAL collaboration [21] measured the ΛK^+ and ΣK beam asymmetries in the region from the threshold to $W = 1906$ MeV. These data are an important addition to the LEPS data on the beam asymmetry [22], covering the energy region from $W = 1950$ MeV to 2300 MeV. Data and fits are shown in Fig. 5.12.

The GRAAL collaboration measured also the recoil polarisation [21] for which data from CLAS [20] had been taken in the region from the threshold up to 2300 MeV (see Fig. 5.13).

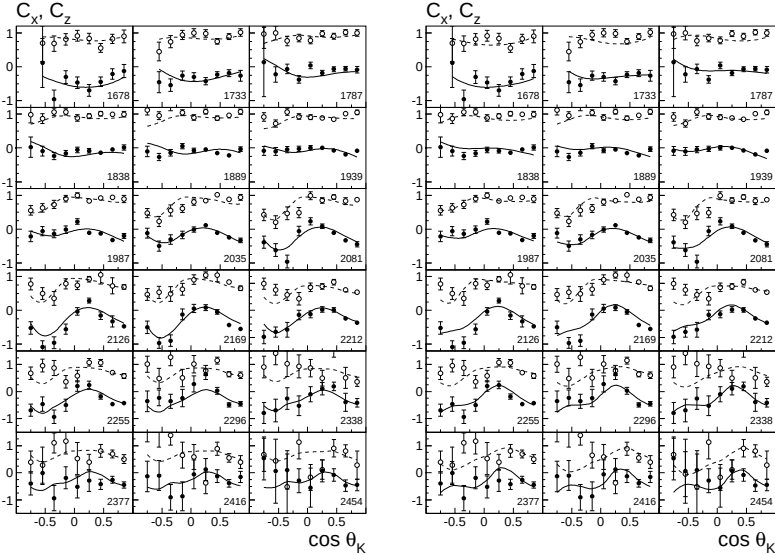


Fig. 5.14 Double polarisation observables C_x (black circle) and C_z (open circle) for $\gamma p \rightarrow \Lambda K^+$ [17]. The solid and dashed curves are results of the PWA fit obtained with solution 1 (left panel) and solution 2 (right panel) for C_x and C_z , respectively.

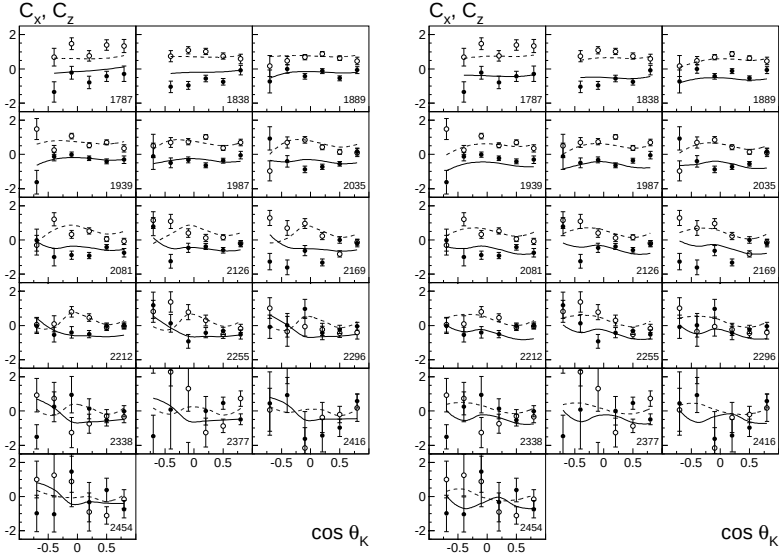


Fig. 5.15 Double polarisation observables C_x (black circle) and C_z (open circle) for $\gamma p \rightarrow \Sigma K$ [17]. The solid and dashed curves are results of our fit obtained with solution 1 (left) and solution 2 (right) for C_x and C_z , respectively.

Figure 5.14 shows the data on C_x and C_z and the fit obtained with solutions 1 and 2. For both observables a very satisfactory agreement between data and fit is achieved. Small deviations show up in two mass slices in the 2.1 GeV mass region. These should, however, not be over-interpreted. $C_x^2 + C_z^2 + P^2$ is constrained by unity; in the corresponding mass- and $\cos \Theta_K$ - bins, C_z^2 and the recoil polarisation are sizable pointing at a statistical fluctuation beyond the physical limits. Of course, the fit should not follow data into not allowed regions.

From the fit, the properties of resonances in the P_{13} -wave were derived. The lowest-mass pole is identified with the established $N(1720)P_{13}$, the second pole with the badly known $N(1900)P_{13}$. A third pole is introduced at about 2200 MeV. It improves the quality of the fit in the high-mass region but its quantum numbers cannot be deduced safely from the present data base.

In the first solution, the double structure in the P_{13} partial wave (see Fig. 5.9a) is due to a strong interference between the first and the second pole. If the structure is fitted to one pole, the pole must have a rather narrow width. The $N(1720)P_{13}$ couples strongly to $\Delta(1232)\pi$ and, in the

second solution, also to the $D_{13}(1520)\pi$ channel. The $D_{13}(1520)\pi$ threshold is close to its mass and creates a double pole structure which makes the definition of helicity amplitudes and of decay partial widths difficult.

$N(1900)P_{13}$ is of special interest for baryon spectroscopy. It belongs to the two-star positive-parity N^* resonances in the 1900–2000 MeV mass interval – $N(1900)P_{13}$, $N(2000)F_{15}$, $N(1990)F_{17}$: they cannot be assigned to quark–diquark oscillations [34], when the diquark is treated as a point-like object with zero spin and isospin. At the present stage of our knowledge on baryon excitations, most four-star and three-star baryon resonances can be interpreted in a simplified model describing baryons as being made up from a diquark and a quark. The $N(2000)F_{15}$ is included in the analysis as well; it is a further two-star N^* resonance which cannot be assigned to quark–diquark oscillations. The evidence for this state from this analysis is, however, weaker. The $N(1840)P_{11}$ state (which we now find at 1880 MeV) could be the missing partner of a super-multiplet of nucleon resonances having – as a leading configuration – an intrinsic total orbital angular momentum $L = 2$ and a total quark spin $S = 3/2$. These angular momenta couple to a series $J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$. Yet in this analysis there is no need to introduce $N(1990)F_{17}$.

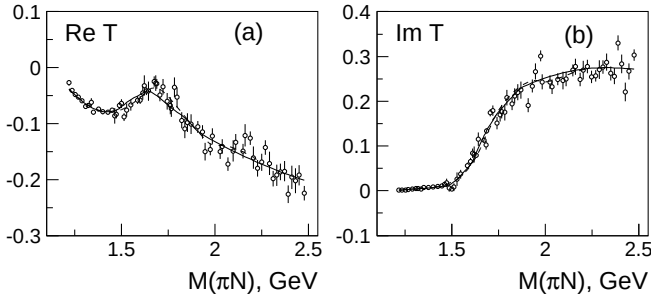


Fig. 5.16 Real (a) and imaginary (b) part of the $\pi N P_{13}$ elastic scattering amplitude [16] and the result of the PWA fit in case of solution 1 (solid curve) and solution 2 (dashed curve).

To check whether elastic data are compatible with the new state, an additional K -matrix pole into the $\pi N \rightarrow \pi N P_{13}$ partial wave with invariant mass ≤ 2.4 GeV was introduced. The K -matrix approach was used for the S_{11} , P_{11} , D_{33} , P_{33} partial waves as well. A satisfactory description of all fitted observables was obtained; as an example we show the elastic scattering data in Fig. 5.16.

The mass and width of $N(1900)P_{13}$ are estimated to be 1915 ± 50 MeV and 180 ± 50 MeV, respectively. This result covers the two K -matrix solutions found in PWA: in the first and second solutions the pole positions are $1870 - i85$ MeV and $1960 - i88$ MeV.

5.8 Analyses of $\gamma p \rightarrow \pi^0 \pi^0 p$ and $\gamma p \rightarrow \pi^0 \eta p$ Reactions

Here we present results of partial wave analyses of the $\gamma p \rightarrow \pi^0 \pi^0 p$ and $\gamma p \rightarrow \pi^0 \eta p$ reactions [5].

(i) $\gamma p \rightarrow \pi^0 \pi^0 p$ reaction.

The left panel in Fig. 5.17 shows the total cross section for $\pi^0 \pi^0$ photoproduction together with the $\Delta\pi$ and $p(\pi\pi)_S$ excitation functions. Two peaks owing to the second ($M_{\gamma p} \sim 1500$ MeV) and third ($M_{\gamma p} \sim 1700$ MeV) resonance regions are immediately identified.

The right panel of Fig. 5.17a, b shows the $p\pi^0$ and $\pi^0\pi^0$ invariant mass and angular distributions after a 1550–1800 MeV cut in the $p\pi^0\pi^0$ mass. The $p\pi^0$ mass distribution reveals the Δ as a contributing isobar. The $\pi^0\pi^0$ mass distribution does not show any significant structure. While $\pi\pi$ decays of resonances belonging to the second resonance region are completely dominated by the $\Delta\pi$ isobar as an intermediate state, the two-pion S -wave provides a significant decay fraction in the third resonance region.

In the combined analysis the Crystal Ball data on the charge exchange reaction $\pi^- p \rightarrow n\pi^0\pi^0$ [28] are useful, even though limited to masses ≤ 1.525 GeV: the data provide also valuable constraints for the third resonance region due to their long low-energy tails. Another important constraint comes from the GRAAL data on the beam asymmetry [35] (see the left panel of Fig. 5.18) and the helicity dependence of the reaction $\gamma p \rightarrow p\pi^0\pi^0$ [36] (see the right panel of Fig. 5.18). These new $p\pi^0\pi^0$ data provide an important information on the $N\pi\pi$ decay modes, at the same time the quality of the fits to the single meson photoproduction data did not worsen significantly due to the constraints given by the $p\pi^0\pi^0$ data.

The masses, widths and branching ratios of the resonances contributing to the $\gamma p \rightarrow \pi^0 \pi^0 p$ reaction are given in in Tables 5.2, 5.3, 5.4.

The P_{11} partial wave in the first ($M_{\gamma p} \sim 1400$ MeV) and second ($M_{\gamma p} \sim 1500$ MeV) resonance regions was found to be a large non-resonance one. Nevertheless, two P_{11} states are needed to describe this partial wave: the Roper resonance and a second one situated in the region 1.84–1.89 GeV/ c^2 . The properties of the $N(1440)P_{11}$ resonance determined in the PWA are as follows:

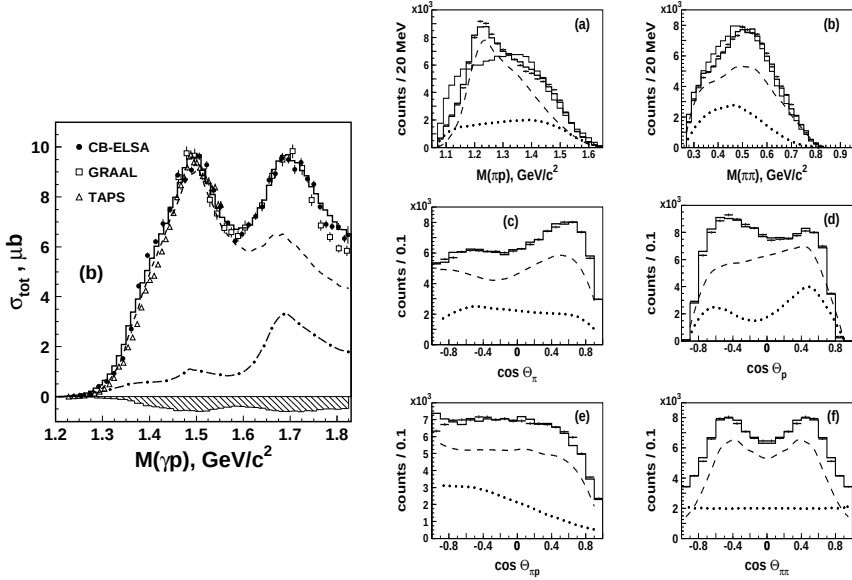


Fig. 5.17 Total cross sections for $\gamma p \rightarrow p\pi^0\pi^0$ (left panel). Solid line: the PWA fit, band below the figure presents systematic errors. Dashed curve stands for the final state $\Delta^+\pi^0 \rightarrow p\pi^0\pi^0$ and dashed-dotted line for the $p(\pi^0\pi^0)_S$ cross section derived from the PWA. Right panel demonstrates mass and angular distributions for $\gamma p \rightarrow p\pi^0\pi^0$ after a 1550–1800 MeV/ c^2 cut in $M_{p\pi^0\pi^0}$: in (a,b) the $p\pi^0$ and $\pi^0\pi^0$ distributions are shown, and (c)–(f) present the $\cos\theta$ -distributions (θ_π is the angle of a π^0 in respect to the incoming photon in the c.m. system, the θ_p is the c.m.s. angle of the proton in respect to the photon, the $\theta_{\pi p}$ is the angle between two pions in the $\pi^0 p$ rest frame, the $\theta_{\pi\pi}$ is the angle between π^0 and p in the $\pi^0\pi^0$ rest frame. Data are represented by crosses, the fit by solid line.

$$\begin{aligned}
 M_{BW} &= 1436 \pm 15 \text{ MeV}, & M_{\text{pole}} &= 1371 \pm 7 \text{ MeV}, \\
 \Gamma_{BW} &= 335 \pm 40 \text{ MeV}, & \Gamma_{\text{pole}} &= 192 \pm 20 \text{ MeV}, \\
 \Gamma_{\pi N} &= 205 \pm 25 \text{ MeV}, & g_{\pi N} &= (0.51 \pm 0.05) \cdot e^{-i(0.61 \pm 0.06)}, \\
 \Gamma_{\sigma N} &= 71 \pm 17 \text{ MeV}, & g_{\sigma N} &= (0.82 \pm 0.16) \cdot e^{-i(0.35 \pm 0.27)}, \\
 \Gamma_{\pi\Delta} &= 59 \pm 15 \text{ MeV}, & g_{\pi\Delta} &= (-0.57 \pm 0.08) \cdot e^{i(0.44 \pm 0.35)}.
 \end{aligned}
 \tag{5.132}$$

Here the left column lists mass, width, partial widths when the $N(1440)P_{11}$ is treated as a standard Breit–Wigner resonance. The right column presents results of the K -matrix fit: it gives pole position and couplings to $N(1440) \rightarrow \pi N$, $N(1440) \rightarrow \sigma N$ and $N(1440) \rightarrow \pi\Delta$ (recall that couplings are determined as residues of the amplitude poles, so they are complex-valued), for more details see [5].

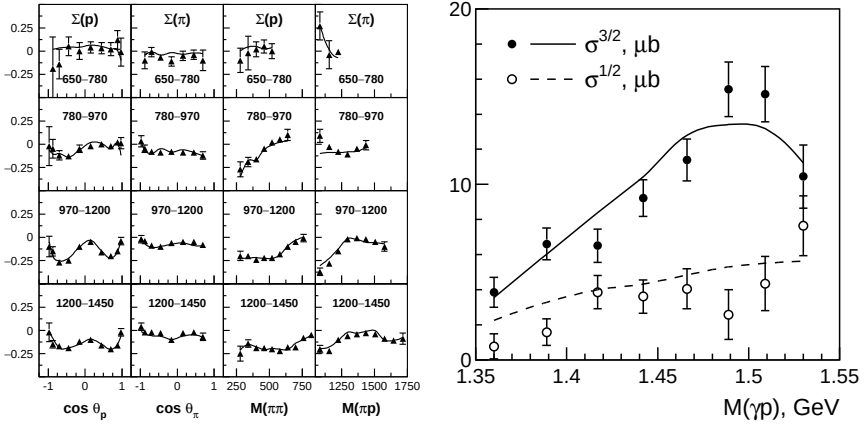


Fig. 5.18 Left panel: the beam asymmetry Σ for the reaction $\gamma p \rightarrow p\pi^0\pi^0$ depending on the proton or π^0 direction with respect to the beam axis (angles Θ_p and Θ_π), and as a function of the $\pi^0\pi^0$ and $p\pi^0$ invariant masses [35] (solid line represents the PWA fit). The numbers given in figures show the photon energy bin. Right panel: the helicity dependence in the reaction $\gamma p \rightarrow p\pi^0\pi^0$ [36] (the lines represent the result of the PWA fit).

Due to its larger phase space, decays into $N\pi$ are more frequent than those into $N\sigma$, even though the latter decay mode provides the largest coupling. For a radial excitation this is not unexpected: about 50% of all $\psi(2S)$ resonances decay into $J/\psi\sigma$, more than 25% of $\Upsilon(2S)$ resonances decay via $\Upsilon(1S)\sigma$ [39]. The large value of $g_{\sigma N}$ may therefore support the interpretation of the Roper resonance as a radial excitation.

In more details we show in Fig. 5.19a,b the elastic P_{11} amplitude for the two-pole solution. The data are well described with the two-pole four-channel (πN , σN , $\Delta\pi$ and $K\Sigma$) K -matrices. As a next step, we introduced a second pole in the Roper region — a pion-induced resonance R and a second photo-induced R'. This attempt failed. The fit reduced the elastic width to the minimal allowed value of 50 MeV; the overall probability of the fit became unacceptable. The resulting elastic amplitude is shown in Fig. 5.19a,b as a dashed line. We did not find any meaningful solution where the Roper region could comprise two resonances.

In [37, 38], no evidence for $N(1710)P_{11}$ was found. The increased sensitivity due to new data encouraged us to introduce a third pole in the P_{11} amplitude. Fig. 5.19c,d shows the result of this fit. A small improvement due to $N(1710)P_{11}$ is observed, and also other data sets are slightly better

Table 5.2 Properties of the resonances contributing to the $\gamma p \rightarrow \pi^0 \pi^0 p$ cross section. The masses and widths are given in MeV, the branching ratios in % and helicity couplings in $\text{GeV}^{-1/2}$. The helicity couplings and phases were calculated as the pole residues denoted as ‘Mass’ and ‘ Γ_{tot} ’. The values of M_{BW} and Γ_{tot}^{BW} for the Breit–Wigner description of resonances are also given.

		$N(1535)S_{11}$	$N(1650)S_{11}$	$N(1520)D_{13}$
Mass		1508^{+10}_{-30}	1645 ± 15	1509 ± 7
	PDG	1495–1515	1640–1680	1505–1515
Γ_{tot}		165 ± 15	187 ± 20	113 ± 12
	PDG	90–250	150–170	110–120
M_{BW}		1548 ± 15	1655 ± 15	1520 ± 10
	PDG	1520–1555	1640–1680	1515–1530
Γ_{tot}^{BW}		170 ± 20	180 ± 20	125 ± 15
	PDG	100–200	145–190	110–135
$A_{1/2}$		0.086 ± 0.025	0.095 ± 0.025	0.007 ± 0.015
	phase	$(20 \pm 15)^\circ$	$(25 \pm 20)^\circ$	–
	PDG	$(5.1 \pm 1.7)^\circ$	$(3.0 \pm 0.9)^\circ$	$-(1.4 \pm 0.5)^\circ$
$A_{3/2}$				0.137 ± 0.012
	phase			$-(5 \pm 5)^\circ$
	PDG			$(9.5 \pm 0.3)^\circ$
Γ_{miss}		–	–	$13 \pm 5\%$
	PDG($N\rho$)	$< 4\%$	4–12 %	15–25 %
$\Gamma_{\pi N}$		$37 \pm 9\%$	$70 \pm 15\%$	$58 \pm 8\%$
	PDG	35–55 %	55–90 %	50–60 %
$\Gamma_{\eta N}$		$40 \pm 10\%$	$15 \pm 6\%$	$0.2 \pm 0.1\%$
	PDG	30–55 %	3–10 %	$0.23 \pm 0.04\%$
$N\sigma$		–	–	$< 4\%$
	PDG		$< 4\%$	$< 8\%$
$\Gamma_{K\Lambda}$		–	$5 \pm 5\%$	–
$\Gamma_{K\Sigma}$		–	–	–
$\Gamma_{\Delta\pi(L < J)}$				$12 \pm 4\%$
$L < J$	PDG			5–12 %
$\Gamma_{\Delta\pi(L > J)}$		$23 \pm 8\%$	$10 \pm 5\%$	$14 \pm 5\%$
$L > J$	PDG	$< 1\%$		10–14 %
$\Gamma_{P_{11}\pi}$				$2 \pm 2\%$
$\Gamma_{D_{13}\pi}$				

described. The parameters of the resonance are not well defined, the pole position is found in the 1580 to 1700 MeV mass range.

The introduction of the $N(1710)P_{11}$ as a third pole changes the $N(1840)P_{11}$ properties. In the two-pole solution, the $N(1840)P_{11}$ resonance is narrow (~ 150 MeV), in the three-pole solution, the $N(1710)P_{11}$ and a ~ 250 MeV wide $N(1840)P_{11}$ resonance interfere to reproduce the

Table 5.3 Properties of the resonances $N(1700)D_{13}$, $N(1675)D_{15}$ and $N(1720)P_{13}$ (notations are as in Table 5.2).

		$N(1700)D_{13}$	$N(1675)D_{15}$	$N(1720)P_{13}$
Mass		1710 ± 15	1639 ± 10	1630 ± 90
	PDG	1630–1730	1655–1665	1660–1690
Γ_{tot}		155 ± 25	180 ± 20	460 ± 80
	PDG	50–150	125–155	115–275
M_{BW}		1740 ± 20	1678 ± 15	1790 ± 100
	PDG	1650–1750	1670–1685	1700–1750
Γ_{tot}^{BW}		180 ± 30	220 ± 25	690 ± 100
	PDG	50–150	140–180	150–300
$A_{1/2}$		0.020 ± 0.016	0.025 ± 0.01	0.15 ± 0.08
	phase	$-(4 \pm 5)^\circ$	$-(7 \pm 5)^\circ$	$-(0 \pm 25)^\circ$
	PDG	$-(1.0 \pm 0.7)^\circ$	$(1.1 \pm 0.5)^\circ$	$(1.0 \pm 1.7)^\circ$
$A_{3/2}$		0.075 ± 0.030	0.044 ± 0.012	0.12 ± 0.08
	phase	$-(6 \pm 8)^\circ$	$-(7 \pm 5)^\circ$	$-(20 \pm 40)^\circ$
	PDG	$-(0.1 \pm 1.4)^\circ$	$(0.9 \pm 0.5)^\circ$	$-(1.1 \pm 1.1)^\circ$
Γ_{miss}		$20 \pm 15 \%$	$20 \pm 8 \%$	-
	PDG($N\rho$)	$< 35 \%$	$< 1-3 \%$	70–85 %
$\Gamma_{\pi N}$		$8^{+8}_{-4} \%$	$30 \pm 8 \%$	$9 \pm 5 \%$
	PDG	5–15 %	40–50 %	10–20 %
$\Gamma_{\eta N}$		$10 \pm 5 \%$	$3 \pm 3 \%$	$10 \pm 7 \%$
	PDG	$0 \pm 1 \%$	$0 \pm 1 \%$	$4 \pm 1 \%$
$N\sigma$		$18 \pm 12 \%$	10 ± 5	$3 \pm 3 \%$
	PDG			-
$\Gamma_{K\Lambda}$		$1 \pm 1 \%$	$3 \pm 3 \%$	$12 \pm 9 \%$
$\Gamma_{K\Sigma}$		$< 1 \%$	$< 1 \%$	$< 1 \%$
$\Gamma_{\Delta\pi(L<J)}$		$10 \pm 5 \%$	$24 \pm 8 \%$	$38 \pm 20 \%$
$L < J$	PDG			-
$\Gamma_{\Delta\pi(L>J)}$		$20 \pm 11 \%$	$< 3 \%$	$6 \pm 6 \%$
$L > J$	PDG			-
$\Gamma_{P_{11}\pi}$		$14 \pm 8 \%$	$< 3 \%$	-
$\Gamma_{D_{13}\pi}$		-	$4 \pm 4 \%$	$24 \pm 20 \%$

structure. Data with polarised photons and protons will hopefully clarify the existence and the properties of these additional resonances. Further P_{11} poles are expected at larger masses.

Table 5.4 Properties of the resonances $N(1680)F_{15}$, $\Delta(1620)S_{31}$ and $\Delta(1700)S_{33}$ (notations are as in Table 5.2).

		$N(1680)F_{15}$	$\Delta(1620)S_{31}$	$\Delta(1700)D_{33}$
Mass		1674 ± 5	1615 ± 25	1610 ± 35
	PDG	1665–1675	1580–1620	1620–1700
Γ_{tot}		95 ± 10	180 ± 35	320 ± 60
	PDG	105–135	100–130	150–250
M_{BW}		1684 ± 8	1650 ± 25	1770 ± 40
	PDG	1675–1690	1615–1675	1670–1770
Γ_{tot}^{BW}		105 ± 8	250 ± 60	630 ± 150
	PDG	120–140	120–180	200–400
$A_{1/2}$		$-(0.012 \pm 0.008)$	0.13 ± 0.05	0.125 ± 0.030
	phase	$-(40 \pm 15)^\circ$	$-(8 \pm 5)^\circ$	$-(15 \pm 10)^\circ$
	PDG	$-(0.9 \pm 0.3)^\circ$	$(1.5 \pm 0.6)^\circ$	$(5.9 \pm 0.9)^\circ$
$A_{3/2}$		0.120 ± 0.015		0.150 ± 0.060
	phase	$-(5 \pm 5)^\circ$		$-(15 \pm 10)^\circ$
	PDG	$(7.6 \pm 0.7)^\circ$		$(4.8 \pm 1.3)^\circ$
Γ_{miss}		$2 \pm 2\%$	$10 \pm 7\%$	$15 \pm 10\%$
	PDG($N\rho$)	3–15 %	7–25 %	30–55
$\Gamma_{\pi N}$		$72 \pm 15\%$	$22 \pm 12\%$	$15 \pm 8\%$
	PDG	60–70 %	10–30 %	10–20 %
$\Gamma_{\eta N}$		$< 1\%$	-	-
	PDG	0 $\pm$ 1 %		
$N\sigma$		$11 \pm 5\%$	-	
	PDG	5–20 %		
$\Gamma_{K\Lambda}$		$< 1\%$		-
$\Gamma_{K\Sigma}$		$< 1\%$		
$\Gamma_{\Delta\pi(L < J)}$		$8 \pm 3\%$	$48 \pm 25\%$	-
$L < J$	PDG	6–14 %	30–60 %	$70 \pm 20\%$
$\Gamma_{\Delta\pi(L > J)}$		$4 \pm 3\%$		30–60 %
$L > J$	PDG	$< 2\%$		
$\Gamma_{P_{11}\pi}$		-	$19 \pm 12\%$	$< 5\%$
$\Gamma_{D_{13}\pi}$		-	-	$< 3\%$

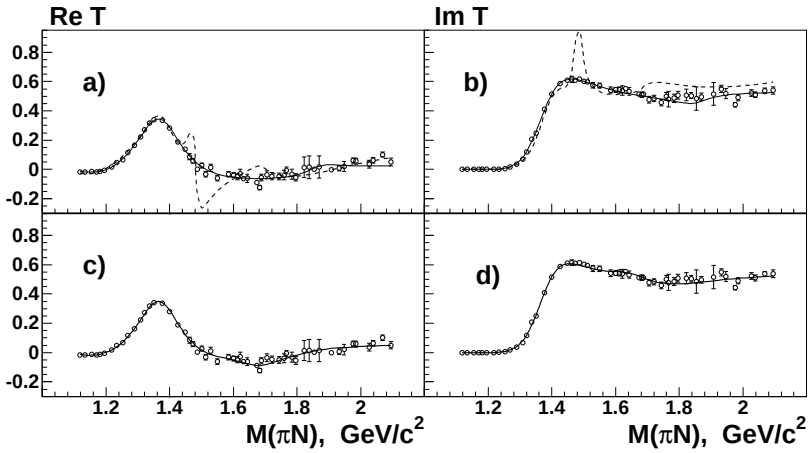


Fig. 5.19 Real (a,c) and imaginary (b,d) part of the πN P_{11} elastic scattering amplitude; data and fit with two (a,b) and three (c,d) K -matrix poles [16]. The dashed line in (a,b) represents a fit in which the Roper resonance is split into two components: the overall likelihood deteriorates to extremely bad values. The fit tries to make one Roper resonance as narrow as possible.

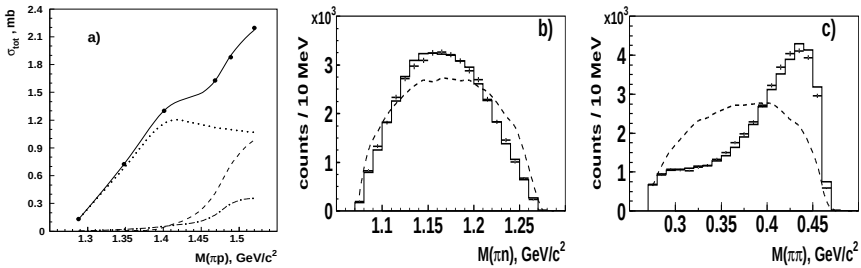


Fig. 5.20 The reaction $\pi^- p \rightarrow n \pi^0 \pi^0$ [28]. (a) Total cross section; the errors are smaller than the dots; the dotted, dashed and dot-dashed lines give the P_{11} , D_{13} and S_{11} contributions, respectively; (b) the $\pi^0 n$ and (c) $\pi^0 \pi^0$ invariant mass distributions for 551 MeV/c: the data (crosses), fit (histogram) and phase space (dashed line) are shown.

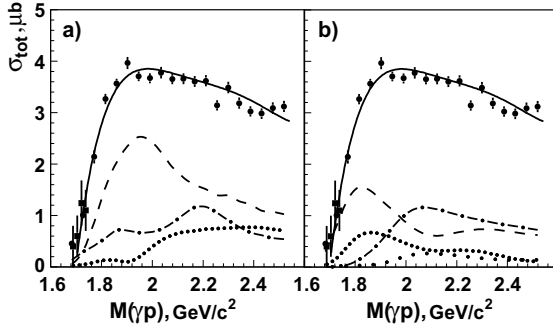


Fig. 5.21 Total cross sections for $\gamma p \rightarrow p\pi^0\eta$. The solid line represents a PWA fit. Excitation functions (a): the dashed curve shows the contribution from the $\Delta(1232)\eta$ intermediate state, the dot-dashed curve the $S_{11}(1535)\pi$, and the dotted curve the $Na_0(980)$ isobar contribution. Partial wave contributions (b): the dashed curve shows the D_{33} partial wave, the dotted curve is due to $\Delta(1232)\eta$, the widely-spaced dotted curve is due to $Na_0(980)$. The dot-dashed line represents the P_{33} contribution.

(ii) $\gamma p \rightarrow \pi^0\eta p$ reaction.

In Fig. 5.21 the total cross section of the $\gamma p \rightarrow \pi^0\eta p$ reaction is displayed. The points with errors give the acceptance-corrected results of the measurement and their statistical errors. The solid curve shows the result of partial wave analysis (PWA).

In Fig. 5.21b, contributions of individual partial waves are shown. The D_{33} partial wave is found to provide the largest contribution. In the figure it is split into its two main subchannels stemming from the $\Delta\eta$ and $N(1535)\pi$ isobars. The second largest contribution, shown as long-dashed line, comes from the P_{33} wave.

The D_{33} partial wave was described within the K -matrix approach. To fix the elastic couplings, the D_{33} πN scattering amplitude was included in the fit. A satisfactory description was obtained with five-channel ($N\pi$, $\Delta(1232)\pi$ (S -wave), $\Delta(1232)\pi$ (D -wave), $\Delta(1232)\eta$, $N(1535)\pi$) and three-pole parametrisation of the K -matrix.

In the D_{33} partial wave there is a four-star resonance in the 1700 MeV mass region [39]. The width of this state is not well defined. Our analysis of the $\gamma p \rightarrow p\pi^0\pi^0$ photoproduction determined its pole position to $(M - i\Gamma/2) = (1615 \pm 50) - i(150 \pm 30)$ MeV. Dominantly, this state decays into $\Delta(1232)\pi$, with a πN branching ratio about 15%. For this mass, the $\Delta(1232)\eta$ branching ratio is found here to be $2.3 \pm 1.0\%$. Due to the fast rising $p\pi\eta$ phase volume this ratio is, however, very sensitive to the precise

mass of the resonance. The result for the second pole in the D_{33} partial wave for the three-pole K -matrix solution is as follows:

$$\begin{array}{ccccc}
 M_{pole} & \Gamma_{pole} & & M_{BW} & \Gamma_{tot}^{BW} \\
 2010 \pm 25 & 440 \pm 90 & & 2035 \pm 25 & 420 \pm 80 \\
 \hline
 & A_{1/2}/A_{3/2} = 1.15 \pm 0.25 & & & \\
 \hline
 \Gamma_{N\pi} & \Gamma_{\Delta\pi(S)} & \Gamma_{\Delta\eta} & \Gamma_{N(1535)\pi} & \Gamma_{\Delta\pi(D)} \\
 8 \pm 3 & 63 \pm 12 & 5 \pm 2 & 2 \pm 1 & 22 \pm 8
 \end{array} \quad (5.133)$$

Masses, widths and partial decay widths are given in MeV. The Breit–Wigner parametrisation results in values denoted as M_{BW} and Γ_{tot}^{BW} .

The third pole in D_{33} -wave is shifted to the region of large masses, it cannot be considered as a reliably determined state.

5.9 Summary

We have demonstrated a relativistically invariant approach which is applied to the analysis of a large number of baryon production data. The new data on $\gamma p \rightarrow p\eta$ reveal the presence of a new state $N(2070)D_{15}$. The data on hyperon photoproduction provide a strong evidence for the existence of two states in the region 1860-1900 MeV with quantum numbers P_{11} and P_{13} . The analysis of the data on double π^0 photoproduction defines the decay properties of baryon states situated below 1750 MeV. In the analysis of $\gamma p \rightarrow p\pi^0\eta$ data a strong evidence was found in favour of the existence of the $\Delta(1940)D_{33}$ resonance.

5.10 Appendix 5.A. Legendre Polynomials and Convolutions of Angular Momentum Operators

Here we present some useful relations which are utilised in analyses of baryon spectra.

5.10.1 Some properties of Legendre polynomials

The recurrent expression for Legendre polynomials is given by

$$P_L(z) = \frac{2L-1}{L} z P_{L-1}(z) - \frac{L-1}{L} P_{L-2}(z). \quad (5.134)$$

The first and second derivatives of the Legendre polynomials can be expressed as

$$\begin{aligned}
 P'_L(z) &= L \frac{P_{L-1}(z) - z P_L(z)}{1 - z^2} = (L+1) \frac{z P_L(z) - P_{L+1}(z)}{1 - z^2}, \\
 P''_L(z) &= \frac{2z P'_L(z) - L(L+1) P_L(z)}{1 - z^2}, \\
 P''_L(z) &= \frac{2P'_{L+1}(z) - (L+1)(L+2) P_L(z)}{1 - z^2}.
 \end{aligned} \tag{5.135}$$

Some other useful expressions given here for convenience are as follows:

$$\begin{aligned}
 P'_{L-1}(z) &= z P'_L(z) - L P_L(z), \\
 P'_{L+1}(z) &= z P'_L(z) + (L+1) P_L(z), \\
 P'_{L+1} - P'_{L-1}(z) &= (2L+1) P_L(z), \\
 P''_{L+1} - P''_{L-1}(z) &= (2L+1) P'_L(z).
 \end{aligned} \tag{5.136}$$

5.10.2 Convolutions of angular momentum operators

In what follows we list the formulae for convolutions of angular momentum operators used in the analysis.

$$\begin{aligned}
 &X_{\mu\alpha_1\ldots\alpha_n}^{(n+1)}(q_\perp) X_{\alpha_1\ldots\alpha_n}^{(n)}(k_\perp) \\
 &= \frac{\alpha_n}{n+1} (\sqrt{k_\perp^2})^n (\sqrt{q_\perp^2})^{n+1} \left[-\frac{k_{1\mu}}{\sqrt{k_\perp^2}} P'_n + \frac{q_{1\mu}}{\sqrt{q_\perp^2}} P'_{n+1} \right],
 \end{aligned} \tag{5.137}$$

$$\begin{aligned}
 &X_{\mu\alpha_2\ldots\alpha_n}^{(n)}(q_\perp) X_{\nu\alpha_2\ldots\alpha_n}^{(n)}(k_\perp) = \frac{\alpha_{n-1}}{n^2} (\sqrt{k_\perp^2})^n (\sqrt{q_\perp^2})^n \left[g_{\mu\nu}^\perp P'_{n-1} \right. \\
 &- \left(\frac{q_\mu^\perp q_\nu^\perp}{q_\perp^2} + \frac{k_\mu^\perp k_\nu^\perp}{k_\perp^2} \right) P''_n + \frac{1}{2} \left(\frac{q_\mu^\perp k_\nu^\perp + k_\mu^\perp q_\nu^\perp}{\sqrt{k_\perp^2} \sqrt{q_\perp^2}} \right) (P'_n + 2z P''_n) \\
 &\left. + \frac{2n-1}{2} \left(\frac{q_\mu^\perp k_\nu^\perp - k_\mu^\perp q_\nu^\perp}{\sqrt{k_\perp^2} \sqrt{q_\perp^2}} \right) P'_n \right],
 \end{aligned} \tag{5.138}$$

$$\begin{aligned}
 &X_{\mu\nu\alpha_1\ldots\alpha_n}^{(n+2)}(q_\perp) X_{\alpha_1\ldots\alpha_n}^{(n)}(k_\perp) = \frac{2}{3} \frac{\alpha_n}{(n+1)(n+2)} (\sqrt{k_\perp^2})^n (\sqrt{q_\perp^2})^{n+2} \\
 &\times \left(\frac{X_{\mu\nu}^{(2)}(q_\perp)}{q_\perp^2} P''_{n+2} + \frac{X_{\mu\nu}^{(2)}(k_\perp)}{k_\perp^2} P''_n \right. \\
 &\left. - \frac{3}{2} \frac{[k_\mu^\perp q_\nu^\perp + k_\nu^\perp q_\mu^\perp - \frac{2}{3} g_{\mu\nu}^\perp (k^\perp q^\perp)]}{\sqrt{k_\perp^2} \sqrt{q_\perp^2}} P''_{n+1} - \right),
 \end{aligned} \tag{5.139}$$

$$\begin{aligned}
& X_{\alpha\gamma_2\ldots\gamma_n}^{(n)}(q_\perp) O_{\mu\beta_2\ldots\beta_n}^{\tau\gamma_2\ldots\gamma_n} X_{\xi\beta_2\ldots\beta_n}^{(n)}(k_\perp) = X_{\alpha\gamma_2\ldots\gamma_n}^{(n)}(q_\perp) \frac{g_{\tau\mu}}{n} X_{\xi\gamma_2\ldots\gamma_n}^{(n)}(k_\perp) \\
& + \frac{n-1}{n} X_{\alpha\mu\gamma_3\ldots\gamma_n}^{(n)}(q_\perp) X_{\xi\tau\gamma_3\ldots\gamma_n}^{(n)}(k_\perp) \\
& - \frac{2(n-1)}{n(2n-1)} X_{\alpha\tau\gamma_3\ldots\gamma_n}^{(n)}(q_\perp) X_{\xi\mu\gamma_3\ldots\gamma_n}^{(n)}(k_\perp). \quad (5.140)
\end{aligned}$$

5.11 Appendix 5.B: Cross Sections and Partial Widths for the Breit–Wigner Resonance Amplitudes

In Chapter 3 (section 3.11.1) we have presented general definitions for the differential cross section, $d\sigma$, for the process $1 + 2 \rightarrow N \text{ particles}$ and the corresponding phase spaces. Here we give general formulae for the case when the transition amplitude is described by the Breit–Wigner resonance.

We consider the production of $N \text{ particles}$ with the momenta q_i from two particles colliding with momenta k_1 and k_2 ; the cross section (see section 3.11.1) reads:

$$d\sigma = \frac{|A|^2 d\phi_N(P; q_1, \dots, q_N)}{4\sqrt{(k_1 k_2)^2 - m_1^2 m_2^2}} = \frac{|A|^2 d\Phi_N(P; q_1, \dots, q_N)}{2|\vec{k}|\sqrt{s}}, \quad (5.141)$$

where A is the transition amplitude $1 + 2 \rightarrow N \text{ particles}$, $P = k_1 + k_2$ ($P^2 = s$), and $\vec{k}$ is the 3-momentum of the initial particle calculated in the centre-of-mass system of the reaction, $|\vec{k}| = \sqrt{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}/4s$. If the polarisation of the particles is not detected, the cross section is calculated by averaging over polarisations of the initial state particles and summing over polarisations of the final state ones, with the following integration over invariant N -particle phase space:

$$d\Phi_N(P; q_1, \dots, q_N) = \frac{1}{2}(2\pi)^4 \delta^4\left(P - \sum_{i=1}^N q_i\right) \prod_{i=1}^N \frac{d^3 q_i}{(2\pi)^3 2q_{0i}}. \quad (5.142)$$

The transition amplitude from the initial state, ' in ', to the final state, ' out ', via a resonance with the total spin J , mass M and width Γ_{tot} has the form:

$$A = \frac{g_{in} Q_{\mu_1 \dots \mu_n}^{in} F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} Q_{\nu_1 \dots \nu_n}^{out} g_{out}}{M^2 - s - iM\Gamma_{tot}}. \quad (5.143)$$

Here $n = J - 1/2$ for the baryon resonance, g_{in} and g_{out} are the initial and final state couplings, Q^{in} and Q^{out} are operators, which describe the production and decay processes, and $F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}$ is the tensor part of the baryon resonance propagator.

The standard formula for the decay of a resonance into N particles is given by

$$M\Gamma = \int |A_{decay}|^2 d\Phi_N(P, q_1 \dots q_N) \quad (5.144)$$

and, as for the cross section, one has to sum over the polarisations of the final state particles.

In the operator representation, the amplitude A_{decay} has the form:

$$A_{decay} = \bar{\Psi}_{\mu_1 \dots \mu_n}^{(i)} Q_{\mu_1 \dots \mu_n} g, \quad (5.145)$$

where $\Psi_{\mu_1 \dots \mu_n}^{(i)}$ is the polarisation tensor of the resonance (conventionally, we call it the polarisation wave function), $Q_{\mu_1 \dots \mu_n}$ is the operator of the transition of the resonance into the final state, and g is the corresponding coupling constant. For example, if $Q = Q^{out}$ and $g = g_{out}$, equation (5.144) provides us with the partial width for the resonance decay into the final state and, if $Q = Q^{in}$ and $g = g_{in}$, for the partial widths for its decay into the initial state.

Recall that the tensor part of the propagator is determined by the polarisation tensor as follows:

$$F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} = \sum_{i=1}^{2J+1} \Psi_{\mu_1 \dots \mu_n}^{(i)} \bar{\Psi}_{\nu_1 \dots \nu_n}^{(i)},$$

with $\bar{\Psi}_{\mu_1 \dots \mu_n}^{(i)} \Psi_{\mu_1 \dots \mu_n}^{(j)} = (-1)^n \delta_{ij}$. (5.146)

Here the summation is performed over all possible polarisations (i) of the resonance state.

Multiplying the amplitude squared by $\Psi_{\alpha_1 \dots \alpha_n}^{(j)} \bar{\Psi}_{\alpha_1 \dots \alpha_n}^{(j)}$ and summing over the polarisations (i), we obtain:

$$\begin{aligned} \Psi_{\alpha_1 \dots \alpha_n}^{(j)} \bar{\Psi}_{\alpha_1 \dots \alpha_n}^{(j)} M\Gamma &= \int d\Phi_N(P; q_1, \dots, q_N) g^2(s) \\ &\times \sum_{i=1}^{2J+1} \Psi_{\alpha_1 \dots \alpha_n}^{(j)} \bar{\Psi}_{\mu_1 \dots \mu_n}^{(i)} Q_{\mu_1 \dots \mu_n} \otimes Q_{\nu_1 \dots \nu_n} \Psi_{\nu_1 \dots \nu_n}^{(i)} \bar{\Psi}_{\alpha_1 \dots \alpha_n}^{(j)}. \end{aligned} \quad (5.147)$$

Due to the orthogonality of the polarisation tensors, $\int \bar{\Psi}_{\mu_1 \dots \mu_n}^{(i)} Q_{\mu_1 \dots \mu_n} \otimes Q_{\nu_1 \dots \nu_n} \Psi_{\nu_1 \dots \nu_n}^{(j)} d\Phi_N(P, q_1 \dots q_N) \sim \delta_{ij}$, the product of the polarisation tensors can be substituted by

$$\Psi_{\nu_1 \dots \nu_n}^{(i)} \bar{\Psi}_{\alpha_1 \dots \alpha_n}^{(j)} \rightarrow \sum_{i=1}^{2J+1} \Psi_{\nu_1 \dots \nu_n}^{(i)} \bar{\Psi}_{\alpha_1 \dots \alpha_n}^{(i)} = F_{\alpha_1 \dots \alpha_n}^{\nu_1 \dots \nu_n}. \quad (5.148)$$

Performing these substitutions in (5.147) and summing over j , we obtain finally:

$$(2J+1)M\Gamma = \int d\Phi_N(P; q_1, \dots, q_N) g^2(s) Q_{\mu_1 \dots \mu_n} \otimes Q_{\nu_1 \dots \nu_n} F_{\mu_1 \dots \mu_n}^{\nu_1 \dots \nu_n}. \quad (5.149)$$

This is the basic equation for the calculation of partial widths of resonances.

The cross section for the Breit–Wigner resonance (the amplitude is given in (5.143)) reads:

$$\begin{aligned} \sigma &= \frac{1}{(2s_1+1)(2s_2+1)} \int \frac{1}{2|\vec{k}|\sqrt{s}} d\Phi_N(P; q_1, \dots, q_N) \\ &\times g_{in}^2 Q_{\mu_1 \dots \mu_n}^{in} \frac{F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} Q_{\nu_1 \dots \nu_n}^{out} \otimes Q_{\alpha_1 \dots \alpha_n}^{out} F_{\beta_1 \dots \beta_n}^{\alpha_1 \dots \alpha_n}}{(M^2 - s)^2 + (M\Gamma_{tot})^2} Q_{\beta_1 \dots \beta_n}^{in} g_{out}^2 \\ &= \frac{g_{in}^2 g_{out}^2}{2|\vec{k}|\sqrt{s}} \frac{Q_{\mu_1 \dots \mu_n}^{in} F_{\beta_1 \dots \beta_n}^{\mu_1 \dots \mu_n} M\Gamma_{out} Q_{\beta_1 \dots \beta_n}^{in}}{(M^2 - s)^2 + (M\Gamma_{tot})^2}. \end{aligned} \quad (5.150)$$

The factor $1/(2s_1+1)(2s_2+1)$ is due to averaging over spins of initial particles, s_1 and s_2 (see Chapter 3, section 3.11.1)

We can rewrite Eq. (5.150) using the partial widths for the initial state particles which depend on the two-body phase space of particles with masses m_1 and m_2 . Recall that $d\Phi_2(P, k_1, k_2) = \rho(s, m_1, m_2) d\Omega/(4\pi)$ and $\rho(s, m_1, m_2) = \sqrt{[(s - (m_1 + m_2)^2)[s - (m_1 - m_2)^2]}/(16\pi s) = |\vec{k}|/(8\pi\sqrt{s})$.

If so, we can use Eq. (5.149) to calculate partial widths for the decays into initial state particles. After the summation over spin variables, one has for the partial width:

$$\sigma = \frac{2J+1}{(2s_1+1)(2s_2+1)} \frac{4\pi}{|\vec{k}|^2} \frac{M^2 \Gamma_{in} \Gamma_{out}}{(M^2 - s)^2 + (M\Gamma_{tot})^2}. \quad (5.151)$$

This is the standard equation for the contribution of a resonance with spin J to the cross section.

5.11.1 The Breit–Wigner resonance and rescattering of particles in the resonance state

The amplitude which describes the rescatterings via a resonance with total spin $J = n + 1/2$ is given by

$$\begin{aligned} A(s) &= g_{in} Q_{\mu_1 \dots \mu_n}^{in} \frac{F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}}{M_0^2 - s} Q_{\nu_1 \dots \nu_n}^{out} g_{out} \\ &+ g_{in} Q_{\mu_1 \dots \mu_n}^{in} \frac{F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}}{M_0^2 - s} \tilde{B}_{\xi_1 \dots \xi_n}^{\nu_1 \dots \nu_n} \frac{F_{\beta_1 \dots \beta_n}^{\xi_1 \dots \xi_n}}{M_0^2 - s} Q_{\beta_1 \dots \beta_n}^{out} g_{out} + \dots \end{aligned} \quad (5.152)$$

As before, we assume that the vertex operators include the polarisation tensors of the initial and final particles.

The imaginary part of the loop diagram for the intermediate state with N particles is given by

$$\text{Im } \tilde{B}_{\xi_1 \dots \xi_n}^{\nu_1 \dots \nu_n} = \int g^2(s) d\Phi_m(P; k_1, \dots, k_N) Q_{\nu_1 \dots \nu_n} \otimes Q_{\xi_1 \dots \xi_n}, \quad (5.153)$$

where $g(s)$ and Q are the coupling and vertex operator, respectively, for the decay of a resonance into the intermediate state. Recall that the definition $Q_{\nu_1 \dots \nu_n} \otimes Q_{\xi_1 \dots \xi_n}$ assumes summation over polarisations of the intermediate particles. In the pure elastic case the intermediate state operator is equal to $Q = Q^{in} = Q^{out}$ but generally, the B -function is equal to the sum of loop diagrams over all possible decay modes.

Let us define the $B(s)$ -function as follows:

$$\begin{aligned} F_{\beta_1 \dots \beta_n}^{\mu_1 \dots \mu_n} \text{Im} B(s) &= F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} \text{Im} \tilde{B}_{\xi_1 \dots \xi_n}^{\nu_1 \dots \nu_n} F_{\beta_1 \dots \beta_n}^{\xi_1 \dots \xi_n} \\ &= F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} \int g^2(s) d\Phi_m(P; k_1, \dots, k_N) Q_{\nu_1 \dots \nu_n} \otimes Q_{\xi_1 \dots \xi_n} F_{\beta_1 \dots \beta_n}^{\xi_1 \dots \xi_n}. \end{aligned} \quad (5.154)$$

Using this equation, one can convolute all tensor factors into one structure, so the amplitude reads:

$$\begin{aligned} A(s) &= g_{in} Q_{\mu_1 \dots \mu_n}^{in} \frac{F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}}{M_0^2 - s} Q_{\nu_1 \dots \nu_n}^{out} g_{out} \left[1 + \frac{B(s)}{M_0^2 - s} + \left(\frac{B(s)}{M_0^2 - s} \right)^2 + \dots \right] \\ &= g_{in} Q_{\mu_1 \dots \mu_n}^{in} \frac{F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}}{M_0^2 - s - B(s)} Q_{\nu_1 \dots \nu_n}^{out} g_{out}. \end{aligned} \quad (5.155)$$

The imaginary part of the B -function defines the width of the state, and we obtain the standard Breit–Wigner expression.

5.11.2 Blatt–Weisskopf form factors

If a resonance with radius r decays into two particles with masses m_1 and m_2 and relative momentum squared $k^2 = [(s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2)]/(4s)$, then the first few expressions for formfactors $F(L, k^2, r)$ are

equal to:

$$\begin{aligned}
 F(L=0, k^2, r) &= 1, \\
 F(L=1, k^2, r) &= \frac{\sqrt{(x+1)}}{r}, \\
 F(L=2, k^2, r) &= \frac{\sqrt{(x^2+3x+9)}}{r^2}, \\
 F(L=3, k^2, r) &= \frac{\sqrt{(x^3+6x^2+45x+225)}}{r^3}, \\
 F(L=4, k^2, r) &= \frac{\sqrt{(x^4+10x^3+135x^2+1575x+11025)}}{r^4},
 \end{aligned} \tag{5.156}$$

where $x = k^2 r^2$.

5.12 Appendix 5.C. Multipoles

Let us consider the transition amplitude $\gamma N \rightarrow \pi N$ when the initial state has the total spin $J = L + 1/2$ and spin $S = 1/2$:

$$\begin{aligned}
 A^+(1/2) &= \bar{u}(q_1) X_{\alpha_1 \dots \alpha_L}^{(L)}(q^\perp) F_{\beta_1 \dots \beta_L}^{\alpha_1 \dots \alpha_L}(P) \gamma_\mu i\gamma_5 X_{\beta_1 \dots \beta_L}^{(L)}(k^\perp) u(p_N) \epsilon_\mu \\
 &\quad \times BW(s).
 \end{aligned} \tag{5.157}$$

Here $BW(s)$ represents the dynamical part of the amplitude. Taking into account the properties of the projection operator, this expression can be rewritten as

$$\begin{aligned}
 &\bar{u}(q_1) X_{\alpha_1 \dots \alpha_L}^{(L)}(q^\perp) T_{\beta_1 \dots \beta_L}^{\alpha_1 \dots \alpha_L} \frac{\sqrt{s} + \hat{P}}{2\sqrt{s}} X_{\beta_1 \dots \beta_L}^{(L)}(k^\perp) \gamma_\mu i\gamma_5 u(p_N) \epsilon_\mu \\
 &= \bar{u}(q_1) \left[\frac{L+1}{2L+1} X_{\alpha_1 \dots \alpha_L}^{(L)}(q^\perp) X_{\alpha_1 \dots \alpha_L}^{(L)}(k^\perp) \right. \\
 &\quad \left. - \frac{L}{2L+1} \sigma_{\alpha\beta} X_{\alpha\alpha_2 \dots \alpha_L}^{(L)}(q^\perp) X_{\beta\alpha_2 \dots \alpha_L}^{(L)}(k^\perp) \right] \frac{\sqrt{s} + \hat{P}}{2\sqrt{s}} \gamma_\mu i\gamma_5 u(p_N) \epsilon_\mu.
 \end{aligned} \tag{5.158}$$

Convoluting the X -operators with external indices (see Appendix 5.A), one obtains:

$$\begin{aligned}
 A^+(1/2) &= \bar{u}(q_1) \frac{L+1}{2L+1} \alpha(L) (\sqrt{q^\perp} \sqrt{k^\perp})^L \left[P_L(z) - \frac{P'_L(z)}{L+1} \sigma_{\alpha\beta} \frac{q_\alpha^\perp k_\beta^\perp}{(\sqrt{q^\perp} \sqrt{k^\perp})} \right] \\
 &\quad \times \frac{\sqrt{s} + \hat{P}}{2\sqrt{s}} \gamma_\mu i\gamma_5 u(p_N) \epsilon_\mu BW(s).
 \end{aligned} \tag{5.159}$$

In the c.m. system we have:

$$\bar{u}(q_1) \frac{\sqrt{s} + \hat{P}}{2\sqrt{s}} \gamma_\mu i\gamma_5 u(p_N) \epsilon_\mu = -\sqrt{\chi_i \chi_f} i\varphi^*(\vec{\epsilon}_i \vec{\sigma}_i) \varphi', \tag{5.160}$$

that leads to

$$A^+(1/2) = -\varphi^* \sqrt{\chi_i \chi_f} \frac{\alpha(L)}{2L+1} \epsilon_i (\sqrt{q^\perp} \sqrt{k^\perp})^L \left[i \sigma_i \left((L+1) P_L(z) \right. \right. \\ \left. \left. + z P'_L(z) \right) + (\vec{\sigma} \vec{q}) \frac{\varepsilon_{ijm} \sigma_j k_m}{|\vec{k}| |\vec{q}|} P'_L(z) \right] \varphi' BW(s). \quad (5.161)$$

Taking into account the properties of the Legendre polynomials (see Appendix 5.A), the amplitude can be compared with equations (5.57), (5.58). One finds the following correspondence between the spin operators and multipoles:

$$E_L^{+(\frac{1}{2})} = (-1)^L \sqrt{\chi_i \chi_f} \frac{\alpha(L)}{2L+1} \frac{(|\vec{k}| |\vec{q}|)^L}{L+1} BW(s), \quad M_L^{+(\frac{1}{2})} = E_L^{+(\frac{1}{2})}. \quad (5.162)$$

Here and below $E_L^{+(\frac{1}{2})}$ and $M_L^{+(\frac{1}{2})}$ multipoles correspond to the decomposition of the spin-1/2 amplitudes.

Recall that the reaction $\gamma N \rightarrow \pi N$ is characterised by two independent γN -operators for $S = 3/2$ and $J \geq 3/2$, while for $J = 1/2$ state there is only one independent operator. For the set of $J = L + 1/2$ states, the second operator reads:

$$A^+(3/2) = \bar{u}(q_1) X_{\alpha_1 \dots \alpha_L}^{(L)}(q^\perp) F_{\mu \beta_2 \dots \beta_L}^{\alpha_1 \dots \alpha_L}(P) \\ \times \gamma_\xi i \gamma_5 X_{\xi \beta_2 \dots \beta_L}^{(L)}(k^\perp) u(p_N) \epsilon_\mu BW(s). \quad (5.163)$$

Using expressions given in Appendix 5.A, one obtains the following multipole decomposition for the spin-3/2 amplitudes:

$$E_L^{+(\frac{3}{2})} = (-1)^L \sqrt{\chi_i \chi_f} \frac{\alpha(L)}{2L+1} \frac{(|\vec{k}| |\vec{q}|)^L}{L+1} BW(s), \quad M_L^{+(\frac{3}{2})} = -\frac{E_L^{+(\frac{3}{2})}}{L}. \quad (5.164)$$

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Chapter 6

Multiparticle Production Processes

The study of multiparticle production processes gives valuable, sometimes unique information about resonances. One should realise, however, that extracting such an information, one may face certain problems which were mentioned in Chapter 4.

The history of studying multiparticle processes and that of hadrons began simultaneously. More than 50 years ago formulae had been written for the production of two nucleons strongly interacting in $^1S_0$ and $^3S_1$ waves (Watson–Migdal formulae [1]) which aims at the description of the processes of the type in Fig. 6.1.

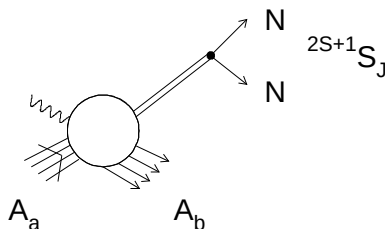


Fig. 6.1 Production of an NN pair in resonance states $^1S_0$ and $^3S_1$ — the diagram corresponds to the Watson–Migdal formula.

The corresponding amplitude reads

$$\Lambda_{a \rightarrow b} \frac{1}{1 - i k_{NN} a_J(k_{NN}^2)}, \quad (6.1)$$

where the block $\Lambda_{a \rightarrow b}$ is related to initial state interactions while the K -matrix factor $[1 - i k_{NN} a_J(k_{NN}^2)]^{-1}$ for two strongly interacting nucleons is singled out.

In the $^3S_1$ wave, equation (6.1) describes both two nucleon and deuteron production, since here the K -matrix factor has a pole singularity on the first (physical) sheet at $k_{NN}^2 = -m\varepsilon_d$ (ε_d is the deuteron binding energy), *i.e.* $1 + \sqrt{m\varepsilon_d} a_1(-m\varepsilon_d) = 0$. Note that the scattering length $a_1(k_{NN}^2)$ is negative, and below the threshold, at $k_{NN}^2 < 0$, on the first sheet one has $k_{NN} = -i\sqrt{|k_{NN}|}$.

In the $^1S_0$ -wave the pole is also present, being located on the second sheet below the NN -threshold ($a_0(k_{NN}^2)$ is positive, and on the second sheet at $k_{NN}^2 < 0$ one has $k_{NN} = i\sqrt{|k_{NN}|}$).

The Watson–Migdal formula was a theoretical forerunner for all subsequent investigations of resonance production in hadron–hadron collisions: in the isobar model for the reaction $NN \rightarrow NN\pi$ [2], in the near-threshold amplitude expansion $NN \rightarrow NN\pi$ over relative momenta of the produced particles [4, 5], in the P -vector model [6].

In fact, we have already discussed an analogous model in Chapter 4 when considering the processes $NN \rightarrow N\Delta$ or, more generally, $NN \rightarrow NN_j^*$. It is essential that in all these processes the block of the production of a resonance is a complex value, for it includes rescatterings in the intermediate state, both elastic and plausible inelastic (see Fig. 6.2) ones.

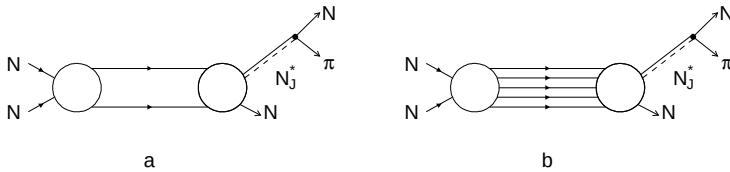


Fig. 6.2 The $NN \rightarrow NN_j^* \rightarrow NN\pi$ process: two-particle (a) and multi-particle (b) intermediate states provide the complex-valued block $\Lambda_{a \rightarrow b}$ entering the isobar model for the considered process.

Of course, one should take into account the complexity in the block $\Lambda_{a \rightarrow b}$: it is extremely important when we deal with the production of several resonances. Still, in the case of single resonance production, when spectra of secondaries are analysed within simplified models (for example, when the background processes are neglected), one may forget about the complexity in $\Lambda_{a \rightarrow b}$, because $\Lambda_{a \rightarrow b} = |\Lambda_{a \rightarrow b}| \exp(i\varphi_\Lambda)$, and the amplitude phase does not participate in fitting to data. But it happens rather frequently that after a while the simplified formulae begin to be accepted in cases where they are unacceptable, keeping the former name for a new model. In the end of the

80's and the beginning of the 90's, the notion "isobar model" meant just a model with real $\Lambda_{a \rightarrow b}$. The use of such "isobar models" — that time it became a traditional delusion — could not but provide mistaken results. As an example, one may recollect the discovery of the tensor state $AX_2(1520)$ in the reaction $p\bar{p}(\text{at rest}) \rightarrow \pi\pi\pi$ [7], while the subsequent analysis proved that it was the scalar state $f_0(1500)$ [8, 9].

As was said above, the account for unitarity and analyticity in the multiparticle amplitudes is of utmost importance, though it is not always possible to perform such an analysis in a completely correct way. Therefore, having in mind the demands of the experiment, we speak here about the existing problems and how to avoid them.

In this chapter we first give a correct representation of the isobar model, comparing it with the K -matrix technique, which generalise the isobar model for multiparticle reactions. In terms of the K -matrix technique we demonstrate an example for the fitting to two-meson spectra in the three-body reaction.

Second, to visualise the process, we consider three-particle reactions and derive equations which take into account the analyticity and the unitarity of the amplitude. We do this for both the comparatively simple case of the S -wave interaction of the produced particles and the much more complicated final state processes.

Third, we analyse three-particle reactions in the region where reggeon exchanges work, *i.e.* at high energies and moderately small momentum transfers. It turned out that there existed also obstacles and traditional delusions in the study of such two-particle spectra. We present analyses of the reactions $\pi N \rightarrow \pi\pi N$, $K\bar{K}N$, $\omega\omega$, $\pi\pi\pi\pi$ at the incident pion momentum about 20–40 GeV/c in terms of reggeon exchanges and discuss the problems appearing this way.

6.1 Three-Particle Production at Intermediate Energies

The analyticity and unitarity constraints on the amplitudes with three particles in the final state are related to the rescatterings of these particles. The rescatterings of three particles have been investigated rather long ago, in particular, when particles are produced near the their threshold — even in this comparatively simple case all characteristic features of the three-particle reactions are seen after imposing the requirements of analyticity and unitarity. In the paper [10], in the framework of the

quantum mechanical approach, two-fold pion rescatterings were considered for the $K \rightarrow 3\pi$ decay. For the same decay the two- and three-fold rescatterings were taken into account within the dispersion, or spectral integration, technique [11]; three-fold rescatterings were also studied in [5, 12]. The method of spectral integral representation, though within the same non-relativistic approach, was applied for the processes with non-zero angular momenta [13].

Let us underline once more that in this field of activity, though non-relativistic, all typical features of amplitudes which are due to analyticity and unitarity are present (see the review [14] for more details).

The relativistic approach for the rescattering processes was applied for the treatment of the triangle diagram singularities [15]. Later the relativistic spectral integral technique was used for the calculation of the final state rescatterings in the $\eta \rightarrow \pi\pi\pi$ decay [16] and for ϕ -meson production in the $p\bar{p}$ annihilation at rest [17].

A relativistic dispersion relation equation for the three-particle production process $\eta \rightarrow \pi\pi\pi$ was suggested in [18]; in this equation all final state two-pion rescatterings are taken into account. Later this type of equation was generalised for the system of amplitudes of the coupled channels [19]: $p\bar{p}(\text{at rest}) \rightarrow \pi\pi\pi, \eta\eta\pi, K\bar{K}\pi$.

Presenting properties of the three-particle amplitudes, we concentrate mainly on the relatively simple case of the production of spinless particles. Realistic analyses of amplitudes are given mainly in the Appendices.

6.1.1 Isobar model

In the isobar model the rescatterings of secondaries are not taken into account but the resonance productions in the final states. We consider the three-particle production amplitude which can be either the decay amplitude of a particle of rather large mass, $h_J \rightarrow h_1 h_2 h_3$, Fig. 6.3a or the reaction of the type of $p\bar{p} \rightarrow h_1 h_2 h_3$ annihilation in the $^{2S+1}L_J$ wave, Fig. 6.3b.

Let us make a comment concerning the name of the model. Initially, the model has been developed for the description of the Δ -isobar production; this was the reason for calling it the isobar model. Further, keeping the same name, this type of model was extended also to other reactions; here we follow this tradition.

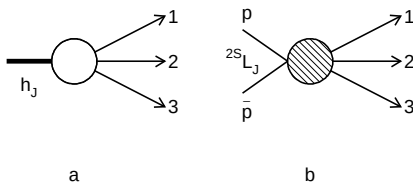


Fig. 6.3 Three-particle production processes: a) $h_J \rightarrow h_1 h_2 h_3$ decay and b) $p\bar{p}(^{2S+1}L_J) \rightarrow h_1 h_2 h_3$ transition.

6.1.1.1 The $(J^P = 0^-)$ -state $\rightarrow P_1 P_2 P_3$ transition

To be illustrative, we consider now a simple case, namely, the decay of a pseudoscalar particle with $J = 0$ into three pseudoscalar particles. Correspondingly, we redenote: $h_1 \rightarrow P_1$, $h_2 \rightarrow P_2$, $h_3 \rightarrow P_3$.

(i) The production of spinless resonances.

In the case of spin-zero produced resonances the amplitude depends on four variables $s = P^2 = (k_1 + k_2 + k_3)^2$ and $s_{i\ell} = p_{i\ell}^2 = (k_i + k_\ell)^2$, three of them being independent because $s + m_1^2 + m_2^2 + m_3^2 = s_{12} + s_{13} + s_{23}$.

Within the isobar model, the amplitude of production of spinless resonances (see Fig. 6.4) reads:

$$A_{P_1 P_2 P_3}^{J=0} = \lambda(s_{12}, s_{13}, s_{23}) + \sum_c \frac{G_c(s, s_{12}) g_c(s_{12})}{M_c^2 - s_{12} - i\Gamma_c(s_{12})M_c} \quad (6.2)$$

$$+ \sum_b \frac{G_b(s, s_{13}) g_b(s_{13})}{M_b^2 - s_{13} - i\Gamma_b(s_{13})M_b} + \sum_a \frac{G_a(s, s_{23}) g_a(s_{23})}{M_a^2 - s_{23} - i\Gamma_a(s_{23})M_a}.$$

Here $\lambda(s_{12}, s_{13}, s_{23})$ is the amplitude block without final-state resonances (the background); it is obvious that it is a complex-valued function. The next terms in (6.2) are the resonance contributions depending on the production (G_a, G_b, G_c) and decay (g_a, g_b, g_c) vertices.

The decay vertices of resonances g_c (the $c \rightarrow P_1 P_2$ decay), g_b (the $b \rightarrow P_1 P_3$ decay) and g_a (the $a \rightarrow P_2 P_3$ decay) are considered as real values. For these vertices one may take into account the dependence on the invariant energy $s_{i\ell}$. If, however, the resonance width is not large, we replace $g_a(s_{23}) \rightarrow g_a(M_a^2)$ etc., with a good accuracy.

In the representation of production vertices G_c, G_b, G_a there is also a freedom: one may take into account the dependence on the invariant energy $s_{i\ell}$, or one can substitute $s_{23} \rightarrow M_a^2$, etc., if this is acceptable by the experimental data. Let us make a principal statement: the vertices G_a ,

G_b , G_c are complex-valued if in the reaction there are other channels. Just the intermediate states between these channels (see Fig. 6.2) lead to the complexities of G_a , G_b , G_c . And the complexities in these vertices can be different.

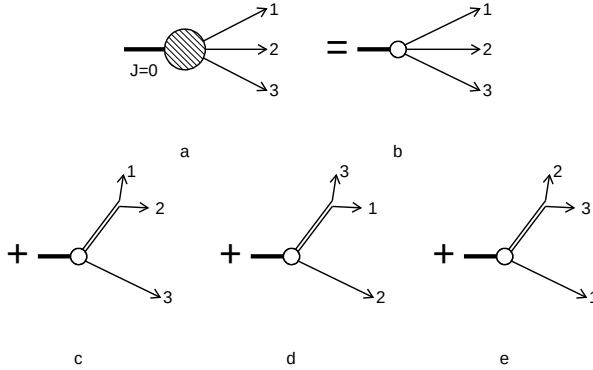


Fig. 6.4 Three-particle production in the isobar model: a) amplitude $A_{P_1 P_2 P_3}^{J=0}$ written as a non-resonance term b) and c,d,e) terms with the production of resonances in different channels.

Finally, let us discuss the s -dependence. If the reaction is analysed in a broad interval of initial energies, the energy dependence of the initial-state vertices G_a , G_b , G_c must be taken into consideration. Moreover, if the resonances appear in the direct channel, the corresponding pole terms in the initial-state vertices should be taken into account. For example:

$$G_a(s, s_{23}) = \sum_{\text{in}} \frac{G_a^{(\text{in})}(s) G_a^{(\text{out})}(s, s_{23})}{M_{\text{in}}^2 - s - i\Gamma_{\text{in}}(s) M_{\text{in}}} + F_{\text{smooth term}}^{(a)}(s, s_{23}). \quad (6.3)$$

The pole term vertex, $G_a^{(\text{out})}(s, s_{23})$, as well as the non-resonance term $F_{\text{smooth term}}^{(a)}(s, s_{23})$, may be complex-valued, provided there are certain intermediate states.

Concerning the widths $\Gamma_a(s_{23})$, $\Gamma_b(s_{13})$, $\Gamma_c(s_{12})$ and $\Gamma_{\text{in}}(s)$, one may raise different hypotheses depending on what resonances we are dealing with. The simplest assumption is that the width is energy-independent; in this case we work with the standard Breit–Wigner resonance. If we want to take account of threshold singularities in resonances, the phase volume of the decaying systems, $\Gamma(s) \rightarrow \rho(s)g^2$, should also be written (we have discussed these points in Chapter 3).

(ii) Production of non-zero spin resonances.

We consider here the case when the initial state has a spin $J = 0$ but the produced resonances have non-zero spins: $j_\ell \neq 0$. Let us explain modifications in this case using, as previously, the last term in (6.2), which corresponds to the process of Fig. 6.4e. At $j_\ell \neq 0$ we should replace

$$\begin{aligned} & \sum_a \frac{G_a(s, s_{23}) g_a(s_{23})}{M_a^2 - s_{23} - i\Gamma_a(s_{23})M_c} \\ & \rightarrow \sum_{a, j_a} \frac{G_a(s, s_{23}) g_a(s_{23})}{M_a^2 - s_{23} - i\Gamma_a(s_{23})M_c} X_{\mu_1 \dots \mu_{j_a}}^{(j_a)}(k_{23}^\perp) X_{\mu_1 \dots \mu_{j_a}}^{(j_a)}(k_1^\perp), \quad (6.4) \end{aligned}$$

where $k_{23} = (k_2 - k_3)/2$, $p_{23} = k_2 + k_3$ and $P = k_1 + k_2 + k_3$. Similar modifications should be carried out in the other terms of the right-hand side of (6.2).

If we consider the annihilation process $p\bar{p}(J^P = 0^-) \rightarrow P_1 P_2 P_3$ then the spin factor in the right-hand side of (6.4) should contain the corresponding spin-dependent term $(\bar{\psi}(-p_2) i\gamma_5 \psi(p_1))$.

The moment-operator expansion used above was applied in analyses of the meson spectra in a number of papers [9, 20, 21, 22]; the results were summarised in [23]. It would be instructive to compare it with procedures suggested in other approaches.

The moment-operator technique [23] is sometimes misleadingly referred as the Zemach expansion method [24]. Comparing the operator $X_{\mu_1 \dots \mu_j}^{(j)}(k^\perp)$ with the corresponding formulae of [24] which use the three-dimensional momentum, $\mathbf{k}_{cm}$, in the c.m. frame of the considered particles, one can see which features are common and which are different in the two approaches. For the operator $X_{\mu_1 \dots \mu_j}^{(j)}(k^\perp)$ written in the c.m. frame ($\mathbf{p} = 0$) the expressions used in the two approaches coincide: at $\mathbf{p} = 0$ the four-momentum $k_\mu^\perp$ has space-like components because $(k^\perp p) = 0$. So in this case the operator $X_{\mu_1 \dots \mu_j}^{(j)}(k^\perp)$, possessing space-like components only, turns into Zemach's operator. However, for the amplitude (6.4) the operators $X_{\mu_1 \dots \mu_{j_a}}^{(j_a)}(k_{23}^\perp)$ and $X_{\mu_1 \dots \mu_{j_a}}^{(j_a)}(k_1^\perp)$ with zero components ($\mu_\ell = 0$) cannot be zero simultaneously. In [24] a special procedure was suggested for such cases, namely: the operator is treated in its own centre-of-mass frame, then a subsequent Lorentz boost transfers it to a relevant frame. But in the method developed in [23] these additional manipulations are unnecessary.

The Lorentz boost should be carried out also upon the three-particle production amplitude considered in terms of spherical wave functions as

well as in the version suggested by [25].

6.1.1.2 The $p\bar{p}(J^P) \longrightarrow P_1 P_2 P_3$ transition for an arbitrary spin state

To be definite, we consider here the reaction $p\bar{p}(J^P) \rightarrow P_1 P_2 P_3$ with $J \geq 0$. As previously, we write down the spin operators for the process of Fig. 6.4e. The bispinor in the initial state of the reaction $p\bar{p}(J^P) \rightarrow P_1 P_2 P_3$ is determined (see Chapter 4) as

$$\left(\bar{\psi}(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{p\bar{p}(SL_{in}J)} (p_1^{\perp P}) \psi(p_1) \right). \quad (6.5)$$

For the final state resonance with spin j_a and angular momentum L of the system $Resonance(j_a) + P_1$, we have for the outgoing mesons

$$|j_a - L| \leq J \leq j_a + L, \quad P = (-1)^{j_a + L + 1}. \quad (6.6)$$

The final state operator is given by the convolution of the final state factors:

$$X_{\nu_1 \dots \nu_{j_a}}^{(j_a)} (k_{23}^{\perp p_{23}}) \otimes X_{\nu'_1 \dots \nu'_L}^{(L)} (k_1^{\perp P}). \quad (6.7)$$

As a result, the convolutions of spin operators for different total momenta $J = j_a + L, j_a + L - 1, j_a + L - 2, \dots$ of the process Fig. 6.4e are as follows:

$$\begin{aligned} J = j_a + L : \\ S_{J=j_a+L}^{(SL_{in}J; j_a L)}(23, 1) &= \left(\bar{\psi}(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{p\bar{p}(SL_{in}J)} (p_1^{\perp P}) \psi(p_1) \right) X_{\mu_1 \dots \mu_{j_a}}^{(j_a)} (k_{23}^{\perp p_{23}}) \\ &\quad \times X_{\mu_{j_a+1} \dots \mu_J}^{(L)} (k_1^{\perp P}), \\ J = j_a + L - 1 : \\ S_{J=j_a+L-1}^{(SL_{in}J; j_a L)}(23, 1) &= \left(\bar{\psi}(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{p\bar{p}(SL_{in}J)} (p_1^{\perp P}) \psi(p_1) \right) X_{\mu_1 \dots \mu_{j_a-1} \nu'}^{(j_a)} (k_{23}^{\perp p_{23}}) \\ &\quad \times X_{\mu_{j_a} \dots \mu_{J-1} \nu''}^{(L)} (k_1^{\perp P}) \varepsilon_{P \nu' \nu'' \mu_J}, \\ J = j_a + L - 2 : \\ S_{J=j_a+L-2}^{(SL_{in}J; j_a L)}(23, 1) &= \left(\bar{\psi}(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{p\bar{p}(SL_{in}J)} (p_1^{\perp P}) \psi(p_1) \right) X_{\mu_1 \dots \mu_{j_a-1} \nu'}^{(j_a)} (k_{23}^{\perp p_{23}}) \\ &\quad \times X_{\mu_{j_a} \dots \mu_J \nu''}^{(L)} (k_1^{\perp P}) g_{\nu' \nu''}, \\ J = j_a + L - 3 : \\ S_{J=j_a+L-3}^{(SL_{in}J; j_a L)}(23, 1) &= \left(\bar{\psi}(-p_2) \hat{Q}_{\mu_1 \dots \mu_J}^{p\bar{p}(SL_{in}J)} (p_1^{\perp P}) \psi(p_1) \right) X_{\mu_1 \dots \mu_{j_a-2} \nu'_1 \nu'_2}^{(j_a)} (k_{23}^{\perp p_{23}}) \\ &\quad \times X_{\mu_{j_a-1} \dots \mu_{J-1} \nu'_1 \nu'_2}^{(L)} (k_1^{\perp P}) g_{\nu'_1 \nu'_2} \varepsilon_{P \nu'_2 \nu'_2 \mu_J}, \end{aligned} \quad (6.8)$$

and so on.

The amplitude for the process shown in Fig. 6.4e is determined at fixed j_a and L by the sum of the following terms:

$$\sum_n S_{J=j_a+L-n}^{(SL_{in}J; j_a L)}(23, 1) A_{J=j_a+L-n}^{(SL_{in}J; j_a L)}(s, s_{23}). \quad (6.9)$$

The amplitudes $A_{J=j_a+L-n}^{(SL_{in}J; j_a L)}(s, s_{23})$ may contain resonances both in the s_{23} -channel (with spin j_a) and in the s -channel (with spin J).

Likewise, we write spin factors and amplitudes for other processes in the right-hand side of Fig. 6.4.

An isobar model of the type considered above has been applied to the analysis of $p\bar{p}$ -annihilation in flight, see [26].

6.1.2 Dispersion integral equation for a three-body system

By now we have a lot of information (millions of events) about the reactions $K \rightarrow \pi\pi\pi$ and $\eta \rightarrow \pi\pi\pi$; LEAR (CERN) accumulated high statistics data on three-meson production from the $\bar{p}p$ annihilation at rest, mainly from ($J^{PC} = 0^{-+}$)-level. The data of the Crystal Barrel Collaboration (LEAR) were successfully analysed (see, for example, [9, 27, 28]) with the aim to search for new meson resonances in the region 1000–1600 MeV.

In this section the dispersion relation N/D -method is presented for a three-body system: the method allows one to take into account final-state two-meson interactions. We consider in detail an illustrative example: the decay of the 0^{-+} -state into three different pseudoscalar mesons.

The first steps in accounting for all two-body final state interactions were made in [29] in a non-relativistic approach for three-nucleon systems. In [30] the two-body interactions were considered in the potential approach (the Faddeev equation).

The relativistic dispersion relation technique was used for the investigation of the final state interaction effects in [15].

A relativistic dispersion relation equation for the amplitude $\eta \rightarrow \pi\pi\pi$ was written in [18]. Later on the method was generalised [19] for the coupled processes $p\bar{p}(\text{at rest}) \rightarrow \pi\pi\pi, \eta\eta\pi, K\bar{K}\pi$: this way a system of coupled equations for decay amplitudes was written. Following [18, 19], we explain here the main points in considering the dispersion relations for a three-particle system. The account of the three-particle final state interactions imposes correct unitarity and analyticity constraints on the amplitude.

6.1.2.1 Two-particle interactions in the 0^- -state $\longrightarrow P_1 P_2 P_3$ decay

As previously, we consider the decay of a pseudoscalar particle ($J_{in}^P = 0^-$) with the mass M and momentum P into three pseudoscalar particles with masses m_1, m_2, m_3 and momenta k_1, k_2, k_3 . There are different contributions to this decay process: those without final state particle interactions (prompt decay, Fig. 6.5a) and decays with subsequent final state interactions (an example is shown in Fig. 6.5b).

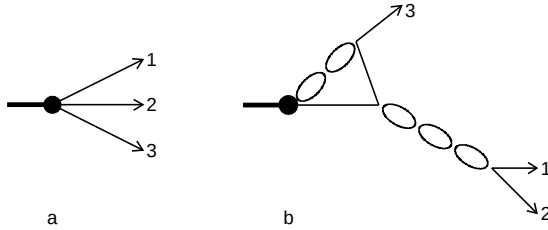


Fig. 6.5 Different types of transitions ($J_{in}^P = 0^-$)-state $\longrightarrow P_1 P_2 P_3$: a) prompt decay, b) decay with subsequent final state interactions.

For the decay amplitude we consider here an equation which takes into account two-particle final state interactions, such as that shown in Fig. 6.5b. First, we consider in detail the S -wave interactions. This case clarifies the main points of the dispersion relation approach for the three-particle interaction amplitude. Then we discuss a scheme for generalising the equations for the case of higher waves.

(i) S -wave interaction.

Let us begin with the S -wave two-particle interactions. The decay amplitude is given by

$$A_{P_1 P_2 P_3}^{(J_{in}=0)}(s_{12}, s_{13}, s_{23}) = \lambda(s_{12}, s_{13}, s_{23}) + A_{12}^{(0)}(s_{12}) + A_{13}^{(0)}(s_{13}) + A_{23}^{(0)}(s_{23}). \quad (6.10)$$

Different terms in (6.10) are illustrated by Fig. 6.6: we have a prompt production amplitude, Fig. 6.6b, and terms $A_{ij}^{(0)}(s_{ij})$ with particles $P_1 P_2$ (Fig. 6.6c), $P_1 P_3$ (Fig. 6.6d) and $P_2 P_3$ (Fig. 6.6e) participating in final state interactions.

To take into account rescatterings of the type shown in Fig. 6.5b, we can write equations for different terms $A_{ij}^{(0)}(s_{ij})$.

The two-particle unitarity condition is explored to derive the integral equation for the amplitude $A_{ij}^{(0)}(s_{ij})$. The idea of the approach suggested

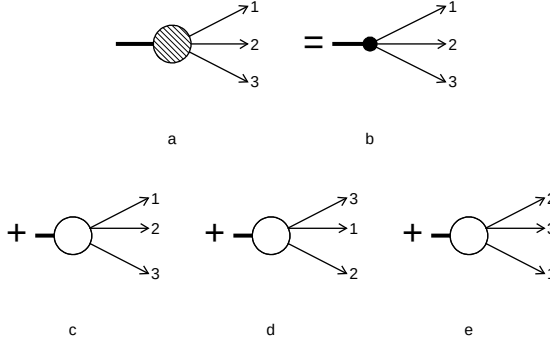


Fig. 6.6 Different terms in the amplitude $A_{P_1 P_2 P_3}^{(J_{in}=0)}(s_{12}, s_{13}, s_{23})$.

in [14] is that one should consider the case of a small external mass $M < m_1 + m_2 + m_3$. A standard spectral integral equation (or a dispersion relation equation) is written in this case for the transitions $h_{in} P_\ell \rightarrow P_i P_j$. Then the analytical continuation is performed over the mass M back to the decay region: this gives a system of equations for decay amplitudes $A_{ij}^{(0)}(s_{ij})$.

So, let us consider the channel of particles 1 and 2, the transition $h_{in} P_3 \rightarrow P_1 P_2$. We write the two-particle unitarity condition for the scattering in this channel with the assumption $(M + m_3) \sim (m_1 + m_2)$.

The discontinuity of the amplitude in the s_{12} -channel equals

$$\begin{aligned} disc_{12} A_{P_1 P_2 P_3}^{J_{in}=0}(s_{12}, s_{13}, s_{23}) &= disc_{12} A_{12}^{(0)}(s_{12}) \\ &= \int d\Phi_{12}(p_{12}; k_1, k_2) \left(\lambda(s_{12}, s_{13}, s_{23}) + A_{12}^{(0)}(s_{12}) + A_{13}^{(0)}(s_{13}) + A_{23}^{(0)}(s_{23}) \right) \\ &\quad \times \left(A_{12 \rightarrow 12}^{(0)}(s_{12}) \right)^* . \end{aligned} \quad (6.11)$$

Here $d\Phi_{12}(p_{12}; k_1, k_2) = (1/2)(2\pi)^{-2} \delta^4(p_{12} - k_1 - k_2) d^4 k_1 d^4 k_2 \delta(m_1^2 - k_1^2) \delta(m_2^2 - k_2^2)$ is the standard phase volume of particles 1 and 2. In (6.11), we should take into account that only $A_{12}^{(0)}(s_{12})$ has a non-zero discontinuity in the channel 12.

But first, let us consider the S -wave two-particle scattering amplitude $A_{P_1 P_2 \rightarrow P_1 P_2}^{(0)}$. It can be written in the dispersion N/D approach with sepa-

rable interaction (see Chapter 3) as a series

$$A_{P_1 P_2 \rightarrow P_1 P_2}^{(0)}(s) = G_0^L(s_{12})G_0^R(s_{12}) + G_0^L(s_{12})B_{12}^{(0)}(s_{12})G_0^R(s_{12}) \quad (6.12) \\ + G_0^L(s_{12})B_{12}^{(0)2}(s_{12})G_0^R(s_{12}) + \dots = \frac{G_0^L(s_{12})G_0^R(s_{12})}{1 - B_{12}^{(0)}(s_{12})},$$

where $G_0^L(s_{12})$ and $G_0^R(s_{12})$ are left and right vertex functions. The loop diagram $B_{12}^{(0)}(s_{12})$ in the dispersion relation representation reads:

$$B_{12}^{(0)}(s_{12}) = \int_{(m_1+m_2)^2}^{\infty} \frac{ds'_{12}}{\pi} \frac{G_0^L(s'_{12})\rho_{12}^{(0)}(s'_{12})G_0^R(s'_{12})}{s'_{12} - s_{12} - i0}, \quad (6.13)$$

where $\rho_{12}^{(0)}(s_{12}) = \sqrt{[s_{12} - (m_1 + m_2)^2][s_{12} - (m_1 - m_2)^2]} / (16\pi s_{12})$ is the two-particle S -wave phase space integrated over the angular variables. The vertex functions contain left-hand singularities related to the t -channel exchange diagrams, while the loop diagram $B_{12}^{(0)}(s_{12})$ has a singularity due to the elastic scattering (the right-hand side singularity). The consideration of the scattering amplitude $A_{P_1 P_2 \rightarrow P_1 P_2}^{(0)}(s_{12})$ does not specify it whether both vertices, $G_0^L(s_{12})$ and $G_0^R(s_{12})$, have left-hand singularities or only one of them (see discussion in Chapter 3). Considering the three-body decay, it is convenient to make use of this freedom. On the first sheet of the decay amplitude, we take into account the threshold singularities at $s_{ij} = (m_i + m_j)^2$, which are associated with the elastic scattering in the subchannel of particles i and j but not those on the left-hand side. This means that the vertex $G_0^R(s_{12})$ should be chosen here as an analytical function. For the sake of simplicity let us put $G_0^R(s_{12}) = 1$ and present the amplitude $P_1 P_2 \rightarrow P_1 P_2$ as

$$A_{P_1 P_2 \rightarrow P_1 P_2}^{(0)}(s_{12}) = G_0^L(s_{12}) \frac{1}{1 - B_{12}^{(0)}(s_{12})} \quad \text{at} \quad G_0^R(s_{12}) = 1. \quad (6.14)$$

Exploring (6.11), let us now return to the equation for the decay amplitude $h_{in} \rightarrow P_1 P_2 P_3$.

As was noted in Chapter 3 (see also [14]), the full set of rescatterings of particles 1 and 2 gives us the factor $(1 - B_0(s_{12}))^{-1}$, so we have from (6.11):

$$A_{12}^{(0)}(s_{12}) = B_{in}^{(0)}(s_{12}) \frac{1}{1 - B_0(s_{12})}. \quad (6.15)$$

The first loop diagram $B_{in}^{(0)}(s_{12})$ is determined as

$$B_{in}^{(0)}(s_{12}) = \int_{(m_1+m_2)^2}^{\infty} \frac{ds'_{12}}{\pi} \frac{disc_{12} B_{in}^{(0)}(s'_{12})}{s'_{12} - s_{12} - i0}, \quad (6.16)$$

where

$$\begin{aligned} & disc_{12} B_{in}^{(0)}(s_{12}) \\ &= \int d\Phi_{12}(p_{12}; k_1, k_2) \left(\lambda(s_{12}, s_{13}, s_{23}) + A_{13}^{(0)}(s_{13}) + A_{23}^{(0)}(s_{23}) \right) G_0^L(s_{12}) \\ &\equiv disc_{12} B_{\lambda-12}^{(0)}(s_{12}) + disc_{12} B_{13-12}^{(0)}(s_{12}) + disc_{12} B_{23-12}^{(0)}(s_{12}). \end{aligned} \quad (6.17)$$

Here we present $disc_{12} B_{in}^{(0)}(s_{12})$ as a sum of three terms because each of them needs a special treatment when $M^2 + i\varepsilon$ is increasing.

It is convenient to perform the phase-space integration in equation (6.17) in the centre-of-mass system of particles 1 and 2 where $\mathbf{k}_1 + \mathbf{k}_2 = 0$. In this frame

$$\begin{aligned} s_{13} &= m_1^2 + m_3^2 + 2k_{10}k_{30} - 2z |\mathbf{k}_1| |\mathbf{k}_3|, \\ s_{23} &= m_2^2 + m_3^2 + 2k_{20}k_{30} + 2z |\mathbf{k}_2| |\mathbf{k}_3|, \end{aligned} \quad (6.18)$$

where $z = \cos \theta_{13}$ and

$$k_{10} = \frac{s_{12} + m_1^2 - m_2^2}{2\sqrt{s_{12}}}, \quad k_{20} = \frac{s_{12} + m_2^2 - m_1^2}{2\sqrt{s_{12}}}, \quad -k_{30} = \frac{s_{12} + m_3^2 - M^2}{2\sqrt{s_{12}}}. \quad (6.19)$$

The minus sign in front of k_{30} reflects the fact that P_3 is an outgoing, not an incoming particle. As usually, $|\mathbf{k}_j| = \sqrt{k_{j0}^2 - m_j^2}$ for $j = 1, 2, 3$, so

$$\begin{aligned} |\mathbf{k}_1| &= |\mathbf{k}_2| = \frac{1}{2\sqrt{s_{12}}} \sqrt{[s_{12} - (m_1 + m_2)^2][s_{12} - (m_1 - m_2)^2]}, \\ |\mathbf{k}_3| &= \frac{1}{2\sqrt{s_{12}}} \sqrt{[M^2 - (\sqrt{s_{12}} + m_3)^2][M^2 - (\sqrt{s_{12}} - m_3)^2]}. \end{aligned} \quad (6.20)$$

In the calculation of $disc_{12} B_0^{(in)}(s_{12})$ all integrations are carried out easily except for the contour integral over dz . It can be rewritten in (6.17) as an integral over ds_{13} or ds_{23} :

$$\int_{-1}^{+1} \frac{dz}{2} \rightarrow \int_{s_{13}(-)}^{s_{13}(+)} \frac{ds_{13}}{4 |\mathbf{k}_1| |\mathbf{k}_3|}, \quad \text{or} \quad \int_{-1}^{+1} \frac{dz}{2} \rightarrow \int_{s_{23}(-)}^{s_{23}(+)} \frac{ds_{23}}{4 |\mathbf{k}_2| |\mathbf{k}_3|}, \quad (6.21)$$

where

$$\begin{aligned} s_{13}(\pm) &= m_1^2 + m_3^2 + 2k_{10}k_{30} \pm 2 \mid \mathbf{k}_1 \mid \mid \mathbf{k}_3 \mid , \\ s_{23}(\pm) &= m_2^2 + m_3^2 + 2k_{20}k_{30} \pm 2 \mid \mathbf{k}_2 \mid \mid \mathbf{k}_3 \mid . \end{aligned} \quad (6.22)$$

The relative location of the integration contours (6.21) and amplitude singularities is the determining point for writing the equation.

Below we use the notation

$$\int_{s_{13}(-)}^{s_{13}(+)} ds_{13} = \int_{C_{13}(s_{12})} ds_{13}. \quad (6.23)$$

One can see from (6.22) that the integration contours $C_{13}(s_{12})$ and $C_{23}(s_{12})$ depend on M^2 and s_{12} , so we should monitor them when $M^2 + i\varepsilon$ increases.

Let us underline again that the idea to consider the decay processes in the dispersion relation approach is the following : we write the equation in the region of the standard scattering *two particles* $\rightarrow$ *two particles* (when $m_1 \sim m_2 \sim m_3 \sim M$) with the subsequent analytical continuation (with $M^2 + i\varepsilon$ at $\varepsilon > 0$) into the decay region, $M > m_1 + m_2 + m_3$, and then $\varepsilon \rightarrow +0$. In this continuation we need to specify what type of singularities (and corresponding type of processes) we take into account and what type of singularities we neglect. Definitely, we take into account right-hand side and left-hand side singularities of the scattering processes $P_i P_j \rightarrow P_i P_j$ (our main aim is to restore the rescattering processes correctly). But singularities of the prompt production amplitude are beyond the field of our interest. In other words, we suppose $\lambda(s_{12}, s_{13}, s_{23})$ to be an analytical function in the region under consideration.

Assuming $\lambda(s_{12}, s_{13}, s_{23})$ to be an analytical function in the region under consideration, we can easily perform analytical continuation of the integral over dz , Eq. (6.21), with $M^2 + i\varepsilon$.

Problems may appear in the integrations of $A_{13}^{(0)}(s_{13})$ and $A_{23}^{(0)}(s_{23})$ owing to the threshold singularities in the amplitudes (at $s_{13} = (m_1 + m_3)^2$ and $s_{23} = (m_2 + m_3)^2$, respectively). However, the analytical continuation over $M^2 + i\varepsilon$ resolves them: one can see in Chapter 4 (Appendix 4.G) the location of the integration contour in the complex- s_{23} plane with respect to the threshold singularity at $s_{23} = (m_2 + m_3)^2$ when $M > m_1 + m_2 + m_3$.

Let us now write the equation for the three-particle production amplitude in more detail. We denote the S -wave projection of $\lambda(s_{12}, s_{13}, s_{23})$

as

$$\langle \lambda(s_{12}, s_{13}, s_{23}) \rangle_{12}^{(0)} = \int_{-1}^{+1} \frac{dz}{2} \lambda(s_{12}, s_{13}, s_{23}), \quad (6.24)$$

and the contour integrals over the amplitudes $A_{13}^{(0)}(s_{13})$ and $A_{23}^{(0)}(s_{23})$ as

$$\langle A_{i3}^{(0)}(s_{i3}) \rangle_{12}^{(0)} = \int_{-1}^{+1} \frac{dz}{2} A_{i3}^{(0)}(s_{i3}) \equiv \int_{C_{i3}(s_{12})} \frac{ds_{i3}}{4|\mathbf{k}_i||\mathbf{k}_3|} A_{i3}^{(0)}(s_{i3}), \quad i = 1, 2. \quad (6.25)$$

Remind once more that the definition of the contours $C_i(s_{12})$ is given in (6.23) while the relative position of the contour $C_2(s_{12})$ and the threshold singularity in the s_{23} -channel is shown in Fig. 4.26. So, we rewrite (6.17) in the form

$$\begin{aligned} disc_{12} B_{in}^{(0)}(s_{12}) &= \left(\langle \lambda(s_{12}, s_{13}, s_{23}) \rangle_{12}^{(0)} + \langle A_{13}^{(0)}(s_{13}) \rangle_{12}^{(0)} + \langle A_{23}^{(0)}(s_{23}) \rangle_{12}^{(0)} \right) \\ &\times \rho_{12}^{(0)}(s_{12}) G_0^L(s_{12}), \end{aligned} \quad (6.26)$$

Equation (6.26) allows us to write the dispersion integral for the loop amplitude $B_{in}^{(0)}(s_{12})$. As a result, we have:

$$A_{12}^{(0)}(s_{12}) = \left(B_{\lambda-12}^{(0)}(s_{12}) + B_{13-12}^{(0)}(s_{12}) + B_{23-12}^{(0)}(s_{12}) \right) \frac{1}{1 - B_{12}^{(0)}(s_{12})} \quad (6.27)$$

where

$$\begin{aligned} B_{\lambda-12}^{(0)}(s_{12}) &= \int_{(m_1+m_2)^2}^{\infty} \frac{ds'_{12}}{\pi} \langle \lambda(s'_{12}, s'_{13}, s'_{23}) \rangle_{12}^{(0)} \frac{\rho_{12}^{(0)}(s'_{12})}{s'_{12} - s_{12} - i0} G_0^L(s'_{12}), \\ B_{i3-12}^{(0)}(s_{12}) &= \int_{(m_1+m_2)^2}^{\infty} \frac{ds'_{12}}{\pi} \langle A_{i3}^{(0)}(s'_{i3}) \rangle_{12}^{(0)} \frac{\rho_{12}^{(0)}(s'_{12})}{s'_{12} - s_{12} - i0} G_0^L(s'_{12}). \end{aligned} \quad (6.28)$$

Let us emphasise that in the integrand (6.28) the energy squared is s'_{12} and hence, calculating $\langle \lambda(s'_{12}, s'_{13}, s'_{23}) \rangle_{12}^{(0)}$ and $\langle A_{i3}^{(0)}(s'_{i3}) \rangle_{12}^{(0)}$, we should use Eqs. (6.18) – (6.23) with the replacement $s_{12} \rightarrow s'_{12}$.

The equation (6.27) is illustrated by Fig. 6.7.

In the same way we can write equations for $A_{13}^{(0)}(s_{13})$ and $A_{23}^{(0)}(s_{23})$. We have a system of three non-homogeneous equations which determine the amplitudes $A_{ij}^{(0)}(s_{ij})$ when $\lambda(s_{12}, s_{13}, s_{23})$ is considered as an input function.

Note that the integration contour $C_i(s_{12})$ in (6.25), see also [14], does not coincide with that of [39] where the corresponding problem was treated starting from the consideration of the three-body channel.

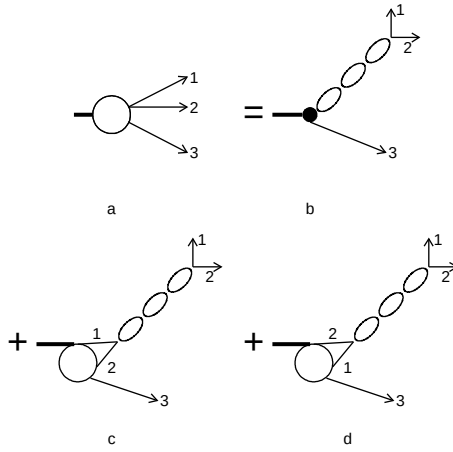


Fig. 6.7 Diagrammatic presentation of Eq. (6.27).

(ii) Final state rescatterings $P_i P_j \rightarrow P_i P_j$ in the $L > 0$ state.

Equations for amplitudes which describe the final state interactions in the transition $(J_{in}^P = 0^-)$ -state $\rightarrow P_1 P_2 P_3$ when rescatterings $P_i P_j \rightarrow P_i P_j$ occur in a state with $L > 0$ can be written in a way analogous to that presented above for $L = 0$. So, we suppose that $P_i P_j \rightarrow P_i P_j$ rescatterings take place in a state with definite orbital momentum L and $L \neq 0$.

The amplitude for the decay $(J_{in}^P = 0^-)$ -state $\rightarrow P_1 P_2 P_3$ (below $L = J$) reads:

$$\begin{aligned}
 A_{P_1 P_2 P_3}^{(J_{in}=0)}(s_{12}, s_{13}, s_{23}) &= \lambda(s_{12}, s_{13}, s_{23}) \\
 &+ A_{12}^{(J)}(s_{12}) X_{\mu_1 \dots \mu_J}^{(J)}(k_3^\perp P) X_{\mu_1 \dots \mu_J}^{(J)}(k_{12}^\perp p_{12}) \\
 &+ A_{13}^{(J)}(s_{13}) X_{\mu_1 \dots \mu_J}^{(J)}(k_2^\perp P) X_{\mu_1 \dots \mu_J}^{(J)}(k_{13}^\perp p_{13}) \\
 &+ A_{23}^{(J)}(s_{23}) X_{\mu_1 \dots \mu_J}^{(J)}(k_1^\perp P) X_{\mu_1 \dots \mu_J}^{(J)}(k_{23}^\perp p_{23}). \quad (6.29)
 \end{aligned}$$

Convolutions of the momentum operators, such as $X_{\mu_1 \dots \mu_J}^{(J)}(k_3^\perp P) X_{\mu_1 \dots \mu_J}^{(J)}(k_{12}^\perp p_{12})$, being functions of s_{ij} do not contain threshold singularities. So we can rewrite (6.29) in a more compact form

$$\begin{aligned}
 A_{P_1 P_2 P_3}^{(J_{in}=0)}(s_{12}, s_{13}, s_{23}) &= \lambda(s_{12}, s_{13}, s_{23}) + A_{12}^{(J-J)}(s_{12}, s_{13}, s_{23}) \\
 &+ A_{13}^{(J-J)}(s_{12}, s_{13}, s_{23}) + A_{23}^{(J-J)}(s_{12}, s_{13}, s_{23}), \quad (6.30)
 \end{aligned}$$

where $A_{12}^{(J-J)}(s_{12}, s_{13}, s_{23}) = A_{12}^{(J)}(s_{12}) X_{\mu_1 \dots \mu_J}^{(J)}(k_3^\perp P) X_{\mu_1 \dots \mu_J}^{(J)}(k_{12}^\perp p_{12})$, and so on. As previously, $\lambda(s_{12}, s_{13}, s_{23})$ is an analytical function of s_{ij} while

the terms $A_{ij}^{(J-J)}(s_{12}, s_{13}, s_{23})$ have threshold singularities of the type $\sqrt{s_{ij} - (m_i + m_j)^2}$ due to final state rescatterings $P_i P_j \rightarrow P_i P_j$.

First, we should introduce a two-particle ($L = J$)-wave scattering amplitude. To be definite, we consider $P_1 P_2 \rightarrow P_1 P_2$. We write the block of a one-fold scattering as

$$A_{(P_1 P_2 \rightarrow P_1 P_2)_{\text{one-fold}}}^{(J)}(s_{12}) = X_{\nu_1 \dots \nu_J}^{(J)}(k_{12}'^{\perp p_{12}}) O_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J}(\perp p_{12}) G_J^L(s_{12}) G_J^R(s_{12}) \\ \times X_{\mu_1 \dots \mu_J}^{(J)}(k_{12}^{\perp p_{12}}). \quad (6.31)$$

The two-fold scattering amplitude reads:

$$A_{(P_1 P_2 \rightarrow P_1 P_2)_{\text{two-fold}}}^{(J)}(s_{12}) = X_{\nu_1 \dots \nu_J}^{(J)}(k_{12}'^{\perp p_{12}}) O_{\nu_1' \dots \nu_J'}^{\nu_1 \dots \nu_J}(\perp p_{12}) G_J^L(s_{12}) \\ \times \left[\int_{(m_1 + m_2)^2}^{\infty} \frac{ds_{12}''}{\pi(s_{12}'' - s_{12} - i0)} G_J^R(s_{12}'') \right. \\ \times \int X_{\nu_1' \dots \nu_J'}^{(J)}(k_{12}''^{\perp p_{12}'}) d\Phi_{12}(p_{12}''; k_1'', k_2'') X_{\nu_1'' \dots \nu_J''}^{(J)}(k_{12}''^{\perp p_{12}'}) G_J^L(s_{12}'') \left. \right] \\ \times G_J^R(s_{12}) O_{\mu_1 \dots \mu_J}^{\nu_1' \dots \nu_J'}(\perp p_{12}) G_J^L(s_{12}) X_{\mu_1 \dots \mu_J}^{(J)}(k_{12}^{\perp p_{12}}). \quad (6.32)$$

In the integrand we can replace $k_{12}''^{\perp p_{12}'} \rightarrow k_{12}''^{\perp p_{12}}$, because in the c.m. frame of particles $P_1 P_2$ one has $p_{12}'' = (\sqrt{s_{12}''}, 0, 0, 0)$ and $p_{12} = (\sqrt{s_{12}}, 0, 0, 0)$. The integration over the phase space gives

$$\int X_{\nu_1' \dots \nu_J'}^{(J)}(k_{12}''^{\perp p_{12}'}) d\Phi_{12}(p_{12}''; k_1'', k_2'') X_{\nu_1'' \dots \nu_J''}^{(J)}(k_{12}''^{\perp p_{12}'}) = O_{\nu_1' \dots \nu_J'}^{\nu_1' \dots \nu_J'}(\perp p_{12}) \\ \times \rho_{12}^{(J)}(s_{12}''). \quad (6.33)$$

Using

$$O_{\nu_1' \dots \nu_J'}^{\nu_1 \dots \nu_J}(\perp p_{12}) O_{\nu_1' \dots \nu_J'}^{\nu_1' \dots \nu_J'}(\perp p_{12}) O_{\mu_1 \dots \mu_J}^{\nu_1' \dots \nu_J'}(\perp p_{12}) = O_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J}(\perp p_{12}), \quad (6.34)$$

we write the two-fold amplitude as follows:

$$A_{(P_1 P_2 \rightarrow P_1 P_2)_{\text{two-fold}}}^{(J)}(s_{12}) = \\ X_{\nu_1 \dots \nu_J}^{(J)}(k_{12}'^{\perp p_{12}}) O_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J}(\perp p_{12}) G_J^L(s_{12}) B_{12}^{(J)}(s_{12}) G_J^R(s_{12}) X_{\mu_1 \dots \mu_J}^{(J)}(k_{12}^{\perp p_{12}}), \quad (6.35)$$

where

$$B_{12}^{(J)}(s_{12}) = \int_{(m_1 + m_2)^2}^{\infty} \frac{ds_{12}''}{\pi(s_{12}'' - s_{12} - i0)} G_J^R(s_{12}'') \rho_{12}^{(J)}(s_{12}'') G_J^L(s_{12}'') \quad (6.36)$$

is the loop diagram.

The full set of rescatterings gives:

$$A_{P_1 P_2 \rightarrow P_1 P_2}^{(J)}(s_{12}) = X_{\nu_1 \dots \nu_J}^{(J)}(k_{12}^{\perp p_{12}}) O_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J}(\perp p_{12}) \\ \times G_J^L(s_{12}) \frac{1}{1 - B_{12}^{(J)}(s_{12})} G_J^R(s_{12}) X_{\mu_1 \dots \mu_J}^{(J)}(k_{12}^{\perp p_{12}}). \quad (6.37)$$

As previously for $J = 0$, we put

$$G_J^R(s_{12}) = 1. \quad (6.38)$$

Finally we write:

$$A_{P_1 P_2 \rightarrow P_1 P_2}^{(J)}(s_{12}) = X_{\mu_1 \dots \mu_J}^{(J)}(k_{12}^{\perp p_{12}}) G_J^L(s_{12}) \frac{1}{1 - B_{12}^{(J)}(s_{12})} X_{\mu_1 \dots \mu_J}^{(J)}(k_{12}^{\perp p_{12}}). \quad (6.39)$$

Let us return now to the equation for the three-particle production amplitude $A_{P_1 P_2 P_3}^{(J_{in}=0)}(s_{12}, s_{13}, s_{23})$ given by (6.30). We write equations for separated terms $A_{ij}^{(J-J)}(s_{12}, s_{13}, s_{23})$. To use Eqs. (6.31)–(6.39) directly, we consider the term with the final state interaction in the channel 12, namely, $A_{12}^{(J-J)}(s_{12}, s_{13}, s_{23})$.

The amplitude $A_{12}^{(J-J)}(s_{12}, s_{13}, s_{23})$ is determined by three terms shown in Fig. 6.7b,c,d.

The term initiated by the prompt production block $\lambda(s_{12}, s_{13}, s_{23})$ is a set of loop diagrams of the type of that in Fig. 6.7b. Therefore this term reads:

$$X_{\mu_1 \dots \mu_J}^{(J)}(k_3^{\perp p_{12}}) B_{\lambda-12}^{(J)}(s_{12}) \frac{1}{1 - B_{12}^{(J)}(s_{12})} X_{\mu_1 \dots \mu_J}^{(J)}(k_{12}^{\perp p_{12}}), \quad (6.40)$$

with

$$B_{\lambda-12}^{(J)}(s_{12}) = \int_{(m_1+m_2)^2}^{\infty} \frac{ds'_{12}}{\pi} \langle \lambda(s'_{12}, s'_{13}, s'_{23}) \rangle_{12}^{(J)} \frac{\rho_{12}^{(J)}(s'_{12})}{s'_{12} - s_{12} - i0} G_0^L(s'_{12}). \quad (6.41)$$

Let us explain Eqs. (6.40), (6.41) in more detail. Similarly to (6.31), we write for the first loop diagram in (6.40) the following representation:

$$X_{\mu_1 \dots \mu_J}^{(J)}(k_3^{\perp p_{12}}) B_{\lambda-12}^{(J)}(s_{12}) X_{\mu_1 \dots \mu_J}^{(J)}(k_{12}^{\perp p_{12}}) = X_{\nu_1 \dots \nu_J}^{(J)}(k_3^{\perp p_{12}}) O_{\nu'_1 \dots \nu'_J}^{\nu_1 \dots \nu_J}(\perp p_{12}) \\ \times \left[\int_{(m_1+m_2)^2}^{\infty} \frac{ds'_{12} \langle \lambda(s'_{12}, s'_{13}, s'_{23}) \rangle_{12}^{(J)}}{\pi(s'_{12} - s_{12} - i0)} \right. \\ \times \int X_{\nu'_1 \dots \nu'_J}^{(J)}(k_{12}^{\perp p_{12}}) d\Phi_{12}(p'_{12}; k'_1, k'_2) X_{\nu''_1 \dots \nu''_J}^{(J)}(k_{12}^{\perp p_{12}}) G_J^L(s'_{12}) \left. \right] \\ \times O_{\mu_1 \dots \mu_J}^{\nu''_1 \dots \nu''_J}(\perp p_{12}) X_{\mu_1 \dots \mu_J}^{(J)}(k_{12}^{\perp p_{12}}). \quad (6.42)$$

Recall that $\langle \lambda(s'_{12}, s'_{13}, s'_{23}) \rangle_{12}^{(J)}$ depends on s'_{12} only. Indeed, the projection $\langle \lambda(s_{12}, s_{13}, s_{23}) \rangle_{12}^{(J)}$ is determined by the following expansion of the non-singular term:

$$\lambda(s_{12}, s_{13}, s_{23}) = \sum_{J'} X_{\nu_1 \dots \nu_{J'}}^{(J')} (k_3^{\perp p_{12}}) \langle \lambda(s_{12}), s_{13}, s_{23} \rangle_{12}^{(J')} X_{\mu_1 \dots \mu_{J'}}^{(J')} (k_{12}^{\perp p_{12}}), \quad (6.43)$$

so we have

$$\begin{aligned} \langle \lambda(s_{12}, s_{13}, s_{23}) \rangle_{12}^{(J)} &= \int d\Phi_{h_{in}3}(p_{12}; P, -k_3) X_{\nu_1 \dots \nu_J}^{(J)} (k_3^{\perp p_{12}}) \lambda(s_{12}, s_{13}, s_{23}) \\ &\times X_{\mu_1 \dots \mu_{J'}}^{(J)} (k_{12}^{\perp p_{12}}) d\Phi_{12}(p_{12}; k_1, k_2) \int d\Phi_{h_{in}3}(p_{12}; P, -k_3) \left(X_{\nu'_1 \dots \nu'_J}^{(J)} (k_3^{\perp p_{12}}) \right)^2 \\ &\times \int d\Phi_{12}(p_{12}; k_1, k_2) \left(X_{\mu'_1 \dots \mu'_J}^{(J)} (k_{12}^{\perp p_{12}}) \right)^2. \end{aligned} \quad (6.44)$$

In the integrand (6.42) the energy squared is s'_{12} . Hence, we should use $\langle \lambda(s'_{12}, s'_{13}, s'_{23}) \rangle_{12}^{(0)}$ in the calculation.

Likewise, we calculate the amplitudes of processes of Fig. 6.7c,d. As a result, we have the equation:

$$\begin{aligned} A_{12}^{(J-J)}(s_{12}, s_{13}, s_{23}) &= X_{\mu_1 \dots \mu_J}^{(J)} (k_3^{\perp p_{12}}) \\ &\times \left(B_{\lambda-12}^{(J)}(s_{12}) + B_{13-12}^{(J)}(s_{12}) + B_{23-12}^{(J)}(s_{12}) \right) \frac{X_{\mu_1 \dots \mu_J}^{(J)} (k_{12}^{\perp p_{12}})}{1 - B_{12}^{(J)}(s_{12})}. \end{aligned} \quad (6.45)$$

Here

$$B_{i3-12}^{(J)}(s_{12}) = \int_{(m_1+m_2)^2}^{\infty} \frac{ds'_{12}}{\pi} \langle A_{i3}^{(J-J)}(s'_{12}, s'_{13}, s'_{23}) \rangle_{12}^{(J)} \frac{\rho_{12}^{(J)}(s'_{12})}{s'_{12} - s_{12} - i0} G_0^L(s'_{12}). \quad (6.46)$$

Let us emphasise once more that in (6.46) the terms $\langle A_{13}^{(J-J)}(s'_{12}, s'_{13}, s'_{23}) \rangle_{12}^{(J')}$ and $\langle A_{23}^{(J-J)}(s'_{12}, s'_{13}, s'_{23}) \rangle_{12}^{(J')}$ depend on s'_{12} only. This is the result of the following expansions:

$$\begin{aligned} A_{13}^{(J-J)}(s'_{12}, s'_{13}, s'_{23}) &= \sum_{J'} X_{\nu_1 \dots \nu_{J'}}^{(J')} (k_3'^{\perp p_{12}}) \langle A_{13}^{(J-J)}(s'_{12}, s'_{13}, s'_{23}) \rangle_{12}^{(J')} \\ &\times X_{\mu_1 \dots \mu_{J'}}^{(J')} (k_{12}'^{\perp p_{12}}), \\ A_{23}^{(J-J)}(s'_{12}, s'_{13}, s'_{23}) &= \sum_{J'} X_{\nu_1 \dots \nu_{J'}}^{(J')} (k_3'^{\perp p_{12}}) \langle A_{23}^{(J-J)}(s'_{12}, s'_{13}, s'_{23}) \rangle_{12}^{(J')} \\ &\times X_{\mu_1 \dots \mu_{J'}}^{(J')} (k_{12}'^{\perp p_{12}}). \end{aligned} \quad (6.47)$$

The integration over the phase space in the calculations of $A_{i3}^{(J-J)}$ ($s'_{12}, s'_{13}, s'_{23}$) is performed in a way analogous to that for $J = 0$. The contour integrals read:

$$\begin{aligned} \langle A_{i3}^{(J-J)}(s'_{12}, s'_{13}, s'_{23}) \rangle_{i3}^{(J)} &= \int_{-1}^{+1} \frac{dz'}{2} A_{i3}^{(J-J)}(s'_{12}, s'_{13}, s'_{23}) \\ &\equiv \int_{C_{i3}(s'_{12})} \frac{ds_{i3}}{4|\mathbf{k}_i||\mathbf{k}_3|} A_{i3}^{(J-J)}(s'_{12}, s'_{13}, s'_{23}), \quad i = 1, 2. \end{aligned} \quad (6.48)$$

with the definition of the contours $C_i(s_{12})$ given in (6.23).

6.1.2.2 Dispersion relation equations for a three-body system with resonance interaction in the two-particle states of the outgoing hadrons

In this section we consider the case when the outgoing particles interact due to two-particle resonances. Such a situation occurs, for example, in the reaction $p\bar{p}(\text{at rest, level } ^1S_0) \rightarrow \pi\pi\pi$: in the 0^{++} -wave of the pion-pion amplitude, there is a set of comparatively narrow resonances while a non-resonance background can be described as a broad resonance. Another possibility to introduce the background contribution in this model is to add pole (resonance) terms beyond (for example, above) the region of application of the amplitude.

In order to avoid cumbersome formulae, we consider, as before, a reaction of the type $h(^1S_0) \rightarrow P_1P_2P_3$, with the S -wave interactions of the outgoing pseudoscalars $P_iP_j \rightarrow P_iP_j$.

(i) Two-particle resonance amplitude.

We start with the dispersion representation of the two-particle amplitude for this particular case. The first resonance term (the one-fold scattering block) of the amplitude $P_iP_j \rightarrow P_iP_j$ can be written in the form

$$\sum_n \frac{g_{ij}^{(n)2}(s_{ij})}{M_n^2 - s_{ij}}, \quad (6.49)$$

where M_n is a non-physical mass of the n -resonance, and vertex $g_{ij}^{(n)}(s_{ij})$ describes its decay into two particles P_iP_j . Experimental data tell us that vertices $g_{ij}^{(n)}(s_{ij})$ can be successfully approximated by the energy dependence: $g_{ij}^{(n)}(s_{ij}) \sim \exp(-s_{ij}/\mu^2)$ with the universal slope $\mu^2 \simeq 0.5 \text{ GeV}^2$. Below we assume this universality:

$$g_{ij}^{(n)}(s_{ij}) = g_{ij}^{(n)} f(s_{ij}) \quad (6.50)$$

where $g_{ij}^{(n)}$ is a constant and $f(s_{ij})$ is a universal form factor of the type $\exp(-s_{ij}/\mu^2)$. If so, the two-fold scattering term of the amplitude contains the universal loop diagram $b(s_{ij})$:

$$A_{(P_i P_j \rightarrow P_i P_j)_{\text{two-fold}}}^{(0)}(s_{ij}) = f(s_{ij}) \sum_n \frac{g_{ij}^{(n)2}}{M_n^2 - s_{ij}} b(s_{ij}) \sum_{n'} \frac{g_{ij}^{(n')2}}{M_{n'}^2 - s_{ij}} f(s_{ij}),$$

$$b(s_{ij}) = \int_{(m_i+m_j)^2}^{\infty} \frac{ds'}{\pi} \frac{\rho_{ij}^{(0)}(s') f^2(s')}{s' - s_{ij} - i0}. \quad (6.51)$$

Summing up the terms with different numbers of loops, one obtains the following expression for the amplitude:

$$A_{(P_i P_j \rightarrow P_i P_j)}^{(0)}(s_{ij}) = \frac{f^2(s_{ij}) \sum_n \frac{g_{ij}^{(n)2}}{M_n^2 - s_{ij}}}{1 - b(s_{ij}) \sum_{n'} \frac{g_{ij}^{(n')2}}{M_{n'}^2 - s_{ij}}}. \quad (6.52)$$

Since the loop diagram has the following real and imaginary parts:

$$b(s_{ij}) = i\rho_{ij}^{(0)}(s_{ij})f^2(s_{ij}) + P \int_{(m_i+m_j)^2}^{\infty} \frac{ds'}{\pi} \frac{\rho_{ij}^{(0)}(s') f^2(s')}{s' - s_{ij}}$$

$$= i\text{Im } b(s_{ij}) + \text{Re } b(s_{ij}), \quad (6.53)$$

the scattering amplitude (6.52) can easily be rewritten in the K-matrix form for the case when an S -wave state contains several resonances.

(ii) Three particle production amplitude $h(^1S_0) \rightarrow P_1 P_2 P_3$.

The decay amplitude is given by an equation of the type of (6.10). In the term $\lambda(s_{12}, s_{13}, s_{23})$, however, we should take into account the prompt production of resonances; it is convenient to consider their widths as well. Correspondingly, we replace

$$\lambda(s_{12}, s_{13}, s_{23}) \rightarrow \Lambda_{12}(s, s_{12}) + \Lambda_{13}(s, s_{13}) + \Lambda_{23}(s, s_{23}), \quad (6.54)$$

$$\Lambda_{ij}(s, s_{ij}) = \sum_n \Lambda_{ij}^{(n)}(s, s_{ij}) \frac{g_{ij}^{(n)}}{M_n^2 - s_{ij}} f(s_{ij}) \frac{1}{1 - b(s_{ij}) \sum_{n'} \frac{g_{ij}^{(n')2}}{M_{n'}^2 - s_{ij}}},$$

and write for the full amplitude:

$$A_{h(^1S_0) \rightarrow P_1 P_2 P_3}(s_{12}, s_{13}, s_{23}) = \Lambda_{12}(s, s_{12}) + \Lambda_{13}(s, s_{13}) + \Lambda_{23}(s, s_{23})$$

$$+ A_{12}(s, s_{12}) + A_{13}(s, s_{13}) + A_{23}(s, s_{23}), \quad (6.55)$$

where the terms $A_{ij}(s, s_{ij})$ describe processes with interactions of all particles, P_1 , P_2 and P_3 . The term $A_{12}(s, s_{12})$ reads:

$$A_{12}(s, s_{12}) = \left(B_{13-12}(s, s_{12}) + B_{23-12}(s, s_{12}) \right) \times \sum_n \frac{g_{12}^{(n)2}}{M_n^2 - s_{12}} f(s_{12}) \frac{1}{1 - b(s_{12}) \sum_{n'} \frac{g_{12}^{(n')2}}{M_{n'}^2 - s_{12}}}, \quad (6.56)$$

$$B_{i3-12}(s_{12}) = \int_{(m_1+m_2)^2}^{\infty} \frac{ds'_{12}}{\pi} \langle \Lambda_{i3}(s, s'_{i3}) + A_{i3}(s, s'_{i3}) \rangle_{12}^{(0)} \frac{\rho_{12}^{(0)}(s'_{12}) f(s'_{12})}{s'_{12} - s_{12} - i0}.$$

Analogous relations for $A_{13}(s, s_{13})$ and $A_{23}(s, s_{23})$ give us three non-homogenous equations for three amplitudes thus solving in principle the problem of construction of the three-body amplitude under the constraints of analyticity and unitarity.

The equations written here require a comment. We realise the convergence of the loop diagrams with the help of cutting vertices or, what is the same, the universal form factor. The convergence of a loop diagram can be realised in other ways as well. For example, in [18, 19] a special cutting function was introduced into the integrands; one may use the subtraction procedure as it is done in [11, 14]. The technical variations are of no importance, the only essential point is that for the convergence of the considered diagrams we have to introduce additional parameters – here it is the form factor slope μ^2 , see Eq. (6.50) and the corresponding discussion.

Miniconclusion

In this section we have presented some characteristic features of the spectral integral equations for three-body systems. The technique can be used both for the determination of levels of compound systems and their wave functions (for instance, in the method of an accounting of the leading singularities [41] – this method was applied to the three-nucleon systems, H_3 and He_3 , and for determination of analytical properties of multiparticle production amplitudes when the produced resonances are studied.

We do not present here formulae for the reactions we have studied — the formulae are rather cumbersome. An example can be found, as it was mention above, in [19] where a set of equations for reactions $p\bar{p} \rightarrow \pi\pi\pi, \pi\eta\eta, \pi K\bar{K}$ was written.

The presented technique may be especially convenient for the study of low-mass singularities in multiparticle production amplitudes. The long-

lasting discussions on the sigma-meson observation, (see [42, 43, 44] and references therein) indicate that this is a problem of current interest.

6.1.3 Description of the three-meson production in the K -matrix approach

A more compact and, hence, a more convenient way for studying resonances in multiparticle processes is the K -matrix technique. Here we present elements of this technique, applying it to the reactions $p\bar{p} \rightarrow \pi\pi\pi, \pi\eta\eta, \pi K \bar{K}$. However, we have to pay a price for the simplifications the K -matrix technique gives us: we cannot take into account in a full scale the left singularities.

For a more detailed explanation we compare, first of all, the scattering amplitude $P_1 P_2 \rightarrow P_1 P_2$ written in spectral integral representation, Eq. (6.52), and that in the K -matrix approach.

6.1.3.1 Resonances in the scattering amplitude $P_1 P_2 \rightarrow P_1 P_2$: spectral integral representation and the K -matrix approach

Let us rewrite the scattering amplitude (6.52) in the K -matrix form. The spectral integral representation amplitude (6.52) looks in the K -matrix form as follows:

$$A_{(P_1 P_2 \rightarrow P_1 P_2)}^{(0)}(s_{12}) = \frac{f^2(s_{12}) \sum_n \frac{g_{12}^{(n)2}}{M_n^2 - s_{12}}}{1 - b(s_{12}) \sum_{n'} \frac{g_{12}^{(n')2}}{M_{n'}^2 - s_{12}}} = \frac{K^{(SI)}(s_{12})}{1 - i\rho_{12}^{(0)}(s_{12})K^{(SI)}(s_{12})},$$

$$K^{(SI)}(s_{12}) = \frac{f^2(s_{12}) \sum_n \frac{g_{12}^{(n)2}}{M_n^2 - s_{12}}}{1 - \text{Re } b(s_{12}) \sum_{n'} \frac{g_{12}^{(n')2}}{M_{n'}^2 - s_{12}}}. \quad (6.57)$$

Here $\text{Re } b(s_{12})$ is the real part of the loop diagram, see (6.53).

Let us now compare $K^{(SI)}(s_{12})$ with a standard representation for K -matrix elements given, for example, in Chapter 3:

$$\frac{f^2(s_{12}) \sum_n \frac{g_{12}^{(n)2}}{M_n^2 - s_{12}}}{1 - \text{Re } b(s_{12}) \sum_{n'} \frac{g_{12}^{(n')2}}{M_{n'}^2 - s_{12}}} \simeq \sum_n \frac{g_{K\text{-matrix}}^{(n)2}}{\mu_n^2 - s_{12}} + f_{K\text{-matrix}}(s_{12}) \quad \text{at } s_{12} > 0. \quad (6.58)$$

In the right-hand side of (6.58) we show a standard representation for the K -matrix element, which contains a set of poles and a smooth term $f_{K\text{-matrix}}(s_{12})$. We see a striking difference in these two representations of the K -matrix elements: the poles of the right-hand side of (6.58) are zeros of the denominator of the K -matrix element given in the left-hand side of (6.58):

$$1 - \text{Re } b(s_{12}) \sum_{n'} \frac{g_{12}^{(n')2}}{M_{n'}^2 - s_{12}} = 0, \quad (6.59)$$

and they do not coincide with the poles introduced in the spectral integrals:

$$\sum_n \frac{g_{K\text{-matrix}}^{(n)2}}{\mu_n^2 - s_{12}}. \quad (6.60)$$

The number of pole terms in (6.59) and (6.60) can also be different. Another obvious difference is the presence of the function $\text{Re } b(s_{12})$ in the left-hand side of (6.60), providing us with the analyticity of the amplitude in the right half-plane of s_{12} . It, however, makes the fitting procedure more complicated.

The simplicity of the description and the economical use of the fitting parameters are the main characteristics of the standard K -matrix technique (see the right-hand side of (6.60)) allowing us to use it in simultaneous fittings of a large number of reactions.

6.1.3.2 Three-meson production in the K -matrix approach

We apply here the K -matrix representation of the amplitude to the description of the production of resonances in the three-particle reactions. The use of the K -matrix approach to the combined analysis of the two-particle and multiparticle processes is based on the fact that the denominator of the K -matrix two-particle amplitude, $[1 - \hat{\rho}\hat{K}]^{-1}$, describes the interactions of mesons in the final state of multiparticle reactions as well.

Let us illustrate this statement using as an example the amplitude of the $p\bar{p}$ annihilation from the $^1S_0$ level: $p\bar{p}(^1S_0) \rightarrow \text{three mesons}$. In the K -matrix approach, the production amplitude for the resonance with the spin $J = 0$ in the channel $(1 + 2)$ reads:

$$A_3(s_{12})_{ca} = \sum_b \left(K_3^{(prompt)}(s_{12}) \right)_{cb} \left(\frac{1}{1 - i\hat{\rho}_{12}\hat{K}_{12}(s_{12})} \right)_{ba}, \quad (6.61)$$

where $c = p\bar{p}(^1S_0)$ and $a, b \in \pi\pi, \eta\eta, K\bar{K}$. The denominator $[1 - i\hat{\rho}_{12}\hat{K}_{12}(s_{12})]^{-1}$ depends on the invariant energy squared of mesons 1 and

2 and it coincides with the denominator of the two-particle amplitude. The factor $\widehat{K}_3^{(prompt)}(s_{12})$ stands for the prompt production of particles and resonances in this channel:

$$\left(K_3^{(prompt)}(s_{12}) \right)_{cb} = \sum_n \frac{\Lambda_c^{(n)} g_b^{(n)}}{\mu_n^2 - s_{12}} + \varphi_{cb}(s_{12}) , \quad (6.62)$$

where $\Lambda_c^{(n)}$ and φ_{cb} are the parameters of the prompt-production amplitude, and $g_b^{(n)}$ and μ_n are the same as in the two-meson scattering amplitude.

The whole amplitude for the production of the ($J = 0$)-resonances is defined by the sum of contributions from all channels:

$$A_3(s_{12}) + A_2(s_{13}) + A_1(s_{23}). \quad (6.63)$$

The amplitudes $A_2(s_{13})$ and $A_1(s_{23})$ are given by formulae similar to (6.61), (6.62) but with different sets of the final and intermediate states.

To take into account the resonances with non-zero spins J , one has to substitute in (6.61)

$$A_3(s_{12}) \rightarrow \sum_J A_3^{(J)}(s_{12}) X_{\mu_1 \mu_2 \dots \mu_J}^{(J)}(k_{12}^{\perp P_{12}}) X_{\mu_1 \mu_2 \dots \mu_J}^{(J)}(k_3^{\perp P}), \quad (6.64)$$

where the K -matrix amplitude $A_3^{(J)}(s_{12})$ is determined by an expression similar to (6.61).

The amplitude expansion with respect to states with different angular momenta has been carried out for the reactions $p\bar{p} \rightarrow \text{three mesons}$, using covariant operators given in the analyses [26, 45, 46, 47, 48, 28].

The pole singularities of the amplitudes are leading singularities, and formula (6.61) makes it possible to single out them in the amplitude $p\bar{p}(^1S_0) \rightarrow \text{three mesons}$. It is useful to compare (6.61) with (6.55) when one neglects the terms containing rescatterings:

$$A_{h(^1S_0) \rightarrow P_1 P_2 P_3}(s_{12}, s_{13}, s_{23}) \simeq \sum_{ij} \Lambda_{ij}(s, s_{ij}) .$$

The next-to-leading (logarithmic) singularities are related to the rescattering of mesons produced by the decaying resonances, in Eq. (6.55) these singularities are in the terms

$$\sum_{ij} A_{ij}(s, s_{ij}).$$

The analysis performed in [45, 47] showed that in the reactions $p\bar{p}(\text{at rest}) \rightarrow \pi^0 \pi^0 \pi^0$, $\pi^0 \pi^0 \eta$, $\pi^0 \eta \eta$ the determination of parameters of

resonances produced in the two-meson channels does not require the explicit consideration of the triangle diagram singularities — it is important to take into account only the complexity of parameters $\Lambda_a^{(n)}$ and φ_{ab} in (6.62) which are due to final-state interactions. Note that it is not a universal rule for the meson production processes in the $p\bar{p}$ annihilation – for example, in the reaction $p\bar{p} \rightarrow \eta\pi^+\pi^-\pi^+\pi^-$ [49], the triangle singularity contribution is important.

Here, to be illustrative, we present the K -matrix fit of the annihilation-at-rest reactions $p\bar{p}, p\bar{n} \rightarrow \pi\pi\pi, \pi\pi\eta, \pi\eta\eta, K\bar{K}\pi$.

6.1.3.3 Results of the K -matrix fit of annihilation reactions at rest $p\bar{p}, p\bar{n}$ into $\pi\pi\pi, \pi\pi\eta, \pi\eta\eta, K\bar{K}\pi$

We present the result of the K -matrix analysis of the following data set:

- (1) Crystal Barrel data on $p\bar{p}(\text{at rest, from liquid } H_2) \rightarrow \pi^0\pi^0\pi^0, \pi^0\pi^0\eta, \pi^0\eta\eta$ [65];
- (2) Crystal Barrel data on proton–antiproton annihilation in gas: $p\bar{p}(\text{at rest, from gaseous } H_2) \rightarrow \pi^0\pi^0\pi^0, \pi^0\pi^0\eta$ [66, 67];
- (3) Crystal Barrel data on proton–antiproton annihilation in liquid: $p\bar{p}(\text{at rest, from liquid } H_2) \rightarrow \pi^+\pi^-\pi^0, K^+K^-\pi^0, K_S K_S \pi^0, K^+ K_S \pi^-$ [66, 67];
- (4) Crystal Barrel data on neutron–antiproton annihilation in liquid deuterium: $n\bar{p}(\text{at rest, from liquid } D_2) \rightarrow \pi^0\pi^0\pi^-, \pi^-\pi^-\pi^+, K_S K^-\pi^0, K_S K_S \pi^-$ [66, 67].

The following two-particle waves were taken into account in this K -matrix analysis [56]:

- (1) 0^{++} ($\pi\pi, K\bar{K}, \eta\eta, \eta\eta', \pi\pi\pi\pi$ channels);
- (2) 1^{--} ($\pi\pi, \pi\pi\pi\pi$ channels);
- (3) 2^{++} ($\pi\pi, K\bar{K}, \eta\eta, \pi\pi\pi\pi$ channels);
- (4) 3^{--} ($\pi\pi, \pi\pi\pi\pi$ channels);
- (5) 4^{++} ($\pi\pi, K\bar{K}, \eta\eta, \pi\pi\pi\pi$ channels)

in the invariant mass range 600–2500 MeV. Note that this analysis is a continuation of an earlier work [32, 33, 34, 28].

The results of the fit are shown in Figs. 6.8–6.18, while the fitting formulae are presented in Appendix 6.A. This fit was performed in [56] simultaneously with fitting to the two-particle amplitudes $\pi\pi \rightarrow \pi\pi, \pi\pi \rightarrow \eta\eta, \pi\pi \rightarrow K\bar{K}$ and $\pi\pi \rightarrow \eta\eta'$ and $\pi K \rightarrow \pi K$ (the results of this fit were described in Chapter 3: in Appendix 3.B we give parameters of amplitudes and characteristics of thus determined resonances).

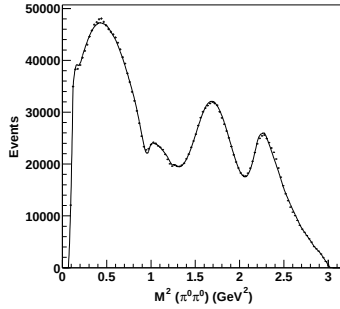


Fig. 6.8 A mass projection of the acceptance-corrected Dalitz plot for the $p\bar{p}$ annihilation into $\pi^0\pi^0\pi^0$ in liquid H_2 . The curve corresponds to Solution II-2.

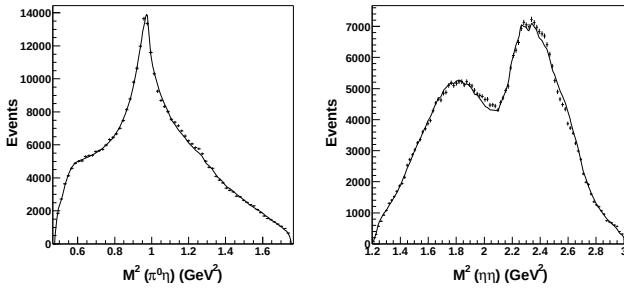


Fig. 6.9 Mass projections of the acceptance-corrected Dalitz plot for the $p\bar{p}$ annihilation into $\pi^0\eta\eta$ in liquid H_2 . Curves correspond to Solution II-2.

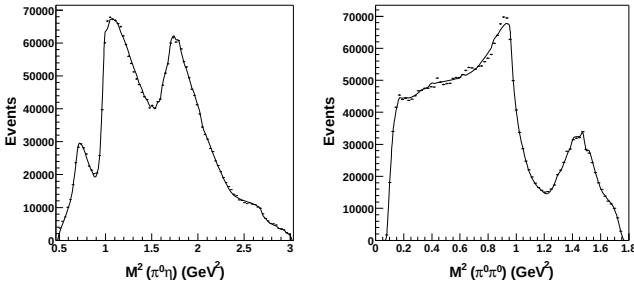


Fig. 6.10 Mass projections of the acceptance-corrected Dalitz plot for the $p\bar{p}$ annihilation into $\pi^0\pi^0\eta$ in liquid H_2 . Curves correspond to Solution II-2.

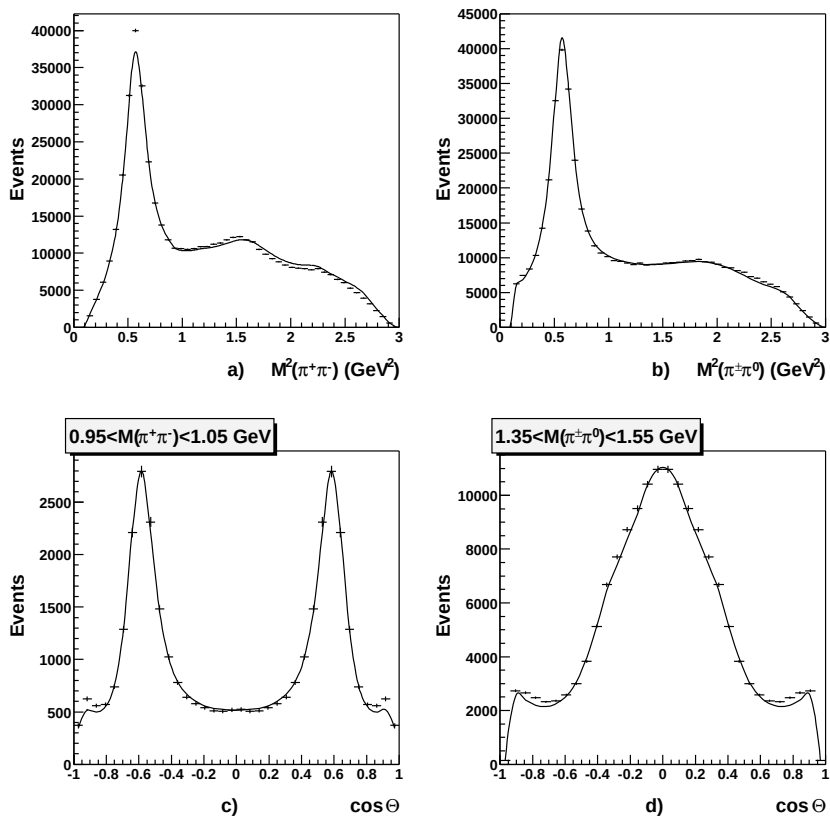


Fig. 6.11 a,b) Mass projections of the acceptance-corrected Dalitz plot for the $p\bar{p}$ annihilation into $\pi^+\pi^0\pi^-$ in liquid H_2 , c) the angle distribution between charged and neutral pions in c.m.s. of $\pi^+\pi^-$ system taken at masses between 0.95 and 1.05 GeV, d) the angle distribution between charged pions in c.m.s. of $\pi^\pm\pi^0$ system taken at masses between 1.35 and 1.55 GeV. Figure 6.12d shows the event distribution along the band with the production of $\rho(1450)$.

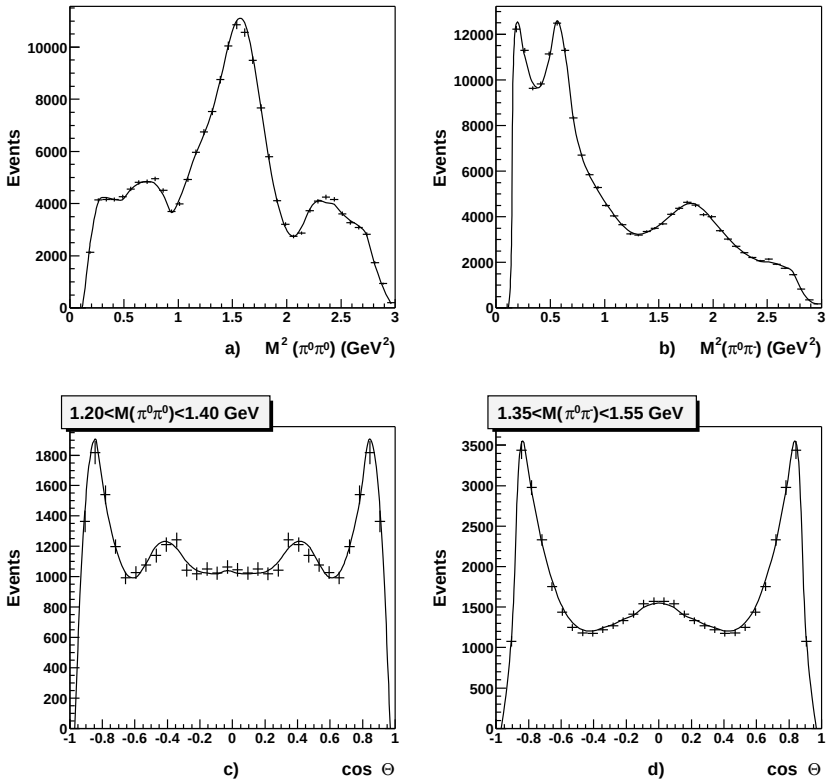


Fig. 6.12 a,b) Mass projections of the acceptance-corrected Dalitz plot for the $n\bar{p}$ annihilation into $\pi^0\pi^0\pi^-$ in liquid D_2 , c) the angle distribution between charged and neutral pions in c.m.s. of $\pi^0\pi^0$ system taken at masses between 1.20 and 1.40 GeV, d) the angle distribution between neutral pions in c.m.s. of $\pi^0\pi^-$ system taken at masses between 1.35 and 1.55 GeV

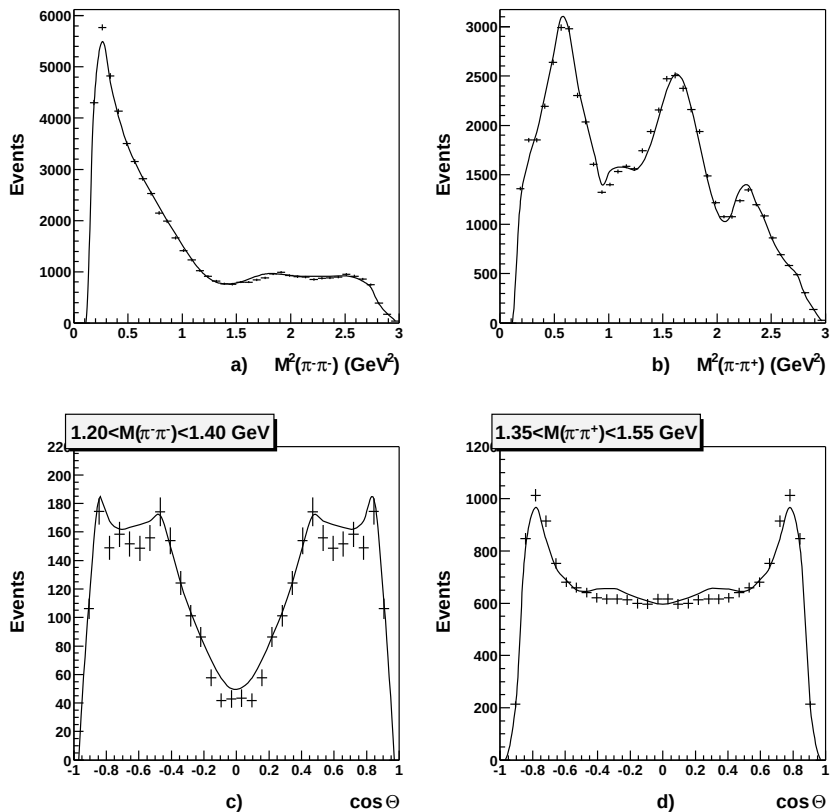


Fig. 6.13 a,b) Mass projections of the acceptance-corrected Dalitz plot for the $n\bar{p}$ annihilation into $\pi^-\pi^-\pi^+$ in liquid D_2 , c) the angle distribution between charged and neutral pions in c.m.s. of $\pi^-\pi^-$ system taken at masses between 1.20 and 1.40 GeV, d) the angle distribution between charged pions in c.m.s. of $\pi^-\pi^+$ system taken at masses between 1.35 and 1.55 GeV.

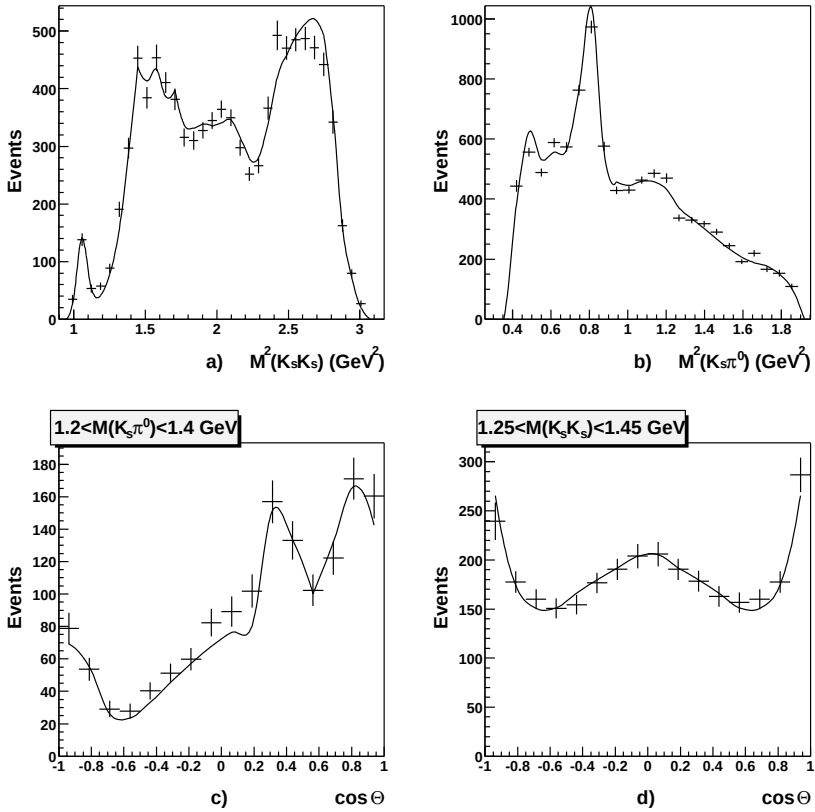


Fig. 6.14 a,b) Mass projections of the acceptance-corrected Dalitz plot for the $p\bar{p}$ annihilation into $K_S K_S \pi^0$ in liquid H_2 , c) the angle distribution between kaons in c.m.s. of $K_S \pi^0$ system taken at masses between 1.20 and 1.40 GeV, d) an angle distribution of the pion in c.m.s. of $K_S K_S$ system taken at masses between 1.25 and 1.45 GeV.

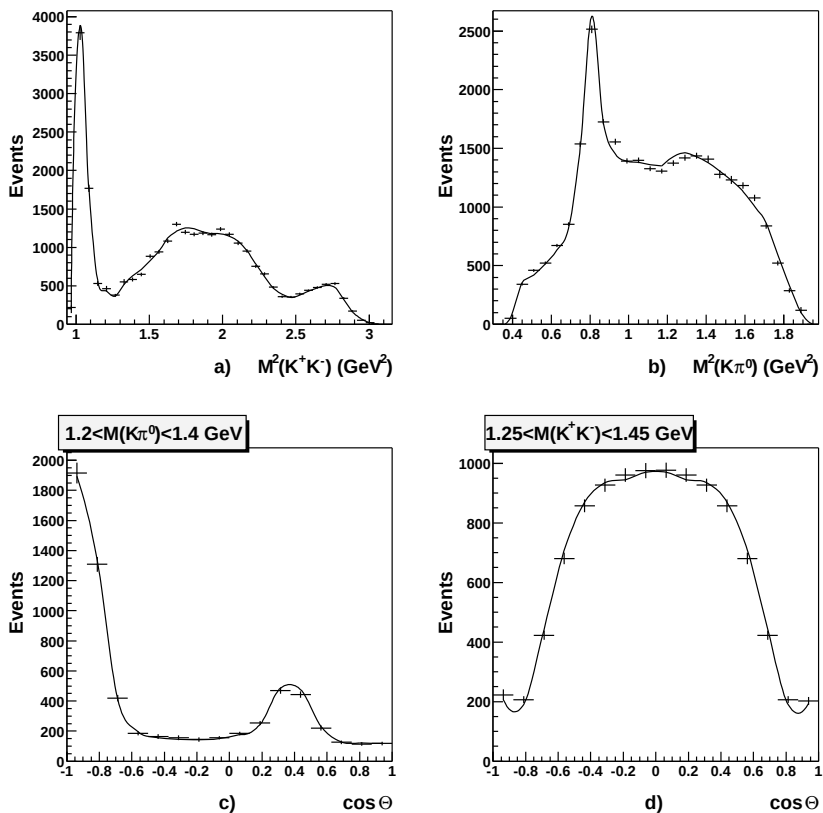


Fig. 6.15 a,b) Mass projections of the acceptance-corrected Dalitz plot for the $p\bar{p}$ annihilation into $K^+K^-\pi^0$ in liquid H_2 , c) the angle distribution between kaons in c.m.s. of $K\pi^0$ system taken at masses between 1.20 and 1.40 GeV , d) the angle distribution of the pion in c.m.s. of K^+K^- system taken at masses between 1.25 and 1.45 GeV .

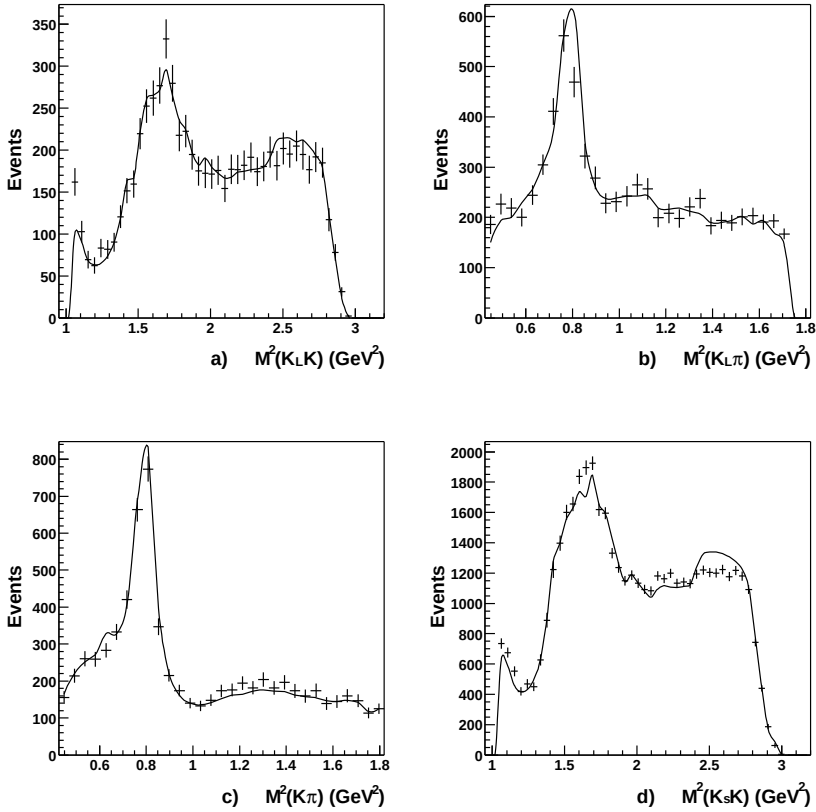


Fig. 6.16 a,b,c) Mass projections of the acceptance-corrected Dalitz plot for the $p\bar{p}$ annihilation into $K_L K^- \pi^+$ ($K_L K^+ \pi^-$) in liquid H_2 , d) $K_S K$ mass projection of the acceptance-corrected Dalitz plot for the $p\bar{p}$ annihilation into $K_S K^- \pi^+$. This reaction has some problems with the acceptance correction and was not used in the analysis. The full curve corresponds to the fit of $p\bar{p} \rightarrow K_L K^- \pi^+$ reaction normalised to the number of $K_S K^- \pi^+$ events.

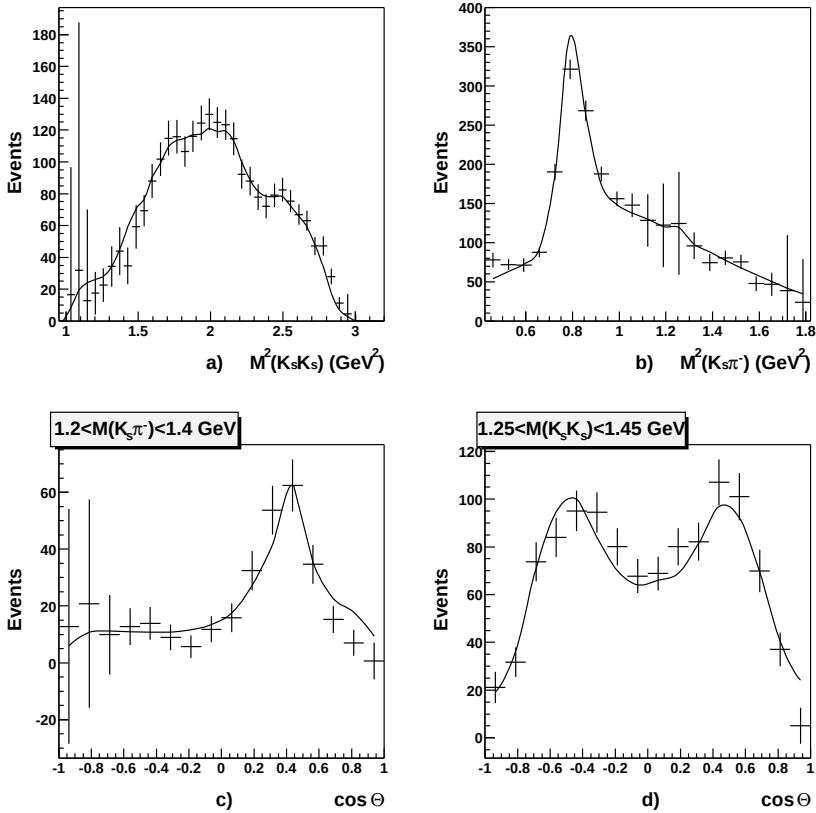


Fig. 6.17 a,b) Mass projections of the acceptance-corrected Dalitz plot for the $p\bar{p}$ annihilation into $K_S K_S \pi^-$ in liquid D_2 , c) the angle distribution between kaons in c.m.s. of $K_S \pi^-$ system taken at masses between 1.20 and 1.40 GeV, d) the angle distribution between kaon and pion in c.m.s. of $K_S K_S$ system taken at masses between 1.25 and 1.45 GeV.

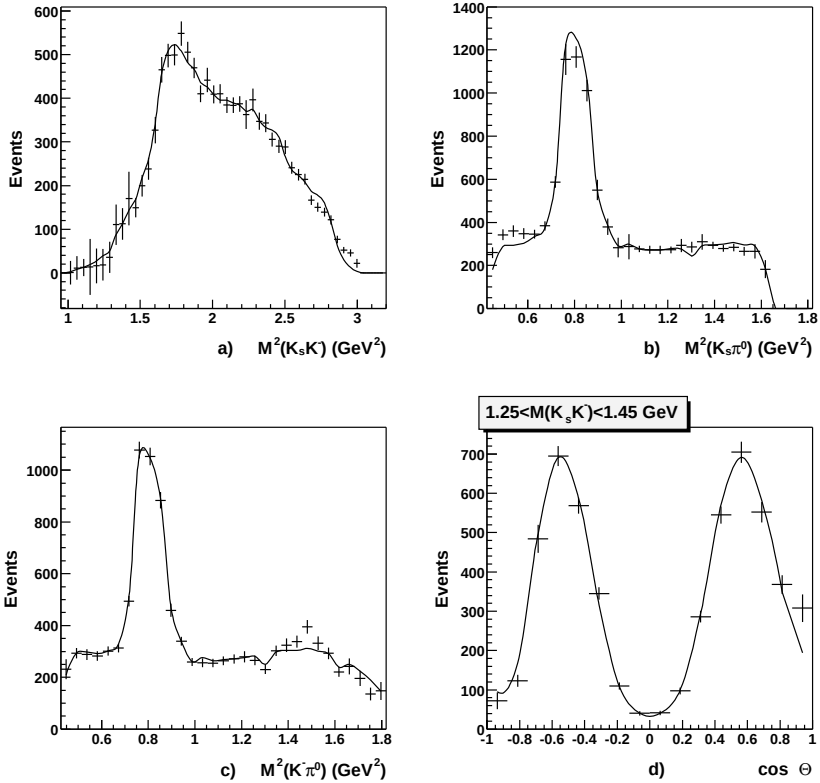


Fig. 6.18 a,b,c) Mass projections of the acceptance-corrected Dalitz plot for the $p\bar{p}$ annihilation into $K_S K^- \pi^0$ in liquid D_2 , d) the angle distribution between K_S and π^0 in c.m.s. of $K_S K^-$ system taken at masses between 1.25 and 1.45 GeV.

6.2 Meson–Nucleon Collisions at High Energies: Peripheral Two-Meson Production in Terms of Reggeon Exchanges

The two-meson production reactions $\pi p \rightarrow \pi\pi n$, $K\bar{K}n$, $\eta\eta n$, $\eta\eta'n$ at high energies and small momentum transfers to the nucleon, t , provide us with a direct information about the amplitudes $\pi\pi \rightarrow \pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$ at $|t| < 0.2 (\text{GeV}/c)^2$ because the π exchange dominates in the case of the produced mesons. At larger $|t|$ there is a change of the regime: at $|t| \gtrsim 0.2 (\text{GeV}/c)^2$ a significant contribution of other reggeons becomes possible (a_1 -exchange, daughter- π and daughter- a_1 exchanges). Despite the not quite proper knowledge of the exchange structure, the study of the two-meson production processes at $|t| \sim 0.5 - 1.5 (\text{GeV}/c)^2$ looks promising, for at such momentum transfers the contribution of the broad resonance (the scalar glueball $f_0(1200 - 1600)$) vanishes, and thus the production of other resonances (such as the $f_0(980)$ and $f_0(1300)$) appears practically without background, which is important for finding their characteristics.

All what we know about the reactions $\pi p \rightarrow \pi\pi n$, $K\bar{K}n$, $\eta\eta n$, $\eta\eta'n$ suggest that the consistent analysis of the peripheral two-meson production in terms of reggeon exchanges can be a good tool for studying meson resonances. Note that the method of investigation of two-meson scattering amplitudes by means of the reggeon exchange expansion of the peripheral two-meson production amplitudes was proposed long ago [64] but was not properly used owing to the lack of data at that time.

The amplitude of the peripheral production of two mesons reads:

$$\left(\bar{\psi}_N(k_3) \hat{G}_R \psi_N(p_2) \right) R(s_{\pi N}, t) \hat{K}_{\pi R(t)}(s) \left[1 - \hat{\rho}(s) \hat{K}(s) \right]^{-1} Q^{(J)}(k_1, k_2), \quad (6.65)$$

This formula is illustrated by Fig. 6.19. Here the factor $(\bar{\psi}_N(k_3) \hat{G}_R \psi_N(p_2))$ stands for the reggeon–nucleon vertex, and $\hat{G}_R$ is the spin operator; $R(s_{\pi N}, t)$ is the reggeon propagator depending on the total energy squared of colliding particles, $s_{\pi N} = (p_1 + p_2)^2$, and the momentum transfer squared $t = (p_2 - k_3)^2$, while the factor $\hat{K}_{\pi R(t)}[1 - i\hat{\rho}(s)\hat{K}(s)]^{-1}$ is related to the block of the two-meson production.

In reactions $\pi p \rightarrow \pi\pi n$, $K\bar{K}n$, $\eta\eta n$, $\eta\eta'n$, the factor $\hat{K}_{\pi R(t)}(s)[1 - i\hat{\rho}(s)\hat{K}(s)]^{-1}$ describes the transitions $\pi R(t) \rightarrow \pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$: the block $\hat{K}_{\pi R(t)}$ is associated to the prompt meson production, and $[1 - i\hat{\rho}(s)\hat{K}(s)]^{-1}$ is a standard factor for meson rescatterings. The prompt-production block

is parametrised in a standard way:

$$\left(\hat{K}_{\pi R(t)} \right)_{\pi R, b} = \sum_n \frac{G_{\pi R}^{(n)}(t) g_b^{(n)}}{\mu_n^2 - s} + f_{\pi R, b}(t, s), \quad (6.66)$$

where $G_{\pi R}^{(n)}(t)$ is the bare state production vertex, and $f_{\pi R, b}$ stands for the background production of mesons, while the parameters $g_b^{(n)}$ and μ_n are the same as in the transition amplitude $\pi\pi \rightarrow \pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$.

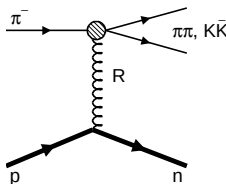


Fig. 6.19 Example of a reaction with the production of two mesons (here $\pi\pi$ and $K\bar{K}$ in π^-p collision) due to reggeon (R) exchange.

Below we shall explain the method of analysis of meson spectra in detail using the reactions $\pi N \rightarrow \pi\pi N$, $K\bar{K}N$, $\eta\eta N$, $\eta\eta' N$, $\pi\pi\pi N$.

6.2.1 *Reggeon exchange technique and the K -matrix analysis of meson spectra in the waves $J^{PC} = 0^{++}$, 1^{--} , 2^{++} , 3^{--} , 4^{++} in high energy reactions $\pi N \rightarrow \text{two mesons} + N$*

Here we present an analysis of the high-energy reactions $\pi^-p \rightarrow \text{mesons} + n$ with the production of mesons in the $J^{PC} = 0^{++}$, 1^{--} , 2^{++} , 3^{--} , 4^{++} states at small and moderate momenta transferred to the nucleon.

The following points are to be emphasised:

- (1) We perform the K -matrix analysis not only for 0^{++} and 2^{++} wave, as in [34, 56], but simultaneously in 1^{--} , 3^{--} , 4^{++} waves as well.
- (2) We use in all reactions the reggeon exchange technique for the description of the t -dependence of the analysed amplitudes. This allows us to perform a partial wave decomposition of the produced meson states without using the published moment expansions (which were done under some simplifying assumptions, it is discussed below in detail) but directly, on the basis of the measured cross sections.
- (3) The mass interval of the analysed meson states is extended till 2500

MeV thus overlapping with the mass region studied in reactions $p\bar{p}(\text{in flight}) \rightarrow \text{mesons}$ [68].

We fix our attention on the reactions measured at incident pion momenta 20 – 50 GeV/c [57, 58, 59, 60, 61, 62]: (i) $\pi^- p \rightarrow \pi^+ \pi^- + n$, (ii) $\pi^- p \rightarrow \pi^0 \pi^0 + n$, (iii) $\pi^- p \rightarrow K_S K_S + n$, (iv) $\pi^- p \rightarrow \eta \eta + n$. At such energies, the mesons in the states $J^{PC} = 0^{++}, 1^{--}, 2^{++}, 3^{--}, 4^{++}$ are produced via t -channel exchange by reggeised mesons belonging to the leading and daughter π , a_1 and ρ trajectories.

But, first of all, let us present notations used in the analysis.

(i) Cross sections for the reactions $\pi N \rightarrow \pi \pi N, K K N, \eta \eta N$.

We consider a process of the Fig. 6.19-type, that is, πN interaction at large momenta of the incoming pion with the production of a two-meson system with a large momentum in the beam direction. This is a peripheral production of two mesons.

The cross section is written as

$$d\sigma = \frac{(2\pi)^4 |A|^2}{8\sqrt{s_{\pi N}} |\vec{p}_2|_{cm(\pi p)}} d\phi(p_1 + p_2, k_1, k_2, k_3),$$

$$d\phi(p_1 + p_2, k_1, k_2, k_3) = (2\pi)^3 d\Phi(P, k_1, k_2) d\Phi(p_1 + p_2, P, k_3) ds, \quad (6.67)$$

where $|\vec{p}_2|_{cm(\pi p)}$ is the pion momentum in the c.m. frame of the incoming hadrons. Taking into account that invariant variables s and t are inherent in the meson peripheral amplitude, we rewrite the phase space in a more convenient form:

$$d\Phi(p_1 + p_2, P, k_3) = \frac{1}{(2\pi)^5} \frac{dt}{8|\vec{p}_2|_{cm(\pi p)} \sqrt{s_{\pi N}}}, \quad t = (k_3 - p_2)^2,$$

$$d\Phi(P, k_1, k_2) = \frac{1}{(2\pi)^5} \rho(s) d\Omega, \quad \rho(s) = \frac{1}{16\pi} \frac{2|\vec{k}_1|_{cm(12)}}{\sqrt{s}}. \quad (6.68)$$

Momentum $|\vec{k}_1|_{cm(12)}$ is calculated in the c.m. frame of the outgoing mesons: in this system one has $P = (M, 0, 0, 0) \equiv (\sqrt{s}, 0, 0, 0)$ and $g_{\mu\nu}^{\perp P} k_{1\nu} = -g_{\mu\nu}^{\perp P} k_{2\nu} = (0, k \sin \Theta \sin \varphi, k \cos \Theta \sin \varphi, k \cos \Theta k)$ while $d\Omega = d(\cos \Theta) d\varphi$. We have:

$$d\sigma = \frac{(2\pi)^4 |A|^2 (2\pi)^3}{8|\vec{p}_2|_{cm(\pi p)} \sqrt{s_{\pi N}} (2\pi)^5} \frac{1}{8|\vec{p}_2|_{cm(\pi p)} \sqrt{s_{\pi N}}} \frac{dt dM^2 d\Phi(P, k_1, k_2)}{8|\vec{p}_2|_{cm(\pi p)} \sqrt{s_{\pi N}}}$$

$$= \frac{|A|^2 \rho(M^2) M dM dt d\Omega}{32(2\pi)^3 |\vec{p}_2|_{cm(\pi p)}^2 s_{\pi N}}, \quad (6.69)$$

with the standard unitarity relation for the amplitude $Im A = \rho(M^2) |A|^2$.

The cross section can be expressed in terms of the spherical functions:

$$\begin{aligned} \frac{d^4\sigma}{dt d\Omega dM} &= N(M, t) I(\Omega) \\ &= N(M, t) \sum_l \left(\langle Y_l^0 \rangle Y_l^0(\Theta, \varphi) + 2 \sum_{m=1}^l \langle Y_l^m \rangle \text{Re} Y_l^m(\Theta, \varphi) \right). \end{aligned} \quad (6.70)$$

The coefficients $N(m, t)$, $\langle Y_l^0 \rangle$, $\langle Y_l^m \rangle$ are subjects of study in the determination of meson resonances.

Before describing the results of analysis based on the reggeon exchange technique, let us comment methods used in other approaches.

(ii) The CERN-Munich approach.

The CERN-Munich model [60] was developed for the analysis of the data on $\pi^- p \rightarrow \pi^+ \pi^- n$ reaction and based partly on the absorption model but mainly on phenomenological observations. The amplitude squared is written as

$$|A|^2 = \left| \sum_{J=0} A_J^0 Y_J^0(\Theta, \varphi) + \sum_{J=1} A_J^- \text{Re} Y_J^1(\Theta, \varphi) \right|^2 + \left| \sum_{J=1} A_J^+ \text{Re} Y_J^1(\Theta, \varphi) \right|^2, \quad (6.71)$$

and additional assumptions are made:

- 1) The helicity-1 amplitudes are equal for natural and unnatural exchanges $A_J^{(-)} = A_J^{(+)}$;
- 2) The ratio of the $A_J^{(-)}$ and the A_J^0 amplitudes is a polynomial over the mass of the two-pion system which does not depend on J up to the total normalisation, $A_J^{(-)} = A_J^0 \left(C_J \sum_{n=0}^3 b_n M^n \right)^{-1}$.

Then the amplitude squared is rewritten in [60] via the density matrices $\rho_{00}^{nm} = A_n^0 A_m^{0*}$, $\rho_{01}^{nm} = A_n^0 A_m^{(-)*}$, $\rho_{11}^{nm} = 2A_n^{(-)} A_m^{(-)}$ as follows:

$$\begin{aligned} |A|^2 &= \sum_{J=0} Y_J^0(\Theta, \varphi) \left(\sum_{n,m} d_{n,m,J}^{0,0,0} \rho_{00}^{nm} + d_{n,m,J}^{1,1,0} \rho_{11}^{nm} \right) \\ &\quad + \sum_{J=0} \text{Re} Y_J^1(\Theta, \varphi) \left(\sum_{n,m} d_{n,m,J}^{1,0,1} \rho_{10}^{nm} + d_{n,m,J}^{0,1,1} \rho_{11}^{nm} \right), \\ d_{n,m,J}^{i,k,l} &= \frac{\int d\Omega \text{Re} Y_n^i(\Theta, \varphi) \text{Re} Y_m^k(\Theta, \varphi) \text{Re} Y_J^l(\Theta, \varphi)}{\int d\Omega \text{Re} Y_J^l(\Theta, \varphi) \text{Re} Y_J^l(\Theta, \varphi)}. \end{aligned} \quad (6.72)$$

Substituting such an amplitude into the cross section, one can directly fit to the moments $\langle Y_J^m \rangle$.

The CERN–Munich approach cannot be applied to large t , it does not work for many other final states.

(iii) GAMS, VES and BNL approaches.

In GAMS [57, 58], VES [61] and BNL [62] approaches, the πN data are decomposed as a sum of amplitudes with an angular dependence defined by spherical functions:

$$|A^2| = \left| \sum_{J=0} A_J^0 Y_J^0(\Theta, \varphi) + \sum_{J=1} A_J^- \sqrt{2} \operatorname{Re} Y_J^1(\Theta, \varphi) \right|^2 + \left| \sum_{J=1} A_J^+ \sqrt{2} \operatorname{Im} Y_J^1(\Theta, \varphi) \right|^2 \quad (6.73)$$

Here the A_J^0 functions are denoted as $S_0, P_0, D_0, F_0 \dots$, the A_J^- functions are defined as $P_-, D_-, F_-, \dots$ and the A_J^+ functions as $P_+, D_+, F_+, \dots$. The equality of the helicity-1 amplitudes with natural and unnatural exchanges is not assumed in these approaches.

However, these approaches are not free from other assumptions like the coherence of some amplitudes or the dominance of the one-pion exchange. In reality the interference of the amplitudes being determined by t -channel exchanges of different particles leads to a more complicated picture than that given by (6.73) which can lead (especially at large t) to a misidentification of the quantum numbers for the produced resonances.

6.2.1.1 The t -channel exchanges of pion trajectories in the reaction $\pi^- p \rightarrow \pi\pi n$

Let us now consider in detail the production of the $\pi\pi$ system in the states with $I = 0$ and $J^{PC} = 0^{++}, 2^{++}$ and show the way of generalisation for higher J .

(i) Amplitude with leading and daughter pion trajectory exchanges.

The amplitude with t -channel pion trajectory exchanges can be written as follows:

$$A_{\pi p \rightarrow \pi\pi n}^{(\pi\text{-trajectories})} = \sum_{R(\pi_j)} A\left(\pi R(\pi_j) \rightarrow \pi\pi\right) R_{\pi_j}(s_{\pi N}, q^2) (\varphi_n^+(\vec{\sigma}\vec{q}_\perp) \varphi_p) g_{pn}^{(\pi_j)}. \quad (6.74)$$

The summation is carried out over the leading and daughter trajectories. Here $A(\pi R(\pi_j) \rightarrow \pi\pi)$ is the transition amplitude for the meson block in

Fig. 6.19, $g_{pn}^{(\pi_j)}$ is the reggeon–NN coupling and $R_{\pi_j}(s_{\pi N}, q^2)$ is the reggeon propagator:

$$R_{\pi_j}(s_{\pi N}, q^2) = \exp\left(-i\frac{\pi}{2}\alpha_{\pi}^{(j)}(q^2)\right) \frac{(s_{\pi N}/s_{\pi N0})^{\alpha_{\pi}^{(j)}(q^2)}}{\sin\left(\frac{\pi}{2}\alpha_{\pi}^{(j)}(q^2)\right) \Gamma\left(\frac{1}{2}\alpha_{\pi}^{(j)}(q^2) + 1\right)}. \quad (6.75)$$

The π -reggeon has a positive signature, $\xi_{\pi} = +1$. Following [71, 70, 69], we use for pion trajectories:

$$\alpha_{\pi}^{(leading)}(q^2) \simeq -0.015 + 0.72q^2, \quad \alpha_{\pi}^{(daughter-1)}(q^2) \simeq -1.10 + 0.72q^2, \quad (6.76)$$

where the slope parameters are given in $(\text{GeV}/c)^{-2}$ units. The normalisation parameter $s_{\pi N0}$ is of the order of $2\text{--}20 \text{ GeV}^2$. To eliminate the poles at $q^2 < 0$ we introduce Gamma-functions in the reggeon propagators (recall that $1/\Gamma(x) = 0$ at $x = 0, -1, -2, \dots$).

For the nucleon–reggeon vertex $\hat{G}_{pn}^{(\pi)}$ we use in the infinite momentum frame the two-component spinors φ_p and φ_n (see Chapter 4 and [69, 72]):

$$g_{\pi}(\bar{\psi}(k_3)\gamma_5\psi(p_2)) \longrightarrow (\varphi_n^+(\vec{\sigma}\vec{q}_{\perp})\varphi_p)g_{pn}^{(\pi)}. \quad (6.77)$$

As to the meson–reggeon vertex, we use the covariant representation [69, 73]. For the production of two pseudoscalar particles (let it be $\pi\pi$ in the considered case), it reads:

$$\begin{aligned} A\left(\pi R(\pi_j) \rightarrow \pi\pi\right) &= \sum_J A_{\pi R(\pi_j) \rightarrow \pi\pi}^{(J)}(s) X_{\mu_1 \dots \mu_J}^{(J)}(p_1^{\perp P}) (-1)^J \\ &\quad \times O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}(\perp P) X_{\nu_1 \dots \nu_J}^{(J)}(k_1^{\perp P}) \xi_J, \\ \xi_J &= \frac{16\pi(2J+1)}{\alpha_J}, \quad \alpha_J = \prod_{n=1}^J \frac{2n-1}{n}. \end{aligned} \quad (6.78)$$

The angular momentum operators are constructed of momenta $p_1^{\perp P}$ and $k_1^{\perp P}$ which are orthogonal to the momentum of the two-pion system. The coefficient ξ_J normalises the angular momentum operators, so that the unitarity condition appears in a simple form (see Appendix 6.B).

(ii) The t -channel π_2 -exchange.

The $R(\pi_j)$ -exchanges dominate the spin flip amplitudes and the amplitudes with $m = 1$, see (6.70), are here suppressed. However, their contributions are visible in the differential cross sections and should be taken into account. The effects appear owing to the interference in the two-meson production amplitude because of the reggeised π_2 exchange in the t -channel.

The corresponding amplitude is written as:

$$\begin{aligned} & \sum_a A_{\alpha\beta} \left(\pi R(\pi_2) \rightarrow \pi\pi \right) \varepsilon_{\alpha\beta}^{(a)} R_{\pi_2}(s_{\pi N}, q^2) \frac{\varepsilon_{\alpha'\beta'}^{(a)+}}{s_{\pi N}^2} \\ & \times X_{\alpha'\beta'}^{(2)}(k_3^{\perp q}) (\varphi_n^+(\vec{\sigma}\vec{q}_{\perp})\varphi_p) g_{pn}^{(\pi_2)}. \end{aligned} \quad (6.79)$$

where $A_{\alpha\beta}(\pi R(\pi_2) \rightarrow \pi\pi)$ is the meson block of the amplitude related to the π_2 -reggeised t -channel transition, $g_{pn}^{(\pi_2)}$ is the reggeon- pn vertex, $R_{\pi_2}(s_{\pi N}, q^2)$ is the reggeon propagator, and $\varepsilon_{\alpha\beta}^{(a)}$ is the polarisation tensor for the 2^{-+} state. Let us remind that k_3 is the momentum of the outgoing nucleon and $k_{3\mu}^{\perp q} = g_{\mu\nu}^{\perp q} k_{3\nu}$ where $g_{\mu\nu}^{\perp q} = g_{\mu\nu} - q_{\mu}q_{\nu}/q^2$.

The π_2 particles are located on the pion trajectories and are described by a similar reggeised propagator. But in the meson block the 2^{-+} state exchange leads to vertices different from that in the 0^{-+} -exchange, so it is convenient to single out these contributions. Therefore, we use for $R_{\pi_2}(s_{\pi N}, q^2)$ the propagator given by (6.75) but eliminating the $\pi(0^{-+})$ -contribution:

$$\begin{aligned} R_{\pi_2}(s_{\pi N}, q^2) &= \exp\left(-i\frac{\pi}{2}\alpha_{\pi}^{(leading)}(q^2)\right) \\ &\times \frac{(s_{\pi N}/s_{\pi N0})\alpha_{\pi}^{(leading)}(q^2)}{\sin\left(\frac{\pi}{2}\alpha_{\pi}^{(leading)}(q^2)\right)\Gamma\left(\frac{1}{2}\alpha_{\pi}^{(leading)}(q^2)-1\right)}. \end{aligned} \quad (6.80)$$

Taking into account that

$$\sum_{a=1}^5 \varepsilon_{\alpha\beta}^{(a)} \varepsilon_{\alpha'\beta'}^{(a)+} = \frac{1}{2} \left(g_{\alpha\alpha'}^{\perp q} g_{\beta\beta'}^{\perp q} + g_{\beta\alpha'}^{\perp q} g_{\alpha\beta'}^{\perp q} - \frac{2}{3} g_{\alpha\beta}^{\perp q} g_{\alpha'\beta'}^{\perp q} \right), \quad (6.81)$$

one obtains:

$$\begin{aligned} & \frac{X_{\alpha'\beta'}^{(2)}(k_3^{\perp q})}{2s_{\pi N}^2} \left(g_{\alpha\alpha'}^{\perp q} g_{\beta\beta'}^{\perp q} + g_{\beta\alpha'}^{\perp q} g_{\alpha\beta'}^{\perp q} - \frac{2}{3} g_{\alpha\beta}^{\perp q} g_{\alpha'\beta'}^{\perp q} \right) \\ &= \frac{3}{2} \frac{k_{3\alpha}^{\perp q} k_{3\beta}^{\perp q}}{s_{\pi N}^2} - \frac{4m_N^2 - q^2}{8s_{\pi N}^2} \left(g_{\alpha\beta} - \frac{q_{\alpha}q_{\beta}}{q^2} \right). \end{aligned} \quad (6.82)$$

In the limit of large momentum of the initial pion the second term in (6.82) is always small and can be neglected, while the convolution of $k_{3\alpha}^{\perp q} k_{3\beta}^{\perp q}$ with the momenta of the meson block results in the term $\sim s_{\pi N}^2$. Hence, the amplitude for π_2 -exchange can be rewritten as follows:

$$\begin{aligned} A_{\pi p \rightarrow \pi\pi n}^{(\pi_2\text{-exchange})} &= \frac{3}{2} A_{\alpha\beta}(\pi R(\pi_2) \rightarrow \pi\pi) \frac{k_{3\alpha}^{\perp q} k_{3\beta}^{\perp q}}{s_{\pi N}^2} R_{\pi_2}(s_{\pi N}, q^2) \\ &\times (\varphi_n^+(\vec{\sigma}\vec{q}_{\perp})\varphi_p) g_{pn}^{(\pi_2)}. \end{aligned} \quad (6.83)$$

A resonance with spin J and fixed parity can be produced owing to the π_2 -exchange with three angular momenta $L = J - 2$, $L = J$ and $L = J + 2$, so we have:

$$\begin{aligned}
 A_{\alpha\beta}(\pi R(\pi_2) \rightarrow \pi\pi) = & \sum_J A_{+2}^{(J)}(s) X_{\alpha\beta\mu_1\dots\mu_J}^{(J+2)}(p_1^{\perp P})(-1)^J \\
 & \times O_{\nu_1\dots\nu_J}^{\mu_1\dots\mu_J}(\perp P) X_{\nu_1\dots\nu_J}^{(J)}(k_1^{\perp P}) \xi_J \\
 & + \sum_J A_0^{(J)}(s) O_{\chi\tau}^{\alpha\beta}(\perp q) X_{\chi\mu_2\dots\mu_J}^{(J)}(p_1^{\perp P})(-1)^J \\
 & \times O_{\nu_1\nu_2\dots\nu_J}^{\tau\mu_2\dots\mu_J}(\perp P) X_{\nu_1\dots\nu_J}^{(J)}(k_1^{\perp P}) \xi_J \\
 & + \sum_J A_{-2}^{(J)}(s) X_{\mu_3\dots\mu_J}^{(J-2)}(p_1^{\perp P})(-1)^J \\
 & \times O_{\nu_1\nu_2\nu_3\dots\nu_J}^{\alpha\beta\mu_3\dots\mu_J}(\perp P) X_{\nu_1\dots\nu_J}^{(J)}(k_1^{\perp P}) \xi_J . \quad (6.84)
 \end{aligned}$$

The sum of the two terms presented in (6.74) and (6.83) gives us an amplitude with a full set of the π_j -meson exchanges. The contribution of this amplitude to the differential cross section expanded over spherical functions, Eq. (6.70), is given in Appendix 6.B.

Let us emphasise an important point: in the K -matrix representation the amplitudes $A_{\pi R(\pi_j) \rightarrow \pi\pi}^{(J)}(s)$ (Eq. (6.78), $j = \text{leading, daughter-1}$) and $A_{+2}^{(J)}(s)$, $A_0^{(J)}(s)$, $A_{-2}^{(J)}(s)$ (Eq. (6.84)) differ only due to the prompt-production K -matrix block, it is the term $\hat{K}_{\pi R(t)}(s)$ in (6.65), while the final state interaction terms, given by the factor $[1 - \hat{\rho}(s)\hat{K}(s)]^{-1}$ in (6.65), are the same for a fixed J .

6.2.1.2 Amplitudes with a_1 -trajectory exchanges

The amplitude with t -channel a_1 -exchanges is a sum of leading and daughter trajectories:

$$\begin{aligned}
 A_{\pi p \rightarrow \pi\pi n}^{(a_1\text{-trajectories})} = & \sum_{a_1^{(j)}} A\left(\pi R(a_1^{(j)}) \rightarrow \pi\pi\right) R_{a_1^{(j)}}(s_{\pi N}, q^2) \times \\
 & \times i\left(\varphi_n^+(\vec{\sigma}\vec{n}_z)\varphi_p\right) g_{pn}^{(a_1j)} , \quad (6.85)
 \end{aligned}$$

where $g_{pn}^{(a_1j)}$ is the reggeon-NN coupling and the reggeon propagator $R_{a_1^{(j)}}(s_{\pi N}, q^2)$ has the form:

$$R_{a_1^{(j)}}(s_{\pi N}, q^2) = i \exp\left(-i\frac{\pi}{2}\alpha_{a_1}^{(j)}(q^2)\right) \frac{(s_{\pi N}/s_{\pi N0})^{\alpha_{a_1}^{(j)}(q^2)}}{\cos\left(\frac{\pi}{2}\alpha_{a_1}^{(j)}(q^2)\right) \Gamma\left(\frac{1}{2}\alpha_{a_1}^{(j)}(q^2) + \frac{1}{2}\right)} . \quad (6.86)$$

Recall that the a_1 trajectories have a negative signature, $\xi_\pi = -1$. Here we take into account the leading and first daughter trajectories which are linear and have a universal slope parameter, $\sim 0.72 \text{ (GeV/c)}^{-2}$ [69, 70, 71]:

$$\alpha_{a_1}^{(leading)}(q^2) \simeq -0.10 + 0.72q^2, \quad \alpha_{a_1}^{(daughter-1)}(q^2) \simeq -1.10 + 0.72q^2. \quad (6.87)$$

As previously, the normalisation parameter $s_{\pi N 0}$ is of the order of 2–20 GeV^2 , and the Gamma-functions in the reggeon propagators are introduced in order to eliminate the poles at $q^2 < 0$.

For the nucleon–reggeon vertex we use two-component spinors in the infinite momentum frame, φ_p and φ_n (see Chapter 4 for detail), the vertex reads: $(\varphi_n^+ i(\vec{\sigma} \vec{n}_z) \varphi_p) g_{pn}^{(a_1)}$ where $\vec{n}_z$ is the unit vector directed along the nucleon momentum in the c.m. frame of colliding particles.

At fixed partial wave $J^{PC} = J^{++}$, the $\pi R(a_1^j)$ channel ($j = \text{leading, daughter-1}$) is characterised by two angular momenta $L = J + 1$, $L = J - 1$, so we have two amplitudes for each J :

$$\begin{aligned} & A\left(\pi R(a_1^{(j)}) \rightarrow \pi\pi\right) \\ &= \sum_J \epsilon_\beta^{(-)} \left[A_{\pi a_1^{(j)} \rightarrow \pi\pi}^{(J+)}(s) X_{\beta\mu_1 \dots \mu_J}^{(J+1)}(p_1^{\perp P}) + A_{\pi a_1^{(j)} \rightarrow \pi\pi}^{(J-)}(s) Z_{\mu_1 \dots \mu_J, \beta}(p_1^{\perp P}) \right] \\ & \times (-1)^J O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}(\perp P) X_{\nu_1 \dots \nu_J}^{(J)}(k_1^{\perp P}), \end{aligned} \quad (6.88)$$

where the polarisation vector $\epsilon_\beta^{(-)} \sim n_\beta$; the GLF-vector n_β [74] was discussed in Chapter 4 (section 4.5.2.2.) – let us remind that in the infinite momentum frame for the nucleon $n_\beta = (1, 0, 0, -1)/2p_z$ with $p_z \rightarrow \infty$.

(i) Calculations in the Godfrey–Jackson system.

In the Godfrey–Jackson system, which is used for the calculation of the meson block (the system of the produced mesons is at rest), we write:

$$\epsilon_\beta^{(-)} = \frac{1}{s_{\pi N}} \left(k_{3\mu} - \frac{q_\mu}{2} \right). \quad (6.89)$$

In the Godfrey–Jackson system the momenta are as follows:

$$\begin{aligned} p_1^{\perp P} &\equiv p_\perp = (0, 0, 0, p), & p^2 &= \frac{(s + m_\pi^2 - t)^2}{4s} - m_\pi^2, \\ k_1^{\perp P} &\equiv k_\perp = (0, k \sin \Theta \cos \varphi, k \sin \Theta \sin \varphi, k \cos \Theta), & k^2 &= \frac{s}{4} - m_\pi^2, \\ q &= (q_0, 0, 0, -p), & q_0 &= \frac{s - m_\pi^2 + t}{2\sqrt{s}}, \end{aligned} \quad (6.90)$$

(recall the notation $A = (A_0, A_x, A_y, A_z)$).

The products of Z and X operators can be written as vectors $V_\beta^{(J+)}$ and $V_\beta^{(J-)}$:

$$\begin{aligned} X_{\beta\mu_1\dots\mu_J}^{(J+1)}(p_\perp)(-1)^J X_{\mu_1\dots\mu_J}(k_\perp) &= \alpha_J(\sqrt{-p_\perp^2})^{J+1}(\sqrt{-k_\perp^2})^J V_\beta^{(J+)}, \\ V_\beta^{(J+)} &= \frac{1}{J+1} \left[P'_{J+1}(z) \frac{p_{\perp\beta}}{\sqrt{-p_\perp^2}} - P'_J(z) \frac{k_{\perp\beta}}{\sqrt{-k_\perp^2}} \right], \\ Z_{\mu_1\dots\mu_J,\beta}(p_\perp)(-1)^J X_{\mu_1\dots\mu_J}^{(J)}(k_\perp) &= \alpha_J(\sqrt{-p_\perp^2})^{J-1}(\sqrt{-k_\perp^2})^J V_\beta^{(J-)}, \\ V_\beta^{(J-)} &= \frac{1}{J} \left[P'_{J-1}(z) \frac{p_{\perp\beta}}{\sqrt{-p_\perp^2}} - P'_J(z) \frac{k_{\perp\beta}}{\sqrt{-k_\perp^2}} \right]. \end{aligned} \quad (6.91)$$

So the convolutions $V_\beta^{(J+)}(k_{3\beta} - q_\beta/2)$, $V_\beta^{(J-)}(k_{3\beta} - q_\beta/2)$ give us the amplitude for the transition $\pi R(a_1^{(j)})$ into two pions (in a GJ-system the momentum $\vec{k}_3$ is usually situated in the (xz) -plane). We write the amplitude in the form

$$\begin{aligned} A\left(\pi R(a_1^{(j)}) \rightarrow \pi\pi\right) &= \sum_J \alpha_J p^{J-1} k^J \\ &\times \left(W_0^{(J)}(s) Y_J^0(\Theta, \varphi) + W_1^{(J)}(s) \text{Re} Y_J^1(\Theta, \varphi) \right), \end{aligned} \quad (6.92)$$

where the coefficients $W_0^{(J)}(s)$, $W_1^{(J)}(s)$ are easily calculated.

(ii) Partial wave decomposition.

As before, the partial wave amplitude $\pi R(a_1^{(j)}) \rightarrow \pi\pi$ with definite J^{++} is presented in the K -matrix form:

$$A_{\pi R(a_1^{(j)}), \pi\pi}^{(L=J\pm 1, J^{++})}(s) = \sum_b K_{\pi R(a_1^{(j)}), b}^{(L=J\pm 1, J^{++})}(s, q^2) \left[\frac{\hat{I}}{\hat{I} - i\hat{\rho}(s)\hat{K}^{(J^{++})}(s)} \right]_{b, \pi\pi}, \quad (6.93)$$

where $K_{\pi R(a_1^{(j)}), b}^{(L=J\pm 1, J^{++})}(s, q^2)$ is the following vector ($b = \pi\pi, K\bar{K}, \eta\eta, \eta\eta', \pi\pi\pi\pi$):

$$\begin{aligned} K_{\pi R(a_1^{(j)}), b}^{(L=J\pm 1, J^{++})}(s, q^2) &= \left(\sum_\alpha \frac{G_{\pi R(a_1^{(j)})}^{(L=J\pm 1, J^{++}, \alpha)}(q^2) g_b^{(J^{++}, \alpha)}}{M_\alpha^2 - s} \right. \\ &\quad \left. + F_{\pi R(a_1^{(j)}), b}^{(J^{++}, J^{++})}(q^2) \frac{1 \text{ GeV}^2 + s_{R0}}{s + s_{R0}} \right) \frac{s - s_A}{s + s_{A0}}. \end{aligned} \quad (6.94)$$

Here $G_{\pi R(a_1^{(j)})}^{(L=J\pm 1, J^{++}, \alpha)}(q^2)$ and $F_{\pi R(a_1^{(j)}), b}^{(J^{++}, J^{++})}(q^2)$ are the q^2 -dependent reggeon form factors.

6.2.1.3 $\pi^- p \rightarrow K \bar{K} n$ reaction with $K \bar{K}$ -exchange by ρ -meson trajectories

In the case of production of the $K \bar{K}$ system the resonance in this channel can have isospins $I = 0$ and $I = 1$, with even spin (production of states of the types ϕ and a_0). Such processes are described by ρ -exchanges.

(i) Amplitude with exchanges by ρ -meson trajectories.

The amplitude with t -channel ρ -meson exchanges is written as follows:

$$A_{\pi p \rightarrow K \bar{K} n}^{(\rho \text{ trajectories})} = \sum_{\rho_j} A\left(\pi R(\rho_j) \rightarrow K \bar{K}\right) R_{\rho_j}(s_{\pi N}, q^2) \hat{g}_{pn}^{(\rho)}, \quad (6.95)$$

where the reggeon propagator $R_{\rho_j}(s_{\pi N}, q^2)$ and the reggeon–nucleon vertex $\hat{g}_{pn}^{(\rho)}$ read, respectively:

$$R_{\rho_j}(s_{\pi N}, q^2) = \exp\left(-i \frac{\pi}{2} \alpha_{\rho}^{(j)}(q^2)\right) \frac{(s_{\pi N}/s_{\pi N 0})^{\alpha_{\rho}^{(j)}(q^2)}}{\sin\left(\frac{\pi}{2} \alpha_{\rho}^{(j)}(q^2)\right) \Gamma\left(\frac{1}{2} \alpha_{\rho}^{(j)}(q^2) + 1\right)},$$

$$\hat{g}_{pn}^{(\rho)} = g_{pn}^{(\rho)}(1)(\varphi_n^+ \varphi_p) + g_{pn}^{(\rho)}(2) \left(\varphi_n^+ \frac{i}{2m_N} (\vec{q}_{\perp} [\vec{n}_z, \vec{\sigma}]) \varphi_p \right). \quad (6.96)$$

The ρ_j -reggeons have positive signatures, $\xi_{\rho} = +1$, being determined by linear trajectories [71, 70, 69]:

$$\alpha_{\rho}^{(\text{leading})}(q^2) \simeq 0.50 + 0.83q^2, \alpha_{\rho}^{(\text{daughter}-1)}(q^2) \simeq -0.75 + 0.83q^2. \quad (6.97)$$

The slope parameters are in $(\text{GeV}/c)^{-2}$ units, the normalisation parameter $s_{\pi N 0} \sim 2 - 20 \text{ GeV}^2$, and the poles in (6.96) at $q^2 < 0$ are cancelled by the poles of Gamma-function. Two vertices in $\hat{g}_{pn}^{(\rho)}$ correspond to charge- and magnetic-type interactions (they are written in the infinite momentum frame of the colliding particles).

The meson–reggeon amplitude can be written as

$$A\left(\pi R(\rho_j) \rightarrow K \bar{K}\right) = \sum_J \varepsilon_{\beta \epsilon(-) p_1 P} Z_{\mu_1 \mu_2 \dots \mu_J, \beta} (p_1^{\perp P}) A_{\pi R_{\rho}(q^2), K \bar{K}}^{(J++)}(s) \\ \times X_{\mu_1 \mu_2 \dots \mu_J}^{(J)}(k_1^{\perp P}) (-1)^J, \quad (6.98)$$

where the polarisation vector $\epsilon_{\beta}^{(-)}$ was introduced in (6.89).

(ii) The Godfrey–Jackson system.

We use the convolution of the Z and X operators in the GJ-system (see notations in (6.90):

$$Z_{\mu_1 \dots \mu_J, \beta}(p_{\perp}) (-1)^J X_{\mu_1 \dots \mu_J}^{(J)}(k_{\perp}) = \frac{\alpha_J}{J} (\sqrt{-p_{\perp}^2})^{J-1} (\sqrt{-k_{\perp}^2})^J \\ \times \left[P'_{J-1}(z) \frac{p_{\perp \beta}}{\sqrt{-p_{\perp}^2}} - P'_J(z) \frac{k_{\perp \beta}}{\sqrt{-k_{\perp}^2}} \right]. \quad (6.99)$$

The convolution of the spin-momentum operators in (6.98) gives:

$$A(\pi\rho_j \rightarrow \pi\pi) = \sum_J \frac{\alpha_J}{J} p^J k^J k_{3x} \sqrt{s} N_{j1} \text{Im } Y_J^1(\Theta, \varphi) A_{\pi R_\rho(q^2), K\bar{K}}^{(J++)}(s). \quad (6.100)$$

Let us remind that in the GJ-system the vector $\vec{k}_3$ is situated in the (xz) -plane.

(iii) Partial wave decomposition.

The amplitude for the transition $\pi R_\pi(q^2) \rightarrow K\bar{K}$ in the K -matrix representation reads:

$$A_{\pi R(\rho_j), K\bar{K}}^{(J++)}(s) = \sum_b K_{\pi R(\rho_j), b}^{(J++)}(s, q^2) \left[\frac{\hat{I}}{\hat{I} - i\hat{\rho}(s)\hat{K}^{(J++)}(s)} \right]_{b, K\bar{K}}, \quad (6.101)$$

where $K_{\pi R(\rho_j), b}^{(J++)}(s, q^2)$ is the following vector ($b = \pi\pi, K\bar{K}, \eta\eta, \eta\eta', \pi\pi\pi\pi$):

$$K_{\pi R(\rho_j), b}^{(J++)}(s, q^2) = \left(\sum_\alpha \frac{G_{\pi R(\rho_j)}^{(J++, \alpha)}(q^2) g_b^{(J++, \alpha)}}{M_\alpha^2 - s} + F_{\pi R(\rho_j), b}^{(J++)}(q^2) \frac{1 \text{ GeV}^2 + s_{R0}}{s + s_{R0}} \right) \frac{s - s_A}{s + s_{A0}}. \quad (6.102)$$

Here $G_{\pi R(\rho_j)}^{(J++, \alpha)}(q^2)$ and $F_{\pi R(\rho_j), b}^{(J++)}(q^2)$ are the reggeon q^2 -dependent form factors.

6.2.2 Results of the K -matrix fit of two-meson systems produced in the peripheral productions

Below we presented fits performed for amplitudes of the following two-meson systems produced in the peripheral three-body reactions $\pi^- p \rightarrow n + \pi\pi, n + \eta\eta, n + \eta\eta', n + K\bar{K}$ and $K^- p \rightarrow n + K^- \pi^+$:

- 1) $\pi^+ \pi^-$ -system, all waves, CERN-Munich data [57],
- 2) $\pi^0 \pi^0$ -system, S-wave, GAMS data [57],
- 3) $\pi^0 \pi^0$ -system, S-wave, E852 data [62],
- 4) $\eta\eta$ -system, S-wave, GAMS data [58],
- 5) $\eta\eta'$ -system, S-wave, GAMS data [58],
- 6) $K\bar{K}$ -system, S-wave, BNL data [59],
- 7) $K^- \pi^+$ -system, S-wave, LASS data [63].

6.2.2.1 The basic formulae

Amplitudes for the π^- - and a_1 -trajectory exchanges can be written as follows:

$$A_{\pi p \rightarrow \pi \pi n}^{(\pi\text{-tra})} = \sum_i A(\pi \pi_i \rightarrow \pi \pi) R_{\pi_j}(s_{\pi N}, q^2) (\varphi_n^+(\vec{\sigma} \vec{p}_\perp) \varphi_p) g_{pn}^{(\pi_i)},$$

$$A_{\pi p \rightarrow \pi \pi n}^{(a_1\text{-tra})} = \sum_i A(\pi a_1^{(i)} \rightarrow \pi \pi) R_{a_1^{(i)}}(s_{\pi N}, q^2) (\varphi_n^+(\vec{\sigma} \vec{n}_z) \varphi_p) g_{pn}^{(a_{1i})}, \quad (6.103)$$

where $A(\pi \pi_i \rightarrow \pi \pi)$ and $A(\pi a_1^{(i)} \rightarrow \pi \pi)$ are the pion-reggeon to two-meson (*e.g.* two-pion) transition amplitudes, $g_{pn}^{(\pi_i)}$ and $g_{pn}^{(a_{1i})}$ are reggeon-NN vertex couplings, and $R(s_{\pi N}, q^2)$ is the reggeon propagator:

$$R_{\pi_i}(s_{\pi N}, q^2) = \exp\left(-i \frac{\pi}{2} \alpha_{\pi}^{(i)}(q^2)\right) \frac{(s_{\pi N}/s_{\pi N0})^{\alpha_{\pi}^{(i)}(q^2)}}{\sin\left(\frac{\pi}{2} \alpha_{\pi}^{(i)}(q^2)\right) \Gamma\left(\frac{1}{2} \alpha_{\pi}^{(i)}(q^2) + 1\right)},$$

$$R_{a_1^{(i)}}(s_{\pi N}, q^2) = i \exp\left(-i \frac{\pi}{2} \alpha_{a_1}^{(i)}(q^2)\right) \frac{(s_{\pi N}/s_{\pi N0})^{\alpha_{a_1}^{(i)}(q^2)}}{\cos\left(\frac{\pi}{2} \alpha_{a_1}^{(i)}(q^2)\right) \Gamma\left(\frac{1}{2} \alpha_{a_1}^{(i)}(q^2) + \frac{1}{2}\right)}. \quad (6.104)$$

The parametrisation of the $\alpha_{\pi}^{(i)}$ and $\alpha_{a_1}^{(i)}$ (here the (i) index counts leading and daughter trajectories) can be found, *e.g.*, in [71, 69]. The normalisation parameter $s_{\pi N0}$ is of the order of 2–20 GeV².

The transition amplitude can be rewritten as:

$$A(\pi \pi_i \rightarrow \pi \pi) = \sum_J A_{\pi \pi_i \rightarrow \pi \pi}^J(s) (2J+1) N_J^0 Y_J^0(\Theta, \varphi) (|\vec{p}| |\vec{k}|)^J, \quad (6.105)$$

$$A(\pi a_1^{(i)} \rightarrow \pi \pi) = \sum_J (2J+1) |\vec{p}|^{J-1} |\vec{k}|^J \left(W_{0i}^{(J)} Y_J^0(\Theta, \varphi) + W_{1i}^{(J)} Re Y_J^1(\Theta, \varphi) \right),$$

where $\vec{p}$ and $\vec{k}$ are vectors of the initial and final pion in the c.m. system of two final mesons, and

$$W_{0i}^{(J)} = -N_{J0} \left(k_{3z} - \frac{|\vec{p}|}{2} \right) \left(|\vec{p}|^2 A_{\pi a_1^{(i)} \rightarrow \pi \pi}^{(J+)} - A_{\pi a_1^{(i)} \rightarrow \pi \pi}^{(J-)} \right),$$

$$W_{1i}^{(J)} = -\frac{N_{J1}}{J(J+1)} k_{3x} \left(|\vec{p}|^2 J A_{\pi a_1^{(i)} \rightarrow \pi \pi}^{(J+)} + (J+1) A_{\pi a_1^{(i)} \rightarrow \pi \pi}^{(J-)} \right). \quad (6.106)$$

Here $A_{\pi a_1^{(i)} \rightarrow \pi \pi}^{(J+)}$ is the amplitude produced in a πa_1 system with orbital momentum $L = J+1$ and $A_{\pi a_1^{(i)} \rightarrow \pi \pi}^{(J-)}$ is the amplitude produced with $L = J-1$.

The leading contribution from the π -exchange trajectory can contribute only to the moments with $m = 0$, while the a_1 -exchange can contribute to

the moments up to $m = 2$. The characteristic feature of the a_1 exchange is that moments with $m = 2$ are suppressed compared to moments with $m = 1$ by the ratio k_{3x}/k_{3z} which is small for the system of two final mesons propagating with a large momentum in the beam direction.

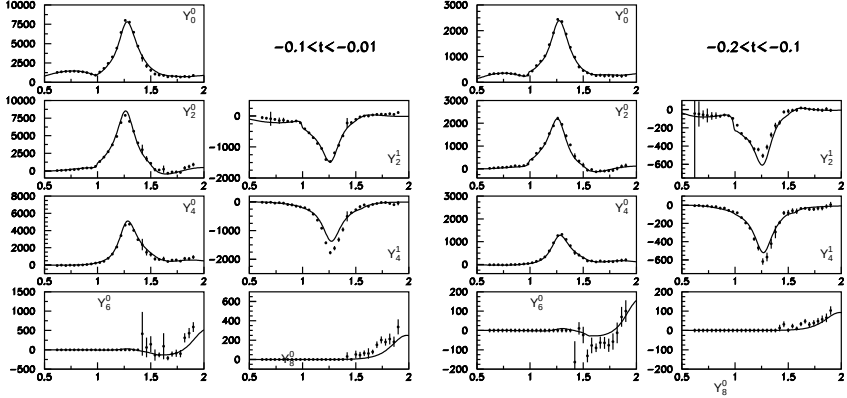


Fig. 6.20 The description of the moments extracted at energy transferred $-0.1 < t < -0.01$ GeV^2 (the left two columns) and $-0.2 < t < -0.1$ GeV^2 (the right two columns).

The amplitudes defined by the π and a_1 exchanges are orthogonal if the nucleon polarisation is not measured. This is due to the fact that the pion trajectory states are defined by the singlet combination of the nucleon spins while the a_1 trajectory states are defined by the triplet combination. This effect is not taken into account for the S-wave contribution in (6.73) which can lead to a misidentification of this wave at large momenta transferred.

The π_2 particle is situated on the pion trajectory and therefore should be described by the reggeised pion exchange. However, the π_2 -exchange has next-to-leading order contributions with spherical functions at $m \geq 1$. The interference of such amplitudes with the pion exchange can be important (especially at small t) and is taken into account in the present analysis.

6.2.2.2 Fit to the data

To reconstruct the total cross section of the reaction $\pi^- p \rightarrow \pi^0 \pi^0 n$ [62] which is not available now we have used Eq. (6.73) from [62] and redecomposed the cross section over moments by applying formulae written above. The two solutions of [62] produced very close results and we included the small differences between them as a systematical error.

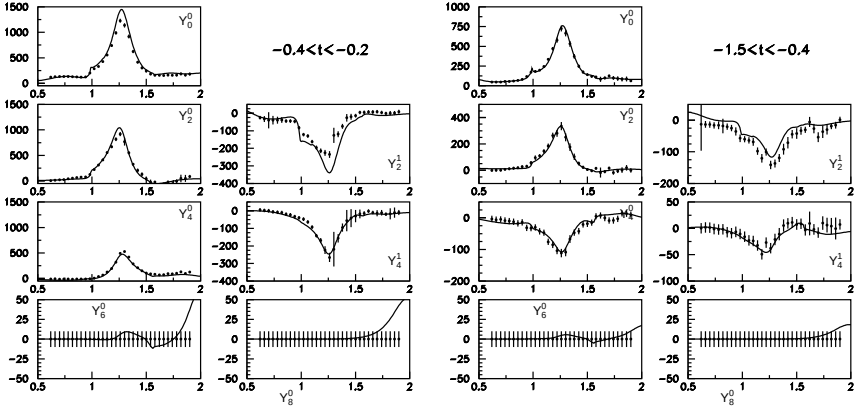


Fig. 6.21 The description of the moments extracted at energy transferred $-0.4 < t < -0.2 \text{ GeV}^2$ (two left columns) and $-1.5 < t < -0.4 \text{ GeV}^2$ (two right columns).

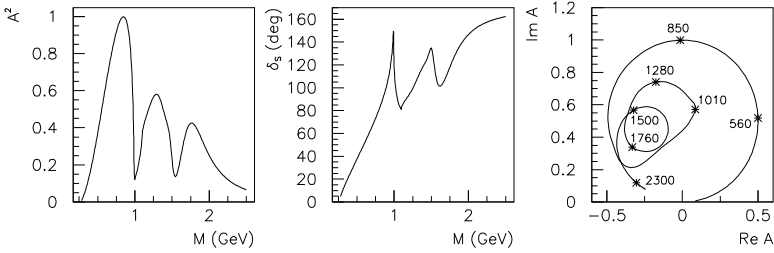


Fig. 6.22 From left to right: a) The $\pi\pi \rightarrow \pi\pi$ S-wave amplitude squared, b) the amplitude phase and c) Argand diagram.

The $\pi^- p \rightarrow \pi^0 \pi^0 n$ data can be described successfully with only π , a_1 and π_2 leading trajectories taken into account. The S-wave was fitted to 5 poles in the 5-channel K -matrix, described in details in the previous sections. The D-wave was fitted to 4 poles in the 4-channel ($\pi\pi$, $K\bar{K}$, $\omega\omega$ and 4π) K -matrix. The position of the first two D-wave poles was found to be $1275 - i98 \text{ MeV}$ and $1525 - i67 \text{ MeV}$ which corresponds to the well-known resonances $f_2(1270)$ and $f_2(1525)$. The third pole has a Flatté-structure near the $\omega\omega$ threshold. Its position was found to be $1530 - i262 \text{ MeV}$ on the sheet above the $\omega\omega$ threshold and $1699 - i216 \text{ MeV}$ on the sheet below the

$\omega\omega$ threshold. For both poles the closest physical region is the beginning of the $\omega\omega$ threshold $M \sim 1570$ MeV, where they form a relatively narrow (220–250 MeV) structure which is called the $f_2(1560)$ state, see Fig. 6.23. The fourth pole cannot be fixed well by the present data.

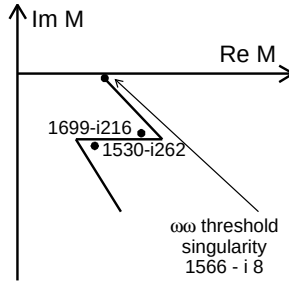


Fig. 6.23 Pole structure of the 2^{++} -amplitude in the region of the $\omega\omega$ -threshold: the resonance $f_2(1560)$.

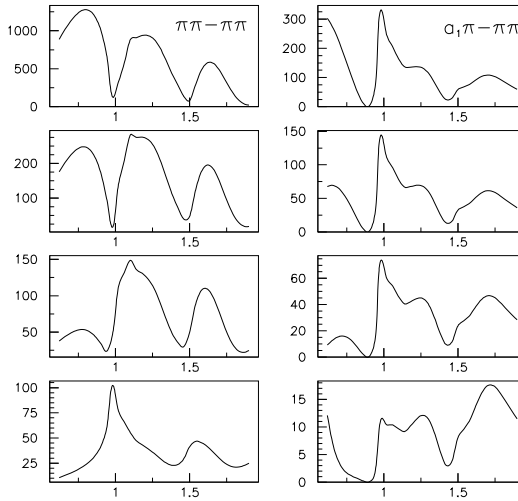


Fig. 6.24 The contribution of S-wave to Y_{00} moment integrated over intervals (from upper line to bottom line) $t < -0.1$, $-0.1 < t < -0.2$, $-0.2 < t < -0.4$ and $-1.5 < t < -0.4$ GeV^2 .

The description of the moments at small $|t|$ is shown in Fig. 6.20 and at large $|t|$ in Fig. 6.21. It is seen that reggeon trajectory exchanges can

describe the moments at all t -intervals rather well already with the simple assumption about the t -dependence of form factor for all partial waves.

The $\pi\pi \rightarrow \pi\pi$ S-wave elastic amplitude is shown in Fig.6.22. The structure of the amplitude is well known, it is defined by the destructive interference of the broad component with $f_0(980)$ and $f_0(1500)$. Neither $f_0(1300)$ nor $f_0(1750)$ provide a strong change of the amplitudes. However, this is hardly a surprise: both these states are relatively broad and very inelastic. The K -matrix parameters found in this solution are given in Table 6.1 (Appendix 6.A).

The S-wave contributions defined by the π and α_1 exchanges integrated over four intervals $t < -0.1$, $-0.1 < t < -0.2$, $-0.2 < t < -0.4$ and $-1.5 < t < -0.4$ GeV^2 are shown in Fig. 6.24. In the S -wave part defined by the π -trajectory exchange there is no significant contribution from $f_0(1370)$. This is probably not a surprise: this resonance rather weakly couples to the $\pi\pi$ channel. In the S-wave amplitude defined by the a_1 exchange the $f_0(1370)$ resonance contributes notably at large t , which means that the large 4π width of this state can be defined by the decay into the $a_1\pi$ system.

The $\pi\pi \rightarrow \pi\pi$ D-wave elastic amplitude is shown in Fig. 6.25. The amplitude squared is dominated by the $f_2(1270)$ state. The $f_2(1560)$ as well as $f_2(1510)$ (included as K -matrix pole coupled dominantly to the $K\bar{K}$ channel) show no structure in the amplitude squared. Due to large inelasticity these contributions produce only a small circle at the high energy tail of $f_2(1270)$. The K -matrix parameters found in this solution are given in Table 6.2 (Appendix 6.A).

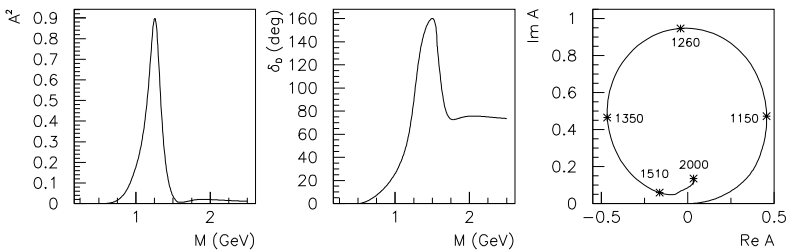


Fig. 6.25 From left to right: The $\pi\pi \rightarrow \pi\pi$ D-wave amplitude squared, the amplitude phase and Argand diagram.

Table 6.1 Masses and couplings (in GeV) for S-wave K -matrix poles (f_0^{bare} states). The II sheet is defined by $\pi\pi$ and 4π cuts, the IV by $\pi\pi$, 4π , $K\bar{K}$ and $\eta\eta$ cuts, and the V sheet by $\pi\pi$, 4π , $K\bar{K}$, $\eta\eta$ and $\eta\eta'$ cuts.

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
M	$0.650^{+.120}_{-.050}$	$1.230^{+.040}_{-.030}$	$1.220^{+.030}_{-.030}$	$1.540^{+.030}_{-.020}$	$1.820^{+.040}_{-.040}$
$g_0^{(\alpha)}$	$0.910^{+.80}_{-.100}$	$0.920^{+.080}_{-.080}$	$0.530^{+.050}_{-.050}$	$0.300^{+.040}_{-.040}$	$0.480^{+.050}_{-.050}$
$g_5^{(\alpha)}$	0	0	$0.940^{+.100}_{-.100}$	$0.570^{+.070}_{-.070}$	$-0.900^{+.070}_{-.070}$
φ_α	$-(70^{+3}_{-15})$	12^{+8}_{-8}	49^{+8}_{-8}	11^{+10}_{-10}	-48^{+10}_{-10}
	$a = \pi\pi$	$a = K\bar{K}$	$a = \eta\eta$	$a = \eta\eta'$	$a = 4\pi$
f_{1a}	$0.060^{+.100}_{-.100}$	$0.150^{+.100}_{-.100}$ $f_{ba} = 0$	$0.300^{+.100}_{-.100}$ $b = 2, 3, 4, 5$	$0.300^{+.100}_{-.100}$	$0.0^{+.060}_{-.060}$
II sheet	Pole position				
	$1.020^{+.008}_{-.008}$ $-i(0.038^{+.008}_{-.008})$				
IV sheet		$1.340^{+.020}_{-.030}$ $-i(0.175^{+.020}_{-.040})$	$1.486^{+.010}_{-.010}$ $-i(0.067^{+.005}_{-.005})$	$1.450^{+.150}_{-.100}$ $-i(0.800^{+.100}_{-.150})$	
V sheet					$1.720^{+.020}_{-.020}$ $-i(0.180^{+.025}_{-.010})$

Table 6.2 Masses and couplings (in GeV) for D -wave K -matrix poles (f_2^{bare} states). The III sheet is defined by $\pi\pi$ and 4π and $K\bar{K}$ cuts, IV sheet by $\pi\pi$, 4π , $K\bar{K}$ and $\omega\omega$ cuts. The values marked by * were fixed in the fit.

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
M	1.254	1.540	1.570	1.940
$g_{\pi\pi}^{(\alpha)}$	0.620	0.05	0.250	-0.6
$g_{K\bar{K}}^{(\alpha)}$	0.250	0.5	0.150	0*
$g_{4\pi}^{(\alpha)}$	0.10	0.05	0.60	0.21
$g_{\omega\omega}^{(\alpha)}$	0*	0*	0.500	-0.5
	$a = \pi\pi$	$a = K\bar{K}$	$a = \omega\omega$	$a = 4\pi$
f_{1a}	0.05	0.15	0*	0*
		$f_{ba} = 0$	$b = 2, 3, 4, 5$	
	Pole position			
III sheet	1.270 -i 0.095	1.525 -i 0.075		
	Pole position			
IV sheet			1.570 -i 0.160	

6.3 Appendix 6.A. Three-meson production

$p\bar{p} \rightarrow \pi\pi\pi, \pi\pi\eta, \pi\eta\eta$

First, we present the formulae for the reactions $p\bar{p} \rightarrow \pi^0\pi^0\pi^0, \pi^0\pi^0\eta, \pi^0\eta\eta$ from the liquid H_2 , when annihilation occurs from the $^1S_0 p\bar{p}$ state and scalar resonances, f_0 and a_0 , are formed in the final state. This is a case which represents well the applied technique of the three-meson production reactions. A full set of amplitude terms taken into account in the analysis [56] (production of vector and tensor resonances, $p\bar{p}$ annihilation from the P -wave states $^3P_1, ^3P_2, ^1P_1$) is constructed in an analogous way.

(i) Production of the S -wave resonances.

For the transition $p\bar{p} (^1S_0) \rightarrow \pi^0\pi^0\pi^0$ with the production of two pions in a (00^{++}) -state, we use the following amplitude:

$$A_{p\bar{p} (^1S_0) \rightarrow \pi^0\pi^0\pi^0} = \left(\bar{\psi}(-q_2) \frac{i\gamma_5}{2\sqrt{2}m_N} \psi(q_1) \right) \quad (6.107)$$

$$\times [A_{p\bar{p} (^1S_0)\pi^0, \pi^0\pi^0}(s_{23}) + A_{p\bar{p} (^1S_0)\pi^0, \pi^0\pi^0}(s_{13}) + A_{p\bar{p} (^1S_0)\pi^0, \pi^0\pi^0}(s_{12})].$$

The four-spinors $\bar{\psi}(-p_2)$ and $\psi(p_1)$ refer to the initial antiproton and proton in the $I^{(2S+1)}L_J = 1^1S_0$ state. For the produced pseudoscalars we denote amplitudes in the left-hand side of (6.107) as $A_{p\bar{p} (^1S_0)P_\ell, P_i P_j}(s_{ij})$.

The amplitudes for the transitions $p\bar{p} (0^1S_0) \rightarrow \eta\pi^0\pi^0, p\bar{p} (1^1S_0) \rightarrow$

$\pi^0\eta\eta$ have a similar form:

$$A_{p\bar{p} (0^1 S_0) \rightarrow \eta\pi^0\pi^0} = \left(\bar{\psi}(-p_2) \frac{i\gamma_5}{2\sqrt{2}m_N} \psi(p_1) \right) \quad (6.108)$$

$$\times [A_{p\bar{p} (0^1 S_0)\eta,\pi^0\pi^0}(s_{23}) + A_{p\bar{p} (0^1 S_0)\pi^0,\eta\pi^0}(s_{13}) + A_{p\bar{p} (0^1 S_0)\pi^0,\eta\pi^0}(s_{12})] ,$$

and

$$A_{p\bar{p} (1^1 S_0) \rightarrow \pi^0\eta\eta} = \left(\bar{\psi}(-p_2) \frac{i\gamma_5}{2\sqrt{2}m_N} \psi(p_1) \right) \quad (6.109)$$

$$\times [A_{p\bar{p} (1^1 S_0)\pi^0,\eta\eta}(s_{23}) + A_{p\bar{p} (1^1 S_0)\eta,\eta\pi^0}(s_{13}) + A_{p\bar{p} (1^1 S_0)\eta,\eta\pi^0}(s_{12})] .$$

For the description of the S -wave interaction of two mesons in the scalar-isoscalar state (index (00)) the following amplitudes are used in (6.107), (6.108) and (6.109):

$$A_{p\bar{p} (I^1 S_0)\pi^0,b}(s_{ij}) = \sum_a \tilde{K}_{p\bar{p}(I^1 S_0)\pi^0,a}^{(00)}(s_{ij}) \left[\hat{I} - i\hat{\rho}_{ij}^{(0)}(s_{ij}) \hat{K}^{(00)}(s_{23}) \right]_{ab}^{-1} . \quad (6.110)$$

Here $b = \pi^0\pi^0, \eta\eta$ and $a = \pi^0\pi^0, \eta\eta, K\bar{K}, \eta\eta', \pi^0\pi^0\pi^0\pi^0$. The K -matrix term responsible for meson scattering is given in Appendix 3.B of Chapter 3. The $\tilde{K}$ -matrix terms which describe the prompt resonance and background meson production in the $p\bar{p}$ annihilation read:

$$\begin{aligned} \tilde{K}_{p\bar{p}(1^1 S_0)\pi^0,a}^{(00)}(s_{23}) = & \left(\sum_{\alpha} \frac{\Lambda_{p\bar{p}(1^1 S_0)\pi^0,a}^{(00,\alpha)} g_a^{(\alpha)}}{M_{\alpha}^2 - s_{23}} \right. \\ & \left. + \phi_{p\bar{p}(1^1 S_0)\pi^0,a}^{(00)} \frac{1 \text{ GeV}^2 + \tilde{s}_0}{s_{23} + \tilde{s}_0} \right) \left(\frac{s_{23} - \tilde{s}_A}{s_{23} + \tilde{s}_{A0}} \right) . \end{aligned} \quad (6.111)$$

The parameters $\Lambda_{p\bar{p}(1^1 S_0)\pi^0,a}^{(00,\alpha)}$ and $\phi_{p\bar{p}(1^1 S_0)\pi^0,a}^{(00)}$ are complex-valued, with different phases due to three-particle interactions. Let us recall: the matter is that in the final state interaction term we take into account the leading (pole) singularities only. The next-to-leading singularities are accounted for effectively, by considering the vertices $p\bar{p} \rightarrow \text{mesons}$ as complex factors.

(ii) Three-meson amplitudes with the production of spin-non-zero resonances.

In the three-meson production processes, the final-state two-meson interactions in other states are taken into account in a way similar to what was considered above.

The invariant part of the production amplitude $A_{p\bar{p} (I^1 S_0,b)}^{(I,tj)}(23)$ for the transition $p\bar{p} (I^1 S_0) \rightarrow 1 + (2+3)_{tj}$, where the indices tj refer to the

isospin and spin of the meson in the channel $b = 2 + 3$, is as follows:

$$A_{p\bar{p} (I^1 S_0)1,b}^{(tj)}(23) = \sum_a \tilde{K}_{p\bar{p} (I^1 S_0)1,a}^{(tj)}(s_{23}) \left[\hat{I} - i \hat{\rho}_{23}^{(j)} \hat{K}^{(tj)}(s_{23}) \right]_{ab}^{-1},$$

$$\tilde{K}_{p\bar{p} (I^1 S_0)1,a}^{(tj)}(s_{23}) = \left(\sum_{\alpha} \frac{\Lambda_{p\bar{p} (I^1 S_0)1}^{(tj,\alpha)} g_a^{(\alpha)}}{M_{\alpha}^2 - s_{23}} + \phi_{p\bar{p} (I^1 S_0)1,a}^{(tj)} \frac{1 \text{ GeV}^2 + \tilde{s}_{tj0}}{s_{23} + \tilde{s}_{tj0}} \right) D_a(s_{23}). \quad (6.112)$$

The parameters $\Lambda_{p\bar{p} (I^1 S_0)1}^{(tj,\alpha)}$, $\phi_{p\bar{p} (I^1 S_0)1,a}^{(tj)}$ may be complex-valued, with different phases due to three-particle interactions.

The K -matrix elements for the scattering amplitudes (which enter the denominator of (6.112)) are determined in the partial waves 02^{++} , 10^{++} , 12^{++} as follows:

(1) *Isoscalar-tensor, 02^{++} , partial wave.*

The D -wave interaction in the isoscalar sector is parametrised by the 4×4 K -matrix where $1 = \pi\pi$, $2 = K\bar{K}$, $3 = \eta\eta$ and $4 = \text{multi-meson states}$:

$$K_{ab}^{(02)}(s) = D_a(s) \left(\sum_{\alpha} \frac{g_a^{(\alpha)} g_b^{(\alpha)}}{M_{\alpha}^2 - s} + f_{ab}^{(02)} \frac{1 \text{ GeV}^2 + s_2}{s + s_2} \right) D_b(s). \quad (6.113)$$

Factor $D_a(s)$ stands for the D -wave centrifugal barrier. We take this factor in the following form:

$$D_a(s) = \frac{k_a^2}{k_a^2 + 3/r_a^2}, \quad a = 1, 2, 3, \quad (6.114)$$

where $k_a = \sqrt{s/4 - m_a^2}$ is the momentum of the decaying meson in the c.m. frame of the resonance. For the multi-meson decay the factor $D_4(s)$ is taken to be 1. The phase space factors we use are the same as those for the isoscalar S -wave channel.

(2) *Isvector-scalar, 10^{++} , and isovector-tensor, 12^{++} , partial waves.*

For the amplitude in the isovector-scalar and isovector-tensor channels we use the 4×4 K -matrix with $1 = \pi\eta$, $2 = K\bar{K}$, $3 = \pi\eta'$ and $4 = \text{multi-meson states}$:

$$K_{ab}^{(1j)}(s) = D_a(s) \left(\sum_{\alpha} \frac{g_a^{(\alpha)} g_b^{(\alpha)}}{M_{\alpha}^2 - s} + f_{ab} \frac{1.5 \text{ GeV}^2 + s_1}{s + s_1} \right) D_b(s). \quad (6.115)$$

Here $j = 0, 2$; the factors $D_a(s)$ are equal to 1 for the 10^{++} amplitude, while for the D -wave partial amplitude the factor $D_a(s)$ is taken in the form

$$D_a(s) = \frac{k_a^2}{k_a^2 + 3/r_3^2}, \quad a = 1, 2, 3, \quad D_4(s) = 1. \quad (6.116)$$

6.4 Appendix 6.B. Reggeon Exchanges in the Two-Meson Production Reactions — Calculation Routine and Some Useful Relations

Here we present calculation details for the method of partial wave analysis of the πN interaction based on the reggeon exchanges. The reggeon exchange approach is a good tool for studying the interference effects in the amplitudes thus providing valuable information about contributions of the resonances with different quantum numbers to the particular partial wave – the calculation details important for understanding this technique.

Kinematics for reggeon exchange amplitudes

For illustration, we consider the reaction $\pi^- p \rightarrow \pi\pi + n$ in the c.m. system of the reaction and present the momenta of the incoming and outgoing particles (below we use the notation $p = (p_0, \vec{p}_\perp, p_z)$ for the four-vectors).

For the incoming particles we have:

$$\begin{aligned} \text{pion momentum : } p_1 &= (p_z + \frac{m_\pi^2}{2p_z}, 0, p_z), \\ \text{proton momentum : } p_2 &= (p_z + \frac{m_N^2}{2p_z}, 0, -p_z), \\ \text{total energy squared : } s_{\pi N} &= (p_1 + p_2)^2. \end{aligned} \quad (6.117)$$

Here we have performed an expansion over the large momentum p_z . Analogously, we write for the outgoing particles:

$$\begin{aligned} \text{total momentum of mesons : } P &= (p_z + \frac{s + m_\pi^2 + 2q_\perp^2}{4p_z}, \vec{q}_\perp, p_z - \frac{s - m_\pi^2}{4p_z}), \\ \text{proton momentum : } k_3 &= (p_z - \frac{s - m_\pi^2 + 2q_\perp^2}{4p_z}, -\vec{q}_\perp, -p_z + \frac{s - m_\pi^2}{4p_z}), \\ \text{meson momenta } (i = 1, 2) : k_i &= (k_{iz} - \frac{m_i^2 + k_{i\perp}^2}{2k_{iz}}, \vec{k}_{i\perp}, k_{iz}), \\ \text{energy squared of mesons : } s &= P^2 = (k_1 + k_2)^2. \end{aligned} \quad (6.118)$$

The momentum squared transferred to the nucleon is comparatively small: $t \equiv q^2 \sim m_N^2 \ll s_{\pi N}$ where

$$q = (-\frac{s + m_\pi^2 + 2q_\perp^2}{4p_z}, -\vec{q}_\perp, \frac{s - m_\pi^2}{4p_z}). \quad (6.119)$$

Neglecting $O(1/p_z^2)$ -terms, one has $q^2 \simeq -q_\perp^2$.

6.4.1 Reggeised pion exchanges

Here we present formulae which lead to differential cross section moment expansion in processes related to reggeised pion exchanges.

6.4.1.1 Calculation routine for the reggeised pion exchange

For meson momenta we use the notations:

$$\begin{aligned}
 P &= p_1 - q = k_1 + k_2, \quad k = \frac{1}{2}(k_1 - k_2), \quad p = \frac{1}{2}(p_1 + q), \\
 p_{\perp\mu} &= \frac{1}{2}(p_1 + q)_{\nu} g_{\nu\mu}^{\perp P} = p_{1\nu} g_{\nu\mu}^{\perp P} = q_{\nu} g_{\nu\mu}^{\perp P} \\
 &= \frac{1}{2} \left[p_{1\mu} \left(1 - \frac{m_{\pi}^2 - q^2}{s} \right) + q_{\mu} \left(1 + \frac{m_{\pi}^2 - q^2}{s} \right) \right], \\
 k_{\perp\mu} &= \frac{1}{2}(k_1 - k_2)_{\nu} g_{\nu\mu}^{\perp P} = k_{1\nu} g_{\nu\mu}^{\perp P} = -k_{2\nu} g_{\nu\mu}^{\perp P} \equiv k_{\mu}. \quad (6.120)
 \end{aligned}$$

Recall that $g_{\mu\nu}^{\perp P} = g_{\mu\nu} - P_{\mu}P_{\nu}/s \equiv g_{\mu\nu}^{\perp}$, and the operators for S - and D -waves are introduced as follows: $X^{(0)}(k) = 1$, $X_{\mu_1\mu_2}^{(2)}(k) = 3/2 (k_{\mu_1}k_{\mu_2} - 1/3 g_{\mu_1\mu_2}^{\perp} k^2)$; for $J > 2$ see Chapter 4. The projection operators, being constructed of metric tensors $g_{\mu\nu}^{\perp}$, obey the relations:

$$\begin{aligned}
 O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}(\perp P) X_{\nu_1 \dots \nu_J}^{(J)}(k_{\perp}) &= X_{\mu_1 \dots \mu_J}^{(J)}(k_{\perp}), \\
 O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}(\perp P) k_{\nu_1} k_{\nu_2} \dots k_{\nu_J} &= \frac{1}{\alpha_J} X_{\mu_1 \dots \mu_J}^{(J)}(k_{\perp}). \quad (6.121)
 \end{aligned}$$

Hence, the product of the two X^J operators results in the Legendre polynomials as follows:

$$\begin{aligned}
 X_{\mu_1 \dots \mu_J}^{(J)}(p_{\perp}) (-1)^J O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}(\perp P) X_{\nu_1 \dots \nu_J}^{(J)}(k_{\perp}) &= \alpha_J (\sqrt{-p_{\perp}^2} \sqrt{-k_{\perp}^2})^J P_J(z), \\
 z &\equiv \frac{(-p_{\perp}^{\perp} k_{\perp}^{\perp})}{\sqrt{-p_{\perp}^2} \sqrt{-k_{\perp}^2}}, \quad (6.122)
 \end{aligned}$$

where $k_{\perp}^2 = k_{\mu}^{\perp} g_{\mu\nu} k_{\nu}^{\perp}$. Then the transition amplitude can be rewritten as:

$$\begin{aligned}
 A(\pi R(\pi_j) \rightarrow \pi\pi) &= 16\pi \sum_J A_{\pi R(\pi_j) \rightarrow \pi\pi}^J(s) (2J+1) N_J^0 Y_J^0(z, \varphi), \quad (6.123) \\
 (\sqrt{-p_{\perp}^2} \sqrt{-k_{\perp}^2})^J Y_J^m(z, \varphi) &= \frac{1}{N_J^m} P_J^m(z) e^{im\varphi}, \quad N_J^m = \sqrt{\frac{4\pi}{2J+1} \frac{(J+m)!}{(J-m)!}}.
 \end{aligned}$$

Let us consider the case of decay amplitudes in the set of channels with two pseudoscalar mesons in the final states. For isosinglet amplitudes these are

$\pi\pi, K\bar{K}, \eta\eta$ and so on — we denote these channels as $f, b, c, \dots$. Then the unitarity condition for the transition amplitude reads:

$$\begin{aligned} \text{Im} X_{\mu_1 \dots \mu_J}^{(J)}(p^\perp) A_{a \rightarrow n}^J(s) (-1)^J O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J} X_{\nu_1 \dots \nu_J}^{(J)}(p'^\perp) \xi_J = \sum_i X_{\mu_1 \dots \mu_J}^{(J)}(p^\perp) \\ \times A_{a \rightarrow b}^J(s) (-1)^J O_{\beta_1 \dots \beta_J}^{\mu_1 \dots \mu_J} \int \frac{d\Omega_b}{4\pi} X_{\beta_1 \dots \beta_J}^{(J)}(k_b^\perp) \frac{\sqrt{-k_{b\perp}^2}}{8\pi\sqrt{s}} X_{\chi_1 \dots \chi_J}^{(J)}(k_b^\perp) \\ \times (-1)^J O_{\nu_1 \dots \nu_J}^{\chi_1 \dots \chi_J} A_{b \rightarrow c}^{J*}(s) X_{\nu_1 \dots \nu_J}^{(J)}(p'^\perp) \xi_J^2. \end{aligned} \quad (6.124)$$

Here $p^\perp$ is the relative momentum in the channel a , $k_b^\perp$ is the relative momentum in the intermediate channel b (Ω_b is its solid angle) and $p'^\perp$ is the relative momentum in the final channel c . Taking into account that

$$\int \frac{d\Omega_b}{4\pi} X_{\beta_1 \dots \beta_J}^{(J)}(k_b^\perp) X_{\chi_1 \dots \chi_J}^{(J)}(k_b^\perp) = \frac{\alpha_J}{2J+1} O_{\chi_1 \dots \chi_J}^{\beta_1 \dots \beta_J} (-1)^J (-k_{b\perp}^2)^J, \quad (6.125)$$

we write the unitarity condition as follows:

$$\text{Im} A_{a \rightarrow c}^J(s) = \sum_b \frac{2\sqrt{-k_{b\perp}^2}}{\sqrt{s}} A_{a \rightarrow b}^J(s) A_{b \rightarrow c}^{J*}(s) (-k_{c\perp}^2)^J. \quad (6.126)$$

In the K -matrix form this condition is satisfied if

$$A_{a \rightarrow c}^J(s) = \sum_b \hat{K}_{ab}^J (I - i\hat{\rho}^J(s) \hat{K}^J)_{bc}^{-1}, \quad (6.127)$$

where $\hat{\rho}$ is a diagonal matrix with elements $\rho_{bb}^J(s) = 2\sqrt{-k_{b\perp}^2} (-k_{b\perp}^2)^J / \sqrt{s}$. Here we parametrise the elements of the K -matrix as follows:

$$\begin{aligned} K_{ab}^J = \sum_\alpha \frac{1}{B_J(-k_{a\perp}^2, r_\alpha)} \left(\frac{g_a^{\alpha(J)} g_c^{\alpha(J)}}{M_\alpha^2 - s} \right) \frac{1}{B_J(-k_{c\perp}^2, r_\alpha)} \\ + \frac{f_{ac}^{(J)}}{B_J(-k_{a\perp}^2, r_0) B_J(-k_{c\perp}^2, r_0)}. \end{aligned} \quad (6.128)$$

In (6.128) the resonance couplings g_c^α are constants, and f_{ac} is a non-resonance transition amplitude. The form factors $B_J(-k_\perp^2, r)$ are introduced to compensate the divergence of the relative momentum factor at large energies. Such form factors are known as the Blatt–Weisskopf factors depending on the radius of the state r_α . For non-resonance transition the radius is taken to be much larger than that for resonance contributions.

In the case of virtual pion exchange the initial-state K -matrix elements are called the P -vector $K_{\pi R(\pi_j) \rightarrow b}^J \equiv P_{\pi R(\pi_j) \rightarrow b}^J$. Following this tradition, we use for the reggeon exchange a similar notation:

$$A_{\pi R(\pi_j) \rightarrow c}^J(s) = \sum_b P_{\pi R(\pi_j) \rightarrow b}^J i \left(I - i\hat{\rho}^J(s) \hat{K}^J \right)_{bc}^{-1}. \quad (6.129)$$

The P -vector is parametrised in the form

$$P_{\pi R(\pi_j) \rightarrow c}^J = \sum_{\alpha} \frac{1}{B_J(-p_{\perp}^2, r_{\alpha})} \left(\frac{G_{\alpha}^{(J)} g_c^{\alpha(J)}}{M_{\alpha}^2 - s} \right) \frac{1}{B_J(-k_{c\perp}^2, r_{\alpha})} + \frac{F_c^{(J)}}{B_J(-p_{\perp}^2, r_0) B_J(-k_{c\perp}^2, r_0)}. \quad (6.130)$$

When the mass of the virtual pion tends to the mass of the real pion, the production couplings $G_{(J)}^{\alpha}$ and $F_c^{(J)}$ should turn to $g_{(J)1}^{\alpha}$ and $f_{1c}^{(J)}$, respectively. So, we parametrise:

$$G_{\alpha}^{(J)} = g_1^{\alpha(J)} + g_{add}^{\alpha(J)}(m_{\pi}^2 - t), \quad F_c^{(J)} = f_{1c}^{(J)} + f_{1c}^{add(J)}(m_{\pi}^2 - t). \quad (6.131)$$

The production of the two amplitudes equals:

$$A(\pi R(\pi_j) \rightarrow \pi\pi) A^*(\pi\pi_k \rightarrow \pi\pi) = (16\pi)^2 \sum_J Y_J^0(z, \varphi) \times \sum_{J_1 J_2} d_{J_1 J_2 J}^{000} A_{\pi R(\pi_j) \rightarrow \pi\pi}^{J_1}(s) A_{\pi\pi_k \rightarrow \pi\pi}^{J_2}(s) (2J_1 + 1)(2J_2 + 1) N_{J_1}^0 N_{J_2}^0, \quad (6.132)$$

where the coefficients d_{nmk}^{ij} are given below. Averaging over the polarisations of the initial nucleons and summing over the polarisation of the final ones, we get $\text{Sp}[(\vec{\sigma}\vec{q}_{\perp})(\vec{\sigma}\vec{q}_{\perp})] \simeq -q^2 = -t$. So we obtain for the total amplitude squared:

$$|A_{\pi p \rightarrow \pi\pi n}^{(\text{pion trajectories})}|^2 = \sum_{R(\pi_j)R(\pi_k)} A(\pi R(\pi_j) \rightarrow \pi\pi) A^*(\pi R(\pi_k) \rightarrow \pi\pi) \times R_{\pi_j}(s_{\pi N}, q^2) R_{\pi_k}^*(s_{\pi N}, q^2) (-t) (g_{pn}^{(\pi)})^2. \quad (6.133)$$

The final expression reads:

$$N(M, t) \langle Y_J^0 \rangle = \frac{\rho(s) \sqrt{s}}{\pi |\vec{p}_2|^2 s_{\pi N}} \sum_{R(\pi_j)R(\pi_k)} R_{\pi_j}(s_{\pi N}, q^2) R_{\pi_k}^*(s_{\pi N}, q^2) (-t) (g_{pn}^{(\pi)})^2 \times \sum_{J_1 J_2} d_{J_1 J_2 J}^{000} A_{\pi\pi_j \rightarrow \pi\pi}^{J_1}(s) A_{\pi R(\pi_k) \rightarrow \pi\pi}^{J_2}(s) (2J_1 + 1)(2J_2 + 1) N_{J_1}^0 N_{J_2}^0. \quad (6.134)$$

(i) Spherical functions.

Let us present here some relations for the spherical functions used in the calculations:

$$Y_l^m(\Theta, \varphi) = \sqrt{\frac{1}{N_{lm}}} P_l^m(z) e^{im\varphi}, \quad N_{lm} = \frac{4\pi}{2l+1} \frac{(n+m)!}{(l-m)!},$$

$$P_l^m(z) = (-1)^m (1-z^2)^{\frac{m}{2}} \frac{d^m}{dz^m} P_l(z), \quad (6.135)$$

where $z = \cos \Theta$. We have the following convolution rule for two spherical functions:

$$Y_n^i(\Theta, \varphi) Y_m^j(\Theta, \varphi) = \sum_{k=0}^{n+m} d_{n,m,k}^{ij} Y_k^{i+j}(\Theta, \varphi) . \quad (6.136)$$

Let us calculate the coefficients $d_{n,m,k}^{ij}$. The coefficients in the expansion of the Legendre polynomials have the form:

$$P_n(z) = \sum_{k=0}^n a_k^n z^k = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{n!} \times \left[z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} z^{n-4} - \dots \right]. \quad (6.137)$$

The reverse expression reads:

$$z^n = \sum_{k=0}^n b_k^n P_k(z), \quad b_k^n = (2k+1) \sum_{m=0}^k a_m^k \frac{(1 - (-1)^{n+m+1})}{n+m+1}. \quad (6.138)$$

For the derivatives of the Legendre polynomial we have:

$$\begin{aligned} \frac{d^i}{dz^i} P_n(z) &= \sum_{k=i}^n a_k^n \frac{k!}{(k-i)!} z^k, \\ \frac{d^\xi}{dz^\xi} P_n(z) \frac{d^\eta}{dz^\eta} P_m(z) &= \sum_{k=0}^{n+m} \frac{d^{\xi+\eta}}{dz^{\xi+\eta}} P_k f_{n,m,k}^{\xi\eta}, \\ f_{n,m,k}^{\xi\eta} &= \sum_{l=k}^{n+m} b_k^l \frac{(l-\xi-\eta)!}{l!} C_{n,m,l}^{\xi\eta}, \\ C_{n,m,l}^{\xi\eta} &= \sum_{i=0}^{\min(n,l)} a_i^n \frac{i!}{(i-\xi)!} a_{l-i}^m \frac{(l-i)!}{(l-i-\eta)!}. \end{aligned} \quad (6.139)$$

The $d_{n,m,k}^{ij}$ coefficients differ from $f_{n,m,k}^{\xi\eta}$ by the normalisation coefficients only:

$$d_{n,m,k}^{ij} = \sqrt{\frac{N_{k,i+j}}{N_{n,i} N_{m,j}}} (-1)^{i+j+k} f_{n,m,k}^{ij}. \quad (6.140)$$

6.4.1.2 Calculations related to the expansion of the differential cross section $\pi p \rightarrow \pi\pi + N$ over spherical functions for the reggeised π_2 -exchange

Here we present formulae which refer to the calculation routine related to the reggeised π_2 -exchange.

The convolution of angular momentum operators can be expressed through Legendre polynomials and their derivatives:

$$X_{\alpha\beta\mu_1\ldots\mu_J}^{(J+2)}(p_\perp)(-1)^J O_{\nu_1\ldots\nu_J}^{\mu_1\ldots\mu_J}(\perp P) X_{\nu_1\ldots\nu_J}^{(J)}(k_\perp) \quad (6.141)$$

$$= \frac{2\alpha_J \left(\sqrt{-k_\perp^2}\right)^J \left(\sqrt{-p_\perp^2}\right)^{J+2}}{3(J+1)(J+2)} \\ \times \left(X_{\mu\nu}^{(2)}(p_\perp) \frac{P_{J+2}''}{-p_\perp^2} + X_{\mu\nu}^{(2)}(k_\perp) \frac{P_J''}{-k_\perp^2} - \frac{3P_{J+1}''}{\sqrt{-k_\perp^2}\sqrt{-p_\perp^2}} k_\mu^\perp p_\nu^\perp \right) O_{\mu\nu}^{\alpha\beta}(\perp q) ,$$

$$O_{\chi\tau}^{\alpha\beta}(\perp q) X_{\chi\mu_2\ldots\mu_J}^{(J)}(p_\perp)(-1)^J O_{\nu_1\nu_2\ldots\nu_J}^{\tau\mu_2\ldots\mu_J}(\perp P) X_{\nu_1\ldots\nu_J}^{(J)}(k_\perp) \quad (6.142)$$

$$= \frac{2\alpha_{J-1}}{3J^2} \left(\sqrt{-k_\perp^2}\right)^J \left(\sqrt{-p_\perp^2}\right)^J \\ \times \left(X_{\mu\nu}^{(2)}(p_\perp) \frac{P_J''}{-p_\perp^2} + X_{\mu\nu}^{(2)}(k_\perp) \frac{P_J''}{-k_\perp^2} - \frac{P_J' + 2zP_J''}{\sqrt{-k_\perp^2}\sqrt{-p_\perp^2}} k_\mu^\perp p_\nu^\perp \right) O_{\mu\nu}^{\alpha\beta}(\perp q) ,$$

$$X_{m\nu_3\ldots\nu_J}^{(J-2)}(p_\perp)(-1)^J O_{\nu_1\nu_2\nu_3\ldots\nu_J}^{\alpha\beta\mu_3\ldots\mu_J}(\perp P) X_{\nu_1\ldots\nu_J}^{(J)}(k_\perp) \quad (6.143)$$

$$= \frac{2\alpha_{J-2} \left(\sqrt{-k_\perp^2}\right)^J \left(\sqrt{-p_\perp^2}\right)^{J-2}}{3(n-1)n} \\ \times \left(X_{\mu\nu}^{(2)}(p_\perp) \frac{P_{J-2}''}{-p_\perp^2} + X_{\mu\nu}^{(2)}(k_\perp) \frac{P_J''}{-k_\perp^2} - \frac{3P_{J-1}''}{\sqrt{-k_\perp^2}\sqrt{-p_\perp^2}} k_\mu^\perp p_\nu^\perp \right) O_{\mu\nu}^{\alpha\beta}(\perp q).$$

Let us remind that $p_\mu^\perp = p_{1\nu} g_{\nu\mu}^{\perp P}$ and $k_\mu^\perp = k_{1\nu} g_{\nu\mu}^{\perp P}$. Therefore the amplitude (6.84) can be rewritten as:

$$A_{\alpha\beta}(\pi R(\pi_2) \rightarrow \pi\pi) = \frac{2}{3} \sum_J \left[\frac{X_{\alpha\beta}^{(2)}(p^\perp)}{-p_\perp^2} \left(C_1^{(J)} P_{J+2}'' A_{+2}^{(J)}(s) + \right. \right. \\ \left. \left. + C_2^{(J)} P_J'' A_0^{(J)}(s) + C_3^{(J)} P_{J-2}'' A_{-2}^{(J)}(s) \right) \right. \\ \left. + \frac{X_{\alpha\beta}^{(2)}(k^\perp)}{-k_\perp^2} P_J'' \left(C_1^{(J)} A_{+2}^{(J)}(s) + C_2^{(J)} A_0^{(J)}(s) + C_3^{(J)} A_{-2}^{(J)}(s) \right) \right. \\ \left. - \frac{O_{\mu\nu}^{\alpha\beta} k_\mu^\perp p_\nu^\perp}{\sqrt{-k_\perp^2}\sqrt{-p_\perp^2}} \left(3C_1^{(J)} P_{J+1}'' A_{+2}^{(J)}(s) + C_2^{(J)} (P_J' + 2zP_J'') A_0^{(J)}(s) \right. \right. \\ \left. \left. + 3C_3^{(J)} P_{J-1}'' A_{-2}^{(J)}(s) \right) \right] , \quad (6.144)$$

where

$$\begin{aligned} C_1^{(J)} &= \frac{16\pi(2J+1)}{(J+1)(J+2)} \left(\sqrt{-k_\perp^2} \right)^J \left(\sqrt{-p_\perp^2} \right)^{J+2}, \\ C_2^{(J)} &= \frac{16\pi(2J+1)}{J(2J-1)} \left(\sqrt{-k_\perp^2} \right)^J \left(\sqrt{-p_\perp^2} \right)^J, \\ C_3^{(J)} &= \frac{16\pi(2J+1)}{(2J-1)(2J-3)} \left(\sqrt{-k_\perp^2} \right)^J \left(\sqrt{-p_\perp^2} \right)^{J-2}. \end{aligned} \quad (6.145)$$

In the amplitude with the $X_{\alpha\beta}^{(2)}(p^\perp)$ structure there is no $m = 1$ component. This amplitude should be taken effectively into account by the π trajectory. The second amplitude has the same angular dependence $P_J''(z)$ and works for resonances with $J \geq 2$. In the first approximation it is reasonable to use the third term only, which has the smallest power of $p_\perp^2$.

The third amplitude has angular dependences:

$$P_{J+1}'', \quad P_J' + 2zP_J'', \quad P_{J-1}''. \quad (6.146)$$

The first and second angular dependences are the same for $J = 1, 2$ and differ only at $n \geq 3$, when the third term appears. Therefore, in the first approximation one can use only the second term which has a lower order of $p_\perp^2$ to fit the data. Thus the π_2 exchange amplitude can be approximated as:

$$\begin{aligned} A_{\alpha\beta}(\pi\pi_2 \rightarrow \pi\pi) &\simeq \frac{2}{3} \sum_J \left[\frac{X_{\alpha\beta}^{(2)}(k^\perp)}{-k_\perp^2} P_J'' C_3^{(J)} A_{-2}^{(J)}(s) \right. \\ &\quad \left. - \frac{O_{\mu\nu}^{\alpha\beta} k_\mu^\perp p_\nu^\perp}{\sqrt{-k_\perp^2} \sqrt{-p_\perp^2}} C_2^{(J)} (P_J' + 2zP_J'') A_0^{(J)}(s) \right]. \end{aligned} \quad (6.147)$$

The convolution of operators in (6.147) with $k_{3\alpha}^q k_{3\beta}^q$ in the GJ system gives:

$$\begin{aligned} k_{3\alpha}^q k_{3\beta}^q X_{\alpha\beta}^{(2)}(k_\perp^2) &= |\vec{k}|^2 k_{3z}^q (k_{3z}^q P_2(z) + 3k_{3x} z \cos \varphi \sin \Theta), \\ k_{3\alpha}^q k_{3\beta}^q O_{\mu\nu}^{\alpha\beta} k_\mu^\perp p_\nu^\perp &= \frac{1}{3} |\vec{k}| |\vec{p}| k_{3z}^q (2k_{3z}^q z + 3k_{3x}^q \cos \varphi \sin \Theta), \end{aligned} \quad (6.148)$$

and the total amplitude (6.83) is equal to:

$$\begin{aligned} A_{\pi p \rightarrow \pi \pi n}^{(\pi_2)} &= \frac{1}{s_{\pi N}^2} \sum_J \left(V_1^J A_{-2}^{(J)}(s) - V_2^J A_0^{(J)}(s) \right) R_{\pi_2}(s_{\pi N}, q^2) \\ &\quad \times (\varphi_n^+(\vec{\sigma} \vec{p}_\perp) \varphi_p) g_{pn}^{(\pi_2)}, \end{aligned} \quad (6.149)$$

where

$$\begin{aligned} V_1^{(J)} &= C_3^{(J)} k_{3z}^q (k_{3z}^q P_2(z) + 3k_{3x} z \cos \varphi \sin \Theta) P_J'', \\ V_2^{(J)} &= \frac{1}{3} C_2^{(J)} k_{3z}^q (P_J' + 2z P_J'') (2k_{3z}^q z + 3k_{3x}^q \cos \varphi \sin \Theta). \end{aligned} \quad (6.150)$$

For $J = 1$ the first vertex is equal to 0; for the second one the expression reads:

$$V_2^{(1)} = \frac{1}{3} C_2^{(1)} k_{3z}^q (2k_{3z}^q Y_1^0 N_1^0 - 3k_{3x}^q \operatorname{Re} Y_1^1 N_1^1). \quad (6.151)$$

Here

$$Y_n^0 = \frac{1}{N_n^0} P_n(z), \quad Y_n^1 = -\frac{1}{N_n^1} \sin \Theta P_n'(z) e^{-i\varphi}. \quad (6.152)$$

For $J = 2$:

$$P_2(z) = \frac{1}{2}(3z^2 - 1), \quad P_2'(z) = 3z, \quad P_2'' = 3. \quad (6.153)$$

Then

$$\begin{aligned} (P_2' + 2z P_2'') 2z &= 18z^2 = 12P_2(z) + 6P_0(z), \\ (P_2' + 2z P_2'') 3 &= 27z = 9P_2'(z), \end{aligned} \quad (6.154)$$

and thus

$$\begin{aligned} V_1^{(2)} &= C_3^{(2)} k_{3z}^q (k_{3z}^q Y_2^0 N_2^0 - k_{3x} \operatorname{Re} Y_2^1 N_2^1), \\ V_2^{(2)} &= \frac{1}{3} C_2^{(2)} k_{3z}^q (12k_{3z}^q Y_2^0 N_2^0 + 6k_{3z}^q Y_0^0 N_0^0 - 9k_{3x}^q \operatorname{Re} Y_2^1 N_2^1). \end{aligned} \quad (6.155)$$

For $J = 3$:

$$P_3(z) = \frac{1}{2}(5z^3 - 3z), \quad P_3'(z) = \frac{3}{2}(5z^2 - 1), \quad P_3'' = 15z. \quad (6.156)$$

Then

$$\begin{aligned} (P_3' + 2z P_3'') 2z &= 3(25z^3 - z) = 30P_3(z) + 42P_1(z), \\ (P_3' + 2z P_3'') 3 &= \frac{9}{2}(25z^2 - 1) = 15P_3'(z) + 18P_1'(z), \\ P_3'' P_2(z) &= \frac{15}{2}(3z^3 - z) = 9P_3(z) + 18P_1(z), \\ P_3'' 3z &= 45z^2 = 6P_3'(z) + 9P_1'(z). \end{aligned} \quad (6.157)$$

Consequently,

$$\begin{aligned} V_1^{(3)} &= C_3^{(3)} k_{3z}^q (9k_{3z}^q Y_3^0 N_3^0 + 18k_{3z}^q Y_1^0 N_1^0 - 6k_{3x} \operatorname{Re} Y_3^1 N_3^1 - 9k_{3x} \operatorname{Re} Y_1^1 N_1^1) \\ V_2^{(3)} &= \frac{1}{3} C_2^{(3)} k_{3z}^q (30k_{3z}^q Y_3^0 N_3^0 + 42k_{3z}^q Y_1^0 N_1^0 - 15k_{3x}^q \operatorname{Re} Y_3^1 N_3^1 \\ &\quad - 18k_{3x}^q \operatorname{Re} Y_1^1 N_1^1). \end{aligned} \quad (6.158)$$

For $J = 4$:

$$\begin{aligned} P_4(z) &= \frac{1}{8}(35z^4 - 30z^2 + 3), & P'_4(z) &= \frac{1}{2}(35z^3 - 15z), \\ P''_4 &= \frac{15}{2}(7z^2 - 1), \end{aligned} \quad (6.159)$$

and

$$\begin{aligned} (P'_4 + 2zP''_4) 2z &= 245z^4 - 45z^2 = 56P_4(z) + 110P_2(z) + 34P_0(z), \\ (P'_4 + 2zP''_4) 3 &= \frac{3}{2}(245z^3 - 45z) = 21P'_4(z) + 30P'_2(z), \\ P''_4 P_2(z) &= \frac{15}{4}(21z^4 - 10z^2 + 1) = 18P_4(z) + 20P_2(z) + 7P_0(z), \\ P''_4 3z &= \frac{45}{2}(7z^3 - z) = 9P'_4(z) + 15P'_2(z). \end{aligned} \quad (6.160)$$

Hence, for $n = 4$:

$$\begin{aligned} V_1^{(4)} &= C_3^{(4)} k_{3z}^q (18k_{3z}^q Y_4^0 N_4^0 + 20k_{3z}^q Y_2^0 N_2^0 + 7k_{3z}^q Y_0^0 N_0^0 \\ &\quad - 9k_{3x} \text{Re } Y_4^1 N_4^1 - 15k_{3x} \text{Re } Y_2^1 N_2^1), \\ V_2^{(4)} &= \frac{C_2^{(4)}}{3} k_{3z}^q [k_{3z}^q (56Y_4^0 N_4^0 + 110Y_2^0 N_2^0 + 34Y_0^0 N_0^0) \\ &\quad - k_{3x}^q (21 \text{Re } Y_4^1 N_4^1 - 30 \text{Re } Y_2^1 N_2^1)] . \end{aligned} \quad (6.161)$$

In a general form, the expression can be written as:

$$\begin{aligned} V_1^{(J)} &= \sum_{n=0}^J C_3^{(J)} k_{3z}^q [k_{3z}^q Y_n^0 R_n^0 (P_2 P_J'') + 3k_{3x}^q \text{Re } Y_n^1 R_n^1 (z P_J'')], \\ V_2^{(J)} &= \sum_{n=0}^J C_2^{(J)} k_{3z}^q \left[\frac{2}{3} k_{3z}^q Y_n^0 R_n^0 (z (P_J' + 2z P_J'')) \right. \\ &\quad \left. + k_{3x}^q \text{Re } Y_n^1 R_n^1 (P_J' + 2z P_J'') \right], \end{aligned} \quad (6.162)$$

where

$$\begin{aligned} R_n^0(f) &= \int \frac{d\Omega}{4\pi} f(z) Y_n^0(z, \Theta), \\ R_n^1(f) &= 2 \int \frac{d\Omega}{4\pi} f(z) \cos \varphi \sin \Theta \text{Re } Y_n^1(z, \Theta). \end{aligned} \quad (6.163)$$

The P -vector amplitudes for π_2 exchanges read:

$$\begin{aligned} A_{-2}^{(J)}(s) &= \hat{P}_{-2}^{(J)} (I - i \hat{\rho}^J(s) \hat{K}^J)^{-1}, \\ A_0^{(J)}(s) &= \hat{P}_0^{(J)} (I - i \hat{\rho}^J(s) \hat{K}^J)^{-1}. \end{aligned} \quad (6.164)$$

The P -vector components are parametrised in the form:

$$\begin{aligned}
 \left(P_{-2}^{(J)}\right)_n &= \sum_{\alpha} \frac{1}{B_{J-2}(-p_{\perp}^2, r_{\alpha})} \left(\frac{G_{-2}^{(J)\alpha} g_n^{\alpha(J)}}{M_{\alpha}^2 - s} \right) \frac{1}{B_J(-k_{n\perp}^2, r_{\alpha})} \\
 &\quad + \frac{F_{(-2)n}^{(J)}}{B_{J-2}(-p_{\perp}^2, r_0) B_J(-k_{n\perp}^2, r_0)}, \\
 \left(P_{(0)}^{(J)}\right)_n &= \sum_{\alpha} \frac{1}{B_J(-p_{\perp}^2, r_{\alpha})} \left(\frac{G_0^{(J)\alpha} g_n^{\alpha(J)}}{M_{\alpha}^2 - s} \right) \frac{1}{B_J(-k_{n\perp}^2, r_{\alpha})} \\
 &\quad + \frac{F_{(0)n}^{(J)}}{B_J(-p_{\perp}^2, r_0) B_J(-k_{n\perp}^2, r_0)} \quad (6.165)
 \end{aligned}$$

The total amplitude of the π_2 exchange can be rewritten as an expansion over spherical functions:

$$A_{\pi p \rightarrow \pi \pi n}^{(\pi_2)} = \sum_{n=0}^N \left(Y_n^0 A_{tot}^{0(n)}(s) + Y_n^1 A_{tot}^{1(n)}(s) \right) R_{\pi_2}(s_{\pi N}, q^2) (\varphi_n^+(\vec{\sigma} \vec{p}_{\perp}) \varphi_p) g_{pn}^{(\pi_2)}, \quad (6.166)$$

where

$$\begin{aligned}
 A_{tot}^{0(n)}(s) &= \frac{1}{s_{\pi N}^2} (k_{3z}^q)^2 \sum_J \left[R_n^0(P_2 P_J'') C_3^{(J)} A_{-2}^{(J)}(s) \right. \\
 &\quad \left. - \frac{2}{3} R_n^0(z(P_J' + 2zP_J'')) C_2^{(J)} A_0^{(J)}(s) \right], \\
 A_{tot}^{1(n)}(s) &= \frac{1}{s_{\pi N}^2} k_{3z}^q k_{3x} \sum_J \left[3R_n^1(P_2 P_J'') C_3^{(J)} A_{-2}^{(J)}(s) \right. \\
 &\quad \left. - R_n^1(z(P_J' + 2zP_J'')) C_2^{(J)} A_0^{(J)}(s) \right]. \quad (6.167)
 \end{aligned}$$

Then the final expression is:

$$\begin{aligned}
 N(M, t) \langle Y_J^0 \rangle &= \frac{\rho(s) \sqrt{s}}{\pi |\vec{p}_2|^2 s_{\pi N}} R_{\pi_2}(s_{\pi N}, q^2) R_{\pi_2}^*(s_{\pi N}, q^2) (-t) (g_{pn}^{(\pi_2)})^2 \\
 &\times \sum_{n,m} \left[d_{n,m,J}^{000} A_{tot}^{0(n)}(s) A_{tot}^{0(m)*}(s) + d_{n,m,J}^{110} A_{tot}^{1(n)}(s) A_{tot}^{1(m)*}(s) \right], \\
 N(M, t) \langle Y_J^1 \rangle &= \frac{\rho(s) \sqrt{s}}{\pi |\vec{p}_2|^2 s_{\pi N}} R_{\pi_2}(s_{\pi N}, q^2) R_{\pi_2}^*(s_{\pi N}, q^2) (-t) (g_{pn}^{(\pi_2)})^2 \\
 &\times \sum_{n,m} \left[d_{n,m,J}^{101} A_{tot}^{1(n)}(s) A_{tot}^{0(m)*}(s) + d_{n,m,J}^{011} A_{tot}^{0(n)}(s) A_{tot}^{1(m)*}(s) \right]. \quad (6.168)
 \end{aligned}$$

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Chapter 7

Photon Induced Hadron Production, Meson Form Factors and Quark Model

In this chapter we consider some typical photon induced hadron production reactions, the spin-orbital operator expansion for these reactions and constraints imposed on the amplitudes by gauge invariance and analyticity. Form factors for mesons treated as $q\bar{q}$ systems are considered in the non-relativistic quark model approach and in terms of the relativistic spectral integral technique.

Photon-photon collisions (with both real photons and virtual ones) resulting in the production of hadrons play an important role in the determination of the quark-gluon content of mesons. We consider the amplitudes of photon-photon collisions first for virtual photons $\gamma^*(q_1)\gamma^*(q_2) \rightarrow \text{hadrons}$ ($q_1^2 \neq 0, q_2^2 \neq 0$), then for real ones ($q_1^2 = 0, q_2^2 = 0$). As in the previous chapters, we carry out a partial-wave expansion of the amplitude using covariant operators of the angular momenta [1]. To be more illustrative, we consider the photoproduction of a nucleon-antinucleon pair ($\gamma^*\gamma^* \rightarrow N\bar{N}$) and of two pseudoscalar particles ($\gamma^*\gamma^* \rightarrow P_1P_2$).

Coming to the case of a real photon we face a phenomenon which is rather important for the consideration of amplitudes of the radiative processes, that is, a decrease of the number of independent operators in the expansion of amplitudes. We show that this decrease is accompanied by the appearance of nilpotent operators. The existence of nilpotent operators leads to ambiguities in the operator expansion of amplitudes of photon-induced reactions. We discuss this problem in detail using as an example the reaction $\gamma\gamma^* \rightarrow \text{scalar state}$ (or, what is equivalent from the point of view of the operator expansion, the decays of the scalar state $S \rightarrow \gamma V$ and the vector state $V \rightarrow \gamma S$).

The process of the e^+e^- -annihilation, $e^+e^- \rightarrow \gamma^* \rightarrow \sum V \rightarrow \text{hadrons}$, is important for delimiting the regions of hard and soft processes. Here

we consider in detail the spin structure of the amplitude in the region of vector meson production. We concentrate our attention on the reaction $e^+e^- \rightarrow \gamma^* \rightarrow \phi(1020) \rightarrow \pi\pi\gamma$: in this process, first, all characteristic features of the discussed angular momentum expansion become apparent and, second, a way to analyse the final state resonance production is seen.

In the quark model description of photon induced reactions we consider meson form factors in the non-relativistic approach (discussing the dipole formula and the additive quark model approximation) and write the form factors in the relativistic double spectral integral representation.

The problem of the nilpotent operators emerges not only in the process $\gamma\gamma^* \rightarrow S$ but also in the reactions with the production of non-zero spin states such as $S \rightarrow \gamma V$, $P \rightarrow \gamma V$, $T \rightarrow \gamma V$, $A \rightarrow \gamma V$. We consider here these processes in terms of double spectral integrals and write the corresponding form factors, supposing that the mesons are quark–antiquark states. Constraints for $q\bar{q}$ wave functions (or for vertices of transitions $meson \rightarrow q\bar{q}$), which guarantee the quark confinement, are discussed.

The e^+e^- -annihilation plays a determinative role in studying the quark components of a photon wave function. Using the reactions $e^+e^- \rightarrow \gamma^* \rightarrow V$ and $e^+e^- \rightarrow \gamma^* \rightarrow u\bar{u}$, $d\bar{d}$, $s\bar{s}$ in soft and hard regions we find the quark–antiquark components of the photon wave function. On this basis we calculate amplitudes for decays $S(0^{++}) \rightarrow \gamma\gamma$, $P(0^{-+}) \rightarrow \gamma\gamma$ and $T(2^{++}) \rightarrow \gamma\gamma$; calculated partial widths are compared with the available experimental data. We briefly discuss also the nucleon form factors: we present quark–nucleon vertices in a general form and give examples of calculations of the nucleon form factors in the non-relativistic and relativistic approaches.

In the end of this chapter we perform the additive quark model calculations of nucleon form factors, both in the spectral integral technique and in non-relativistic approach. The calculations of nucleon (generally speaking – baryon) form factors are important for the systematisation and classification of states, in particular, for the region of high excitations. Still, the spectral integral technique for three-body systems is not duely developed now.

In Appendix 7.C we present a brief comment to the alternative approach — the QCD sum rules.

7.1 A System of Two Vector Particles

To give a complete presentation, we consider, first, the spin operator structure of a two-vector system in general. We suppose here that the initial state may be both a system of two different or two identical vector particles. Let us start with the case of two different vector particles.

7.1.1 General structure of spin-orbital operators for the system of two vector mesons

Consider a system of two different vector particles ($V_1 V_2$) which, thus, do not obey the symmetry condition. Let the momenta of these particles be q_1 and q_2 where $q_1^2 \neq 0$ and $q_2^2 \neq 0$. We denote the polarisation vectors of V_1 and V_2 as $\epsilon_\alpha^{(1)a}$ and $\epsilon_\beta^{(2)b}$; they satisfy the constraints $\epsilon_\alpha^{(1)a} q_{1\alpha} = \epsilon_\beta^{(2)b} q_{2\beta} = 0$ being characterised by three independent components ($a = 1, 2, 3$ and $b = 1, 2, 3$).

To describe the initial state, we use also the momenta

$$p = q_1 + q_2, \quad q = \frac{1}{2}(q_1 - q_2). \quad (7.1)$$

For a two-body system we introduce, as usual, the relative momentum $q^\perp$ which is orthogonal to the total momentum p : $(q^\perp p) = 0$. With the metric tensor $g_{\mu\nu}^\perp \equiv g_{\mu\nu}^\perp = g_{\mu\nu} - p_\mu p_\nu / p^2$, we write:

$$q_\mu^\perp = q_\nu g_{\nu\mu}^\perp = q_{1\nu} g_{\nu\mu}^\perp = -q_{2\nu} g_{\nu\mu}^\perp = q_\mu - \frac{q_1^2 - q_2^2}{2p^2} p_\mu. \quad (7.2)$$

We work also with the metric tensors which separate spaces orthogonal either to q_1 or to q_2 :

$$g_{\mu\nu}^{\perp q_n} = g_{\mu\nu} - \frac{q_{n\mu} q_{n\nu}}{q_n^2}, \quad n = 1, 2. \quad (7.3)$$

The vector particle has a spin $S_V = 1$ and hence, the spin of the initial system can take three different values: $S = 0, 1, 2$. At a fixed angular momentum L we have nine states:

$$\begin{aligned} S = 0 : \quad & L = J, \\ S = 1 : \quad & L = J + 1, J, J - 1, \\ S = 2 : \quad & L = J + 2, J + 1, J, J - 1, J - 2. \end{aligned} \quad (7.4)$$

Since the polarisation vectors $\epsilon^{(n)a}$ ($n = 1, 2$ and $a = 1, 2, 3$) are orthogonal to the momenta of the vector particles ($\epsilon^{(n)a} q_n = 0$), we can write the identity:

$$\epsilon_\alpha^{(n)a} = \epsilon_{\alpha'}^{(n)a} g_{\alpha'\alpha}^{\perp q_n}, \quad (7.5)$$

which is used below in the construction of wave functions.

Let us introduce spin wave functions for the vector particles with $S = |\mathbf{S}_{V_1} + \mathbf{S}_{V_2}| = 0, 1, 2$:

$$\begin{aligned} \mathbf{S}^{ab} &= (\epsilon^{(1)a} \epsilon^{(2)b}), \\ \mathbf{P}_\mu^{ab} &= \varepsilon_{\mu \epsilon^{(1)a} \epsilon^{(2)b} p} \equiv \varepsilon_{\mu \nu_1 \nu_2 \nu_3} \epsilon_{\nu_1}^{(1)a} \epsilon_{\nu_2}^{(2)b} p_{\nu_3}, \\ \mathbf{T}_{\mu_1 \mu_2}^{ab} &= \frac{1}{2} \left(\epsilon_{\mu_1}^{(1)a} \epsilon_{\mu_2}^{(2)b} + \epsilon_{\mu_2}^{(2)b} \epsilon_{\mu_1}^{(1)a} - \frac{g_{\mu_1 \xi}^{\perp q_1} g_{\xi \mu_2}^{\perp q_2} + g_{\mu_1 \xi}^{\perp q_2} g_{\xi \mu_2}^{\perp q_1}}{g_{\xi' \xi''}^{\perp q_1} g_{\xi' \xi''}^{\perp q_2}} (\epsilon^{(1)a} \epsilon^{(2)b}) \right). \end{aligned} \quad (7.6)$$

The spin state functions $\mathbf{S}^{ab}$ and $\mathbf{T}_{\mu_1 \mu_2}^{ab}$ are even while $\mathbf{P}_\mu^{ab}$ is odd under the permutation of particles 1 and 2 (simultaneous permutation $1a \rightleftharpoons 2b$ and $q_1 \rightleftharpoons q_2$).

For fixed J , the spin-orbital wave functions read:

$$\begin{aligned} \hat{Q}_{\mu_1 \mu_2 \dots \mu_J}^{V_1^a V_2^b (S=0, L=J, J)}(q) &= \mathbf{S}^{ab} X_{\mu_1 \dots \mu_J}^{(J)}(q^\perp) \\ \hat{Q}_{\mu_1 \dots \mu_J}^{V_1^a V_2^b (S=1, L=J+1, J)}(q) &= \mathbf{P}_\mu^{ab} X_{\mu_1 \dots \mu_J \mu}^{(J+1)}(q^\perp) \\ \hat{Q}_{\mu_1 \dots \mu_J}^{V_1^a V_2^b (S=1, L=J, J)}(q) &= \mathbf{P}_\mu^{ab} \varepsilon_{\mu \nu_1 \nu_2 p} Z_{\nu_1 \mu_1 \dots \mu_J, \nu_2}^{(J)}(q^\perp) \\ \hat{Q}_{\mu_1 \dots \mu_J}^{V_1^a V_2^b (S=1, L=J-1, J)}(q) &= \mathbf{P}_\mu^{ab} Z_{\mu_1 \dots \mu_J, \mu}^{(J-1)}(q^\perp) \\ \hat{Q}_{\mu_1 \dots \mu_J}^{V_1^a V_2^b (S=2, L=J+2, J)}(q) &= \mathbf{T}_{\nu_1 \nu_2}^{ab} X_{\mu_1 \dots \mu_J \nu_1 \nu_2}^{(J+2)}(q^\perp) \\ \hat{Q}_{\mu_1 \dots \mu_J}^{V_1^a V_2^b (S=2, L=J+1, J)}(q) &= \mathbf{T}_{\nu_1 \nu_2}^{ab} \varepsilon_{\nu_1 \nu_3 \nu_4 p} Z_{\nu_2 \nu_4 \mu_1 \dots \mu_J, \nu_3}^{(J+1)}(q^\perp) \\ \hat{Q}_{\mu_1 \dots \mu_J}^{V_1^a V_2^b (S=2, L=J, J)}(q) &= \mathbf{T}_{\mu'_1 \nu}^{ab} X_{\nu \mu'_2 \dots \mu'_J}^{(J)}(q^\perp) O_{\mu_1 \mu_2 \dots \mu_J}^{\mu'_1 \mu'_2 \dots \mu'_J}(\perp p) \\ \hat{Q}_{\mu_1 \dots \mu_J}^{V_1^a V_2^b (S=2, L=J-1, J)}(q) &= \mathbf{T}_{\nu_1 \mu'_1}^{ab} \varepsilon_{\nu_1 \nu_2 \nu_3 p} Z_{\nu_2 \mu'_2 \dots \mu'_J, \nu_3}^{(J-1)}(q^\perp) O_{\mu_1 \mu_2 \dots \mu_J}^{\mu'_1 \mu'_2 \dots \mu'_J}(\perp p) \\ \hat{Q}_{\mu_1 \dots \mu_J}^{V_1^a V_2^b (S=2, L=J-2, J)}(q) &= \mathbf{T}_{\mu'_1 \mu'_2}^{ab} X_{\mu'_3 \dots \mu'_J}^{(J-2)}(q^\perp) O_{\mu_1 \mu_2 \dots \mu_J}^{\mu'_1 \mu'_2 \dots \mu'_J}(\perp p) \end{aligned} \quad (7.7)$$

The general form of the operators $X_{\mu_1 \dots \mu_J}^{(J)}(q^\perp)$, $Z_{\mu_1 \dots \mu_J, \nu}^{(J-1)}(q^\perp)$ and $O_{\mu_1 \mu_2 \dots \mu_J}^{\mu'_1 \mu'_2 \dots \mu'_J}(\perp p)$ is introduced in Chapter 4 (Appendix 4.A).

For performing the calculation of form factors in the spectral integral technique, it is convenient to introduce spin operators. To do that, the spin wave functions $\mathbf{S}^{ab}$, $\mathbf{P}_\mu^{ab}$, $\mathbf{T}_{\mu_1 \mu_2}^{ab}$ are multiplied by $\epsilon_\alpha^{(1)a} \epsilon_\beta^{(2)b}$, and the summing is carried out over all three independent and orthogonal polarisation states ($a = 1, 2, 3$ and $b = 1, 2, 3$). In this procedure we use the completeness and normalisation conditions:

$$(\epsilon_\alpha^{(n)a*} \epsilon_\alpha^{(n)a'}) = -\delta_{aa'}, \quad \sum_{a=1,2,3} \epsilon_\alpha^{(n)a} \epsilon_\beta^{(n)a+} = -g_{\alpha\beta}^{\perp q_n}, \quad n = 1, 2. \quad (7.8)$$

Let us present the spin-orbital wave functions and the corresponding operators for states with $L \leq 4$ and $J \leq 2$:

L	$\hat{Q}_{\mu_1 \dots \mu_J}^{V_1^a V_2^b(S, L, J)}(q)$	$S_{\alpha\beta, \mu_1 \dots \mu_J}^{V_1 V_2(S, L, J)}(q_1, q_2)$
S	$\hat{Q}_{\mu_1 \mu_2}^{V_1^a V_2^b(0, 0, 0)} = \mathbf{S}^{ab}$ $\hat{Q}_{\mu}^{V_1^a V_2^b(1, 0, 1)} = \mathbf{P}_{\mu}^{ab}$ $\hat{Q}_{\mu_1 \mu_2}^{V_1^a V_2^b(2, 0, 2)} = \mathbf{T}_{\mu_1 \mu_2}^{ab}$	$S^{\alpha\beta}$ $P_{\mu}^{\alpha\beta}$ $T_{\mu_1 \mu_2}^{\alpha\beta}$
P	$\hat{Q}_{\mu}^{V_1^a V_2^b(0, 1, 1)} = \mathbf{S}^{ab} q_{\mu}^{\perp}$ $\hat{Q}_{\nu}^{V_1^a V_2^b(1, 1, 0)} = \mathbf{P}_{\nu}^{ab} q_{\nu}^{\perp}$ $\hat{Q}_{\mu}^{V_1^a V_2^b(1, 1, 1)} = \mathbf{P}_{\nu_1}^{ab} \varepsilon_{\nu_1 \nu_2 \nu_3 p} Z_{\mu \nu_2, \nu_3}^{(1)}(q^{\perp})$ $\hat{Q}_{\mu_1 \mu_2}^{V_1^a V_2^b(1, 1, 2)} = \mathbf{P}_{\nu}^{ab} Z_{\mu_1 \mu_2, \nu}^{(1)}(q^{\perp})$	$S^{\alpha\beta} q_{\mu}^{\perp}$ $P_{\nu}^{\alpha\beta} q_{\nu}^{\perp}$ $P_{\nu_1}^{\alpha\beta} \varepsilon_{\nu_1 \nu_2 \nu_3 p} Z_{\mu \nu_2, \nu_3}^{(1)}(q^{\perp})$ $P_{\nu}^{\alpha\beta} Z_{\mu_1 \mu_2, \nu}^{(1)}(q^{\perp})$
D	$\hat{Q}_{\mu_1 \mu_2}^{V_1^a V_2^b(0, 2, 2)} = \mathbf{S}^{ab} X_{\mu_1 \mu_2}^{(2)}(q^{\perp})$ $\hat{Q}_{\mu}^{V_1^a V_2^b(1, 2, 1)} = \mathbf{P}_{\nu}^{ab} X_{\nu \mu}^{(2)}(q^{\perp})$ $\hat{Q}_{\mu}^{V_1^a V_2^b(2, 2, 0)} = \mathbf{T}_{\nu_1 \nu_2}^{ab} X_{\nu_1 \nu_2}^{(2)}(q^{\perp})$ $\hat{Q}_{\mu}^{V_1^a V_2^b(2, 2, 1)} = \mathbf{T}_{\nu_1 \nu_2}^{ab} \varepsilon_{\nu_1 \nu_3 \nu_4 p} Z_{\nu_2 \nu_3 \mu, \nu_4}^{(2)}(q^{\perp})$ $\hat{Q}_{\mu_1 \mu_2}^{V_1^a V_2^b(2, 2, 2)} = \mathbf{T}_{\nu_1 \nu_2}^{ab} X_{\nu_2 \nu_3}^{(2)}(q^{\perp}) O_{\mu_1 \mu_2}^{\nu_1 \nu_3}(\perp p)$	$S^{\alpha\beta} X_{\mu_1 \mu_2}^{(2)}(q^{\perp})$ $P_{\nu}^{\alpha\beta} X_{\nu \mu}^{(2)}(q^{\perp})$ $T_{\nu_1 \nu_2}^{\alpha\beta} X_{\nu_1 \nu_2}^{(2)}(q^{\perp})$ $T_{\nu_1 \nu_2}^{\alpha\beta} \varepsilon_{\nu_1 \nu_3 \nu_4 p} Z_{\nu_2 \nu_3 \mu, \nu_4}^{(2)}(q^{\perp})$ $T_{\nu_1 \nu_2}^{\alpha\beta} X_{\nu_2 \nu_3}^{(2)}(q^{\perp}) O_{\mu_1 \mu_2}^{\nu_1 \nu_3}(\perp p)$
F	$\hat{Q}_{\mu_1 \mu_2}^{V_1^a V_2^b(1, 3, 2)} = \mathbf{P}_{\nu}^{ab} X_{\nu \mu_1 \mu_2}^{(3)}(q^{\perp})$ $\hat{Q}_{\mu}^{V_1^a V_2^b(1, 3, 1)} = \mathbf{T}_{\nu_1 \nu_2}^{ab} X_{\nu_1 \nu_2 \mu}^{(3)}(q^{\perp})$ $\hat{Q}_{\mu_1 \mu_2}^{V_1^a V_2^b(2, 3, 2)} = \mathbf{T}_{\nu_1 \nu_2}^{ab} \varepsilon_{\nu_1 \nu_3 \nu_4 p} \times Z_{\nu_2 \nu_3 \mu_1 \mu_2, \nu_4}^{(3)}(q^{\perp})$	$P_{\nu}^{\alpha\beta} X_{\nu \mu_1 \mu_2}^{(3)}(q^{\perp})$ $T_{\nu_1 \nu_2}^{\alpha\beta} X_{\nu_1 \nu_2 \mu}^{(3)}(q^{\perp})$ $T_{\nu_1 \nu_2}^{\alpha\beta} \varepsilon_{\nu_1 \nu_3 \nu_4 p} \times Z_{\nu_2 \nu_3 \mu_1 \mu_2, \nu_4}^{(3)}(q^{\perp})$
G	$\hat{Q}_{\mu_1 \mu_2}^{V_1^a V_2^b(1, 4, 2)} = \mathbf{T}_{\nu_1 \nu_2}^{ab} X_{\nu_1 \nu_2 \mu_1 \mu_2}^{(4)}(q^{\perp})$	$T_{\nu_1 \nu_2}^{\alpha\beta} X_{\nu_1 \nu_2 \mu_1 \mu_2}^{(4)}(q^{\perp})$

(7.9)

where the spin operators are:

$$\begin{aligned}
 \mathbf{S}^{ab} &\rightarrow S^{\alpha\beta}(q_1, q_2) = g_{\alpha\xi}^{\perp q_1} g_{\xi\beta}^{\perp q_2}, \\
 \mathbf{P}_{\mu}^{ab} &\rightarrow P_{\mu}^{\alpha\beta}(q_1, q_2) = \varepsilon_{\mu\xi\xi'\nu} g_{\xi\alpha}^{\perp q_1} g_{\xi'\beta}^{\perp q_2} p_{\nu}, \\
 \mathbf{T}_{\mu_1 \mu_2}^{ab} &\rightarrow T_{\mu_1 \mu_2}^{\alpha\beta}(q_1, q_2) = \frac{1}{2} \left(g_{\mu_1 \alpha}^{\perp q_1} g_{\mu_2 \beta}^{\perp q_2} + g_{\mu_1 \beta}^{\perp q_2} g_{\mu_2 \alpha}^{\perp q_1} \right. \\
 &\quad \left. - \frac{g_{\mu_1 \xi}^{\perp q_1} g_{\xi \mu_2}^{\perp q_2} + g_{\mu_1 \xi}^{\perp q_2} g_{\xi \mu_2}^{\perp q_1}}{g_{\xi' \xi''}^{\perp q_1} g_{\xi' \xi''}^{\perp q_2}} g_{\alpha \xi}^{\perp q_1} g_{\xi \beta}^{\perp q_2} \right). \quad (7.10)
 \end{aligned}$$

The operators $S^{\alpha\beta}(q_1, q_2)$ and $T_{\mu_1 \mu_2}^{\alpha\beta}(q_1, q_2)$ are, of course, even, while the operator $P_{\mu}^{\alpha\beta}(q_1, q_2)$ is odd under the simultaneous permutation $\alpha \rightleftharpoons \beta$ and $q_1 \rightleftharpoons q_2$. Besides, the operator $T_{\mu_1 \mu_2}^{\alpha\beta}(q_1, q_2)$ is even under the permutation $\mu_1 \rightleftharpoons \mu_2$.

Multiplying the operators $S_{\alpha\beta, \mu_1 \dots \mu_J}^{V_1 V_2(S, L, J)}$ by $\epsilon_{\alpha}^{(1)a}$ and $\epsilon_{\beta}^{(2)b}$, we obtain the expressions given in the second column (spin-orbital operators $\hat{Q}_{\mu_1 \dots \mu_J}^{V_1^a V_2^b(S, L, J)}$).

7.1.2 Transitions $\gamma^*\gamma^* \rightarrow \text{hadrons}$

Let us turn now to a two-photon system — it is a system of two identical particles. Considering the $\gamma^*\gamma^*$ system as the initial one, we present, as an example, formulae for the reactions $\gamma^*\gamma^* \rightarrow N\bar{N}$ and $\gamma^*\gamma^* \rightarrow P_1P_2$.

7.1.2.1 Structure of the spin-orbital operators for a $\gamma^*(q_1)\gamma^*(q_2)$ system when $q_1^2 \neq 0, q_2^2 \neq 0$

Let us consider a $\gamma^*\gamma^*$ -collision, see Fig. 7.1. The system $\gamma^*\gamma^*$ is symmetrical under the permutation of photons. This decreases the number of possible spin-orbital states. For even L and J we have:

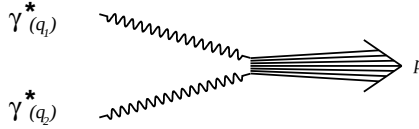


Fig. 7.1 The production of a beam of particles with the total momentum $p = q_1 + q_2$ by two vector particles (virtual photons).

$$\begin{aligned}
 \hat{Q}_{\mu_1\mu_2\ldots\mu_J}^{\gamma_{1a}^*\gamma_{2b}^*}(S=0, L=J, J)(q) &= \mathbf{S}^{ab} X_{\mu_1\ldots\mu_J}^{(J)}(q^\perp), \\
 \hat{Q}_{\mu_1\ldots\mu_J}^{\gamma_{1a}^*\gamma_{2b}^*}(S=2, L=J+2, J)(q) &= \mathbf{T}_{\alpha\beta}^{ab} X_{\mu_1\ldots\mu_J\alpha\beta}^{(J+2)}(q^\perp), \\
 \hat{Q}_{\mu_1\ldots\mu_J}^{\gamma_{1a}^*\gamma_{2b}^*}(S=2, L=J, J)(q) &= \mathbf{T}_{\mu'_1\alpha}^{ab} X_{\alpha\mu'_2\ldots\mu'_J}^{(J)}(q^\perp) O_{\mu_1\mu_2\ldots\mu_J}^{\mu'_1\mu'_2\ldots\mu'_J}(\perp p), \\
 \hat{Q}_{\mu_1\ldots\mu_J}^{\gamma_{1a}^*\gamma_{2b}^*}(S=2, L=J-2, J)(q) &= \mathbf{T}_{\mu'_1\mu'_2}^{ab} X_{\mu'_3\ldots\mu'_J}^{(J-2)}(q^\perp) O_{\mu_1\mu_2\ldots\mu_J}^{\mu'_1\mu'_2\ldots\mu'_J}(\perp p), \quad (7.11)
 \end{aligned}$$

for even L and odd J :

$$\begin{aligned}
 \hat{Q}_{\mu_1\ldots\mu_J}^{\gamma_{1a}^*\gamma_{2b}^*}(S=2, L=J+1, J)(q) &= \mathbf{T}_{\alpha\beta}^{ab} \varepsilon_{\alpha\nu_1\nu_2p} Z_{\nu_2\beta\mu_1\ldots\mu_J, \nu_1}^{(J+1)}(q^\perp), \quad (7.12) \\
 \hat{Q}_{\mu_1\ldots\mu_J}^{\gamma_{1a}^*\gamma_{2b}^*}(S=2, L=J-1, J)(q) &= \mathbf{T}_{\alpha\mu'_1}^{ab} \varepsilon_{\alpha\nu_1\nu_2p} Z_{\nu_1\mu'_2\ldots\mu'_J, \nu_2}^{(J-1)}(q^\perp) O_{\mu_1\mu_2\ldots\mu_J}^{\mu'_1\mu'_2\ldots\mu'_J}(\perp p),
 \end{aligned}$$

for odd L and even J :

$$\begin{aligned}
 \hat{Q}_{\mu_1\ldots\mu_J}^{\gamma_{1a}^*\gamma_{2b}^*}(S=1, L=J+1, J)(q) &= \mathbf{P}_\alpha^{ab} X_{\mu_1\ldots\mu_J\alpha}^{(J+1)}(q^\perp), \\
 \hat{Q}_{\mu_1\ldots\mu_J}^{\gamma_{1a}^*\gamma_{2b}^*}(S=1, L=J-1, J)(q) &= \mathbf{P}_\alpha^{ab} Z_{\mu_1\ldots\mu_J, \alpha}^{(J-1)}(q^\perp), \quad (7.13)
 \end{aligned}$$

and for odd L and J :

$$\hat{Q}_{\mu_1\ldots\mu_J}^{\gamma_{1a}^*\gamma_{2b}^*}(S=1, L=J, J)(q) = \mathbf{P}_\alpha^{ab} \varepsilon_{\alpha\nu_1\nu_2p} Z_{\nu_1\mu_1\ldots\mu_J, \nu_2}^{(J)}(q^\perp). \quad (7.14)$$

Let us remind once more that the operators $\mathbf{S}^{ab}$ and $\mathbf{T}_{\alpha\beta}^{ab}$ are even under the permutation of the particles 1 and 2, while the operator $\mathbf{P}_\alpha^{ab}$ is odd.

As an example, let us consider the production of two hadrons: of a nucleon–antinucleon pair and of two pseudoscalar mesons, see Fig. 7.2. The system of two photons can produce hadrons with isospins $I = 0, 1, 2$. The $N\bar{N}$ system is characterised by two isospins $I = 0, 1$ while the $P_1 P_2$ system may have all three isotopic states $I = 0, 1, 2$.

7.1.2.2 The production of the nucleon–antinucleon pair $\gamma^*(q_1)\gamma^*(q_2) \rightarrow N(k_1)\bar{N}(k_2)$

The formulae of the K -matrix representation which were elaborated in Chapter 4 for the $N\bar{N}$ scattering amplitude can be used here to take into account the final state $N\bar{N}$ interaction. We write the amplitude $\gamma_a^*(q_1)\gamma_b^*(q_2) \rightarrow N(k_1)\bar{N}(k_2)$ as

$$M_{\gamma_a^*\gamma_b^* \rightarrow N\bar{N}}(s, t, u) = \sum_{J, S, S', L, L', I} \left(\bar{\psi}(k_1) \hat{Q}_{\mu_1 \dots \mu_J}^{N\bar{N}(S', L', J)}(k) \psi(-k_2) \right) \\ \times \hat{Q}_{\mu_1 \dots \mu_J}^{\gamma_a^* \gamma_b^* (S, L, J)}(q) A_{\gamma^* \gamma^* \rightarrow N\bar{N}(I)}^{(S, L, L', J)}(s). \quad (7.15)$$

It is essential to distinguish between two cases: when in the $N\bar{N}$ system there is $J = L'$ and when $J = L' \pm 1$.

(i) Partial wave amplitudes $\gamma^*\gamma^* \rightarrow N\bar{N}$ for $J = L'$.

In the considered case for the amplitude with $I = 0, 1$, $A_{\gamma^*\gamma^* \rightarrow N\bar{N}(I)}^{(S, S', L, L', J)}(s)$, the s -channel unitarity condition gives:

$$A_{\gamma^*\gamma^* \rightarrow N\bar{N}(I)}^{(S, S', L, L'=J, J)}(s) = \frac{G_{\gamma^*\gamma^* \rightarrow N\bar{N}(I)}^{(S, S', L, L'=J, J)}(s)}{1 - i\rho^{(S', L'=J, J)}(s) K_{N\bar{N}(I) \rightarrow N\bar{N}(I)}^{(S', L'=J, J)}(s)}, \quad (7.16)$$

where $G_{\gamma^*\gamma^* \rightarrow N\bar{N}(I)}^{(S, S', L, L'=J, J)}(s)$ is the block for $N\bar{N}$ production, $K_{N\bar{N}(I) \rightarrow N\bar{N}(I)}^{(S', L'=J, J)}(s)$ is the K -matrix element of the $N\bar{N}$ scattering amplitude, and the phase space for $N\bar{N}$ is determined as

$$\rho_{N\bar{N}}^{(S', L'=J, J)}(s) = \frac{1}{2J+1} \int d\Phi_2(k_1, k_2) \\ \times \text{Sp} \left(\hat{Q}_{\mu_1 \dots \mu_J}^{(S', L', J)}(k) (-\hat{k}_2 + m_N) \hat{Q}_{\mu_1 \dots \mu_J}^{(S', L', J)}(k) (\hat{k}_1 + m_N) \right). \quad (7.17)$$

Let us note that here we have only one-channel rescatterings of the $N\bar{N}$ state.

(ii) Partial wave amplitudes for $S = 1$ and $J = L \pm 1$.

In this case we have two-channel rescatterings of the $N\bar{N}$ system. We take into account only the mixing in the channel of strongly interacting particles

(in $N\bar{N}$ system), so in this case we have four partial wave amplitudes which form the 2×2 matrix:

$$\hat{A}_{\gamma^*\gamma^* \rightarrow N\bar{N}}^{(S=1, L=J\pm 1, J)}(s) = \begin{vmatrix} A_{\gamma^*\gamma^* \rightarrow N\bar{N}}^{(S=1, J-1 \rightarrow J-1, J)}(s), & A_{\gamma^*\gamma^* \rightarrow N\bar{N}}^{(S=1, J-1 \rightarrow J+1, J)}(s) \\ A_{\gamma^*\gamma^* \rightarrow N\bar{N}}^{(S=1, J+1 \rightarrow J-1, J)}(s), & A_{\gamma^*\gamma^* \rightarrow N\bar{N}}^{(S=1, J+1 \rightarrow J+1, J)}(s) \end{vmatrix}. \quad (7.18)$$

The K -matrix representation reads

$$\begin{aligned} \hat{A}_{\gamma^*\gamma^* \rightarrow N\bar{N}}^{(S=1, L=J\pm 1, J)}(s) &= \hat{G}_{\gamma^*\gamma^* \rightarrow N\bar{N}}^{(S=1, L=J\pm 1, J)}(s) \\ &\times \left[I - i \hat{\rho}_{N\bar{N}}^{(S=1, L=J\pm 1, J)}(s) \hat{K}_{N\bar{N}(I) \rightarrow N\bar{N}(I)}^{(S=1, L=J\pm 1, J)}(s) \right]^{-1}, \end{aligned} \quad (7.19)$$

with the following definitions:

$$\begin{aligned} \hat{K}_{N\bar{N}(I) \rightarrow N\bar{N}(I)}^{(S=1, L=J\pm 1, J)}(s) &= \begin{vmatrix} K_{N\bar{N}(I) \rightarrow N\bar{N}(I)}^{(S=1, J-1 \rightarrow J-1, J)}(s), & K_{N\bar{N}(I) \rightarrow N\bar{N}(I)}^{(S=1, J-1 \rightarrow J+1, J)}(s) \\ K_{N\bar{N}(I) \rightarrow N\bar{N}(I)}^{(S=1, J+1 \rightarrow J-1, J)}(s), & K_{N\bar{N}(I) \rightarrow N\bar{N}(I)}^{(S=1, J+1 \rightarrow J+1, J)}(s) \end{vmatrix}, \\ \hat{\rho}_{N\bar{N}}^{(S=1, L=J\pm 1, J)}(s) &= \begin{vmatrix} \rho_{N\bar{N}}^{(S=1, J-1 \rightarrow J-1, J)}(s), & \rho_{N\bar{N}}^{(S=1, J-1 \rightarrow J+1, J)}(s) \\ \rho_{N\bar{N}}^{(S=1, J+1 \rightarrow J-1, J)}(s), & \rho_{N\bar{N}}^{(S=1, J+1 \rightarrow J+1, J)}(s) \end{vmatrix}. \end{aligned} \quad (7.20)$$

The phase space factors $\rho_{N\bar{N}}^{(S, L \rightarrow L', J)}(s)$ are determined in Chapter 4 (section 4.4). Let us remind that the matrices $\hat{\rho}_I^{(S=1, L=J\pm 1, J)}(s)$ and $\hat{K}_{N\bar{N}(I) \rightarrow N\bar{N}(I)}^{(S=1, L=J\pm 1, J)}(s)$ are symmetrical:
 $\rho_{N\bar{N}}^{(S=1, J-1 \rightarrow J+1, J)}(s) = \rho_{N\bar{N}}^{(S=1, J+1 \rightarrow J-1, J)}(s)$ and $K_{N\bar{N}(I) \rightarrow N\bar{N}(I)}^{(S=1, J-1 \rightarrow J+1, J)}(s) = K_{N\bar{N}(I) \rightarrow N\bar{N}(I)}^{(S=1, J+1 \rightarrow J-1, J)}(s)$.

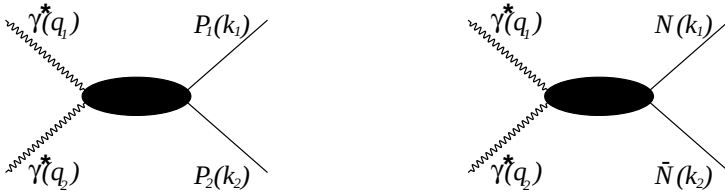


Fig. 7.2 The production of two pseudoscalars ($P_1 P_2$) and a nucleon-antinucleon pair ($N\bar{N}$) by virtual photons.

7.1.2.3 The production of two pseudoscalar mesons

$$\gamma^*(q_1)\gamma^*(q_2) \rightarrow P_1(k_1)P_2(k_2)$$

A two-photon system can, in general, produce hadrons in isotopic states with $I = 0, 1, 2$. Correspondingly, we discuss here the transitions $\gamma^*\gamma^* \rightarrow \pi^+\pi^-(I = 0, 2)$, $\pi^0\pi^0(I = 0, 2)$, $\eta\eta(I = 0)$, $\eta\eta'(I = 0)$, $K\bar{K}(I = 0, 1)$.

The channels with $I = 0$ are connected and, consequently, should be considered simultaneously. Unitarity condition can be fulfilled, for example, in the framework of the K -matrix formalism. Let us consider first the ($I = 0$)-amplitude:

$$M_{\gamma_a^* \gamma_b^* \rightarrow P_1 P_2}(s, t, u) = \sum_{J, S, L, L'} \hat{Q}_{\mu_1 \dots \mu_J}^{P_1 P_2(L', J)}(k) \hat{Q}_{\mu_1 \dots \mu_J}^{\gamma_a^* \gamma_b^*(S, L, J)}(q) A_{\gamma^* \gamma^* \rightarrow P_1 P_2}^{(S, L, L', J)}(s),$$

$$\hat{Q}_{\mu_1 \dots \mu_J}^{P_1 P_2}(k) = X_{\mu_1 \dots \mu_J}^{(J)}(k^\perp), \quad (7.21)$$

where $p = q_1 + q_2 = k_1 + k_2$, $s = p^2$ and $k_\mu^\perp = g_{\mu\nu}^\perp k_\nu$, $k = \frac{1}{2}(k_1 - k_2)$. In the K -matrix representation the amplitude $A_{\gamma^* \gamma^* \rightarrow P_1 P_2}^{(S, L, L', J)}(s)$ reads

$$A_{\gamma^* \gamma^* \rightarrow P_1 P_2}^{(S, L, L'=J, J)}(s) = \sum_b \hat{G}_{\gamma^* \gamma^* \rightarrow b}^{(S, L, L'=J, J)}(s) \left[\frac{1}{1 - i\hat{\rho}^{(0J)}(s)\hat{K}^{(0J)}(s)} \right]_{b, (P_1 P_2)} \quad (7.22)$$

The K -matrix for the ($IJ^{PC} = 0J^{++}$)-state was considered in detail in Chapter 3, see also [2]. The analysis of the (00^{++})-wave was given in Appendix 3.B. A graphical representation of (7.22) is shown in Fig. 7.3:

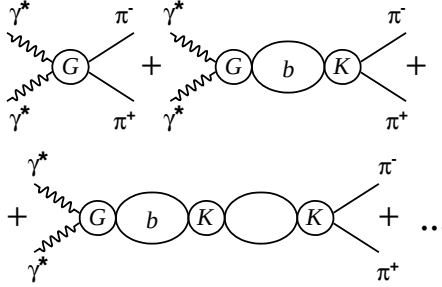


Fig. 7.3 K -matrix representation of the production of $\pi^+ \pi^-$ state: the block of photo-production G and subsequent rescatterings of the mesons ($b = \pi\pi, \eta\eta, \eta\eta', K\bar{K}, \dots$), see Eq. (7.22).

here $b = \pi^+ \pi^-, \pi^0 \pi^0, \eta\eta, \eta\eta', K\bar{K}$. The K -matrix technique makes it possible to take into account higher hadron states such as $\sigma\sigma, \rho\rho$, etc. The diagonal matrix of the phase space $\hat{\rho}^{(0J)}(s)$ is given in Chapter 3.

7.1.3 Quark structure of meson production processes

Let us now describe the production of two mesons $\gamma^* \gamma^* \rightarrow P_1 P_2$ in terms of quark diagrams. The leading contributions in the $1/N$ expansion [3] correspond to planar diagrams; two of the simplest ones (quark skeleton without a gluonic net) are shown in Fig. 7.4.

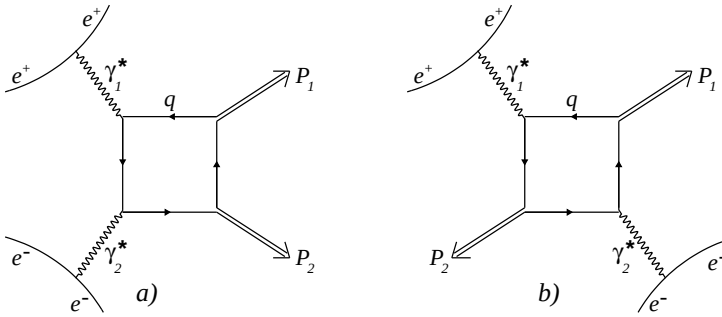


Fig. 7.4 Production of the pseudoscalar mesons, P_1P_2 , in $\gamma^*\gamma^*$ collisions. Primary planar quark diagrams: a) the (s,t) -channel box-diagram (with imaginary parts in s - and t -channels), b) the (t,u) -channel box diagram.

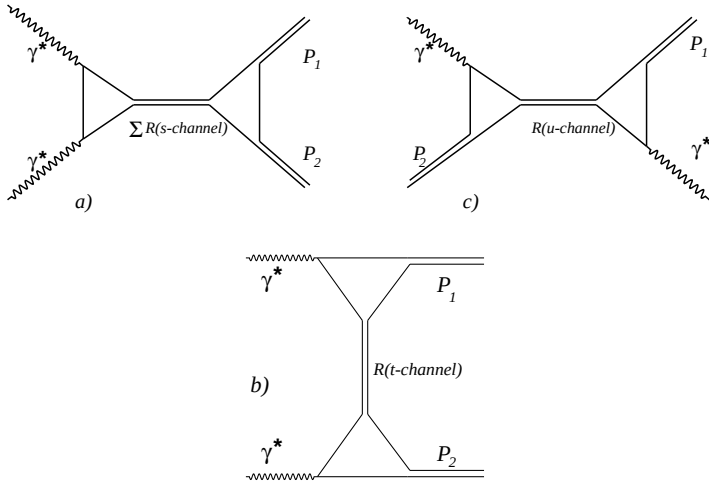


Fig. 7.5 The box diagrams rewritten in the language of resonance exchanges in the channels s , t and u .

These diagrams differ essentially from each other. The diagram Fig. 7.4a is saturated by s - and t -channel resonances, Fig. 7.4b by t - and u -channel resonances (see Fig. 7.5). Hence, resonances in the P_1P_2 system can come from processes in Fig. 7.4a only, and their isotopic spins are $I = 0$ and $I = 1$. In processes shown in Fig. 7.4b the P_1P_2 system can have an isospin $I = 2$. In Fig. 7.6, examples of processes are presented where the final particles may have isospins $I = 2$. In Fig. 7.6a this is the

production of two pions with an ω -exchange in the u -channel. In Fig. 7.6b a more complicated process is shown: photons in the s -channel produce two ρ -mesons (due to a σ -meson exchange in the u -channel) so the $\rho\rho$ system may have an isospin $I = 2$. After the ρ -meson decay, with a subsequent rescattering of the produced π mesons, we arrive at a $\rho\pi\pi$ system. This five-point loop diagram has a pole singularity which can imitate a resonance in the $\rho\pi\pi$ state with the isospin $I = 2$.

Having in mind similar effects, we have to be rather careful when investigating resonances in many-particle systems.

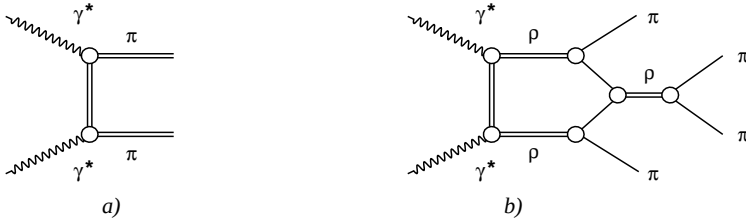


Fig. 7.6 Diagrams of the Fig. 4b-type written in terms of hadrons: they contribute to the $I = 2$ state.

7.2 Nilpotent Operators — Production of Scalar States

Here we consider the amplitudes of the processes $\gamma^*\gamma^* \rightarrow 0^{++}$, $\gamma\gamma^* \rightarrow 0^{++}$ and $\gamma\gamma \rightarrow 0^{++}$, and demonstrate, using this simple example, the problems which appear when we handle real photons.

7.2.1 Gauge invariance and orthogonality of the operators

It was shown in the previous section that the initial state in the process $\gamma_a^*(q_1)\gamma_b^*(q_2) \rightarrow S$ is characterised by two wave functions (with $L = 0$ and $L = 2$) and, correspondingly, by two structures:

$$\begin{aligned} L = 0 : \quad & \epsilon_{\alpha}^{(1)a} \left[g_{\alpha\xi}^{\perp q_1} g_{\xi\beta}^{\perp q_2} \right] \epsilon_{\beta}^{(2)b}, \\ L = 2 : \quad & \epsilon_{\alpha}^{(1)a} \left[g_{\alpha\xi_1}^{\perp q_1} g_{\beta\xi_2}^{\perp q_2} X_{\xi_1\xi_2}^{(2)}(q^{\perp}) \right] \epsilon_{\beta}^{(2)b}. \end{aligned} \quad (7.23)$$

Here the operators are written in the square brackets.

Instead of the operators (7.23), it is convenient to make use of a different set of operators allowing us to carry out a smooth transition to the case of

real photons. These operators, $g_{\alpha\beta}^{\perp\perp}(q_1, q_2)$ and $L_{\alpha\beta}(q_1, q_2)$, read:

$$g_{\alpha\beta}^{\perp\perp}(q_1, q_2) = g_{\alpha\beta} + \frac{q_1^2}{(q_2 q_1)^2 - q_2^2 q_1^2} q_{2\alpha} q_{2\beta} \\ + \frac{q_2^2}{(q_2 q_1)^2 - q_2^2 q_1^2} q_{1\alpha} q_{1\beta} - \frac{(q_2 q_1)}{(q_2 q_1)^2 - q_2^2 q_1^2} (q_{1\alpha} q_{2\beta} + q_{2\alpha} q_{1\beta}), \quad (7.24)$$

and

$$L_{\alpha\beta}(q_1, q_2) = \frac{q_1^2}{(q_2 q_1)^2 - q_2^2 q_1^2} q_{2\alpha} q_{2\beta} + \frac{q_2^2}{(q_2 q_1)^2 - q_2^2 q_1^2} q_{1\alpha} q_{1\beta} \\ - \frac{(q_2 q_1)}{(q_2 q_1)^2 - q_2^2 q_1^2} q_{1\alpha} q_{2\beta} - \frac{q_2^2 q_1^2}{[(q_2 q_1)^2 - q_2^2 q_1^2](q_2 q_1)} q_{2\alpha} q_{1\beta}. \quad (7.25)$$

As is easy to see, these operators obey gauge invariance and are orthogonal to each other:

$$q_{1\alpha} g_{\alpha\beta}^{\perp\perp}(q_1, q_2) = 0, \quad g_{\alpha\beta}^{\perp\perp}(q_1, q_2) q_{2\beta} = 0, \\ q_{1\alpha} L_{\alpha\beta}(q_1, q_2) = 0, \quad L_{\alpha\beta}(q_1, q_2) q_{2\beta} = 0, \\ g_{\alpha\beta}^{\perp\perp}(q_1, q_2) L_{\alpha\beta}(q_1, q_2) = 0. \quad (7.26)$$

Both operators, $g_{\alpha\beta}^{\perp\perp}(q_1, q_2)$ and $L_{\alpha\beta}(q_1, q_2)$ are symmetrical under the simultaneous change $(q_1 \rightleftharpoons q_2)$ and $(\alpha \rightleftharpoons \beta)$. Still, the operator $g_{\alpha\beta}^{\perp\perp}(q_1, q_2)$ satisfies a more rigid symmetry condition: it is symmetrical at $(q_1 \rightleftharpoons q_2)$ only.

The transition amplitude $\gamma^*(q_1)\gamma^*(q_2) \rightarrow S$ reads:

$$A_{\alpha\beta}^{(\gamma^* \gamma^* \rightarrow S)} = g_{\alpha\beta}^{\perp\perp}(q_1, q_2) F_t(q_1^2, q_2^2, p^2) + L_{\alpha\beta}(q_1, q_2) F_\ell(q_1^2, q_2^2, p^2). \quad (7.27)$$

The operators $g_{\alpha\beta}^{\perp\perp}(q_1, q_2)$ and $L_{\alpha\beta}(q_1, q_2)$ are singular. To avoid false kinematical singularities in the amplitude $A_{\alpha\beta}^{(\gamma^* \gamma^* \rightarrow S)}$, the poles in $g_{\alpha\beta}^{\perp\perp}(q_1, q_2)$ and $L_{\alpha\beta}(q_1, q_2)$ should be cancelled by zeros of the amplitude.

Let us turn now our attention to a specific feature of the operators with fixed angular momentum given in (7.23). The operators (7.23), $g_{\alpha\xi}^{\perp q_1} g_{\xi\beta}^{\perp q_2}$ and $g_{\alpha\xi_1}^{\perp q_1} g_{\xi_2\beta}^{\perp q_2} X_{\xi_1\xi_2}^{(2)}(q^\perp)$ are not orthogonal to each other. Indeed, the convolution of these operators is equal to:

$$(g_{\alpha\xi}^{\perp q_1} g_{\xi\beta}^{\perp q_2}) (g_{\alpha\xi_1}^{\perp q_1} X_{\xi_1\xi_2}^{(2)}(q^\perp) g_{\xi_2\beta}^{\perp q_2}) = -\frac{q_1^4}{3q_1^2 q_2^2} (p^2 + q_1^2 + q_2^2). \quad (7.28)$$

The non-orthogonality of the operators (7.23) is due to the fact that for their construction we have used the identity (7.5) which makes the operators gauge invariant. Indeed, in (7.23) the convolution has been performed

with the help of the metric tensors $g_{\alpha\xi_1}^{\perp q_1}$ and $g_{\beta\xi_2}^{\perp q_2}$ which work in three-dimensional space. Had we operated with the four-dimensional metric tensor, namely, had we substituted in (7.28) $g_{\alpha\xi_1}^{\perp q_1} \rightarrow g_{\alpha\xi_1}$ and $g_{\beta\xi_2}^{\perp q_2} \rightarrow g_{\beta\xi_2}$, we would have orthogonal S - and D -wave operators. But the metric tensors $g_{\alpha\xi_1}^{\perp q_1}$ and $g_{\beta\xi_2}^{\perp q_2}$ in (7.23) allow us to fulfil the gauge invariance — in this way, just due to the gauge invariance, the orthogonality in the S - and D -wave operators (7.23) is broken. Let us emphasise that in the spectral integral representation of form factors of the composite systems the orthogonal operators are needed to avoid double counting. This is the reason why further we deal with the orthogonal operators represented by formulae (7.24) and (7.25).

7.2.2 Transition amplitude $\gamma\gamma^* \rightarrow S$ when one of the photons is real

For the transition amplitude $\gamma\gamma^* \rightarrow S$ with a real photon (below $q_1 \equiv q$ with $q_1^2 \equiv q^2 = 0$), we write:

$$A_{\alpha\beta}^{(\gamma\gamma^* \rightarrow S)} = g_{\alpha\beta}^{\perp\perp}(q, q_2) F_\ell(0, q_2^2, p^2) + L_{\alpha\beta}(q, q_2) F_\ell(0, q_2^2, p^2), \quad (7.29)$$

This representation is, however, not unique, below we discuss ambiguities in the representation of the amplitude $A_{\alpha\beta}^{(\gamma\gamma^* \rightarrow S)}$.

7.2.2.1 Ambiguities in the representation of the spin operator

This reaction is determined actually by one form factor because $L_{\alpha\beta}(q, q_2)$ at $q^2 = 0$ is a nilpotent operator [4]. We have for $q_1^2 \equiv q^2 = 0$:

$$g_{\alpha\beta}^{\perp\perp}(q, q_2) = g_{\alpha\beta} + \frac{q_2^2}{(qq_2)^2} q_\alpha q_\beta - \frac{1}{(qq_2)} (q_\alpha q_{2\beta} + q_{2\alpha} q_\beta). \quad (7.30)$$

and

$$L_{\alpha\beta}^{(0)}(q, q_2) \equiv L_{\alpha\beta}(q, q_2) = \frac{q_2^2}{(qq_2)^2} q_\alpha q_\beta - \frac{1}{(qq_2)} q_\alpha q_{2\beta}. \quad (7.31)$$

It is seen directly that the operator $L_{\alpha\beta}^{(0)}(q, q_2)$ obeys the nilpotent requirement:

$$L_{\alpha\beta}^{(0)}(q, q_2) L_{\alpha\beta}^{(0)}(q, q_2) = 0, \quad (7.32)$$

the index (0) is introduced in (7.31) to emphasise that the norm of the operator is equal to zero.

Below we write for the transition amplitude $\gamma\gamma^* \rightarrow S$ with a real photon

$$A_{\alpha\beta}^{(\gamma\gamma^* \rightarrow S)} = g_{\alpha\beta}^{\perp\perp}(q, q_2) F_{\gamma\gamma^* \rightarrow S}(q_2^2, p^2), \quad (7.33)$$

by putting $F_\ell(0, q_2^2, p^2) = 0$ and redefining $F_t(0, q_2^2, p^2) \rightarrow F_{\gamma\gamma^* \rightarrow S}(q_2^2, p^2)$.

Sometimes another spin operator is used in (7.33):

$$g_{\alpha\beta}^{\perp\perp}(q, q_2) \longrightarrow \tilde{S}_{\alpha\beta}(q, q_2) = g_{\alpha\beta} - \frac{q_\alpha q_{2\beta}}{(qq_2)}, \quad (7.34)$$

which equals

$$\tilde{S}_{\alpha\beta}(q, q_2) = g_{\alpha\beta}^{\perp\perp}(q, q_2) - L_{\alpha\beta}^{(0)}(q, q_2). \quad (7.35)$$

Then

$$A_{\alpha\beta}^{(\gamma\gamma^* \rightarrow S)} = \tilde{S}_{\alpha\beta}(q, q_2) F_{\gamma\gamma^* \rightarrow S}(q_2^2, p^2), \quad (7.36)$$

Generally speaking, one can use the spin operator constructing any linear combination of $g_{\alpha\beta}^{\perp\perp}(q, q_2)$ and $L_{\alpha\beta}^{(0)}(q, q_2)$:

$$S_{\alpha\beta}^{(\gamma\gamma^* \rightarrow S)}(q, q_2) = g_{\alpha\beta}^{\perp\perp}(q, q_2) + C(p^2, q_2^2) L_{\alpha\beta}^{(0)}(q, q_2). \quad (7.37)$$

Any of these operators may be equally applied to equation (7.33) for the presentation of the transition amplitude $\gamma\gamma^* \rightarrow S$ with a real photon.

7.2.2.2 Analytical properties of the amplitude for the emission of a real photon

Let us discuss the analytical properties of the amplitude with a real photon, namely, the cancellation of kinematical singularities. In a general form the amplitude $A_{\alpha\beta}^{(\gamma\gamma^* \rightarrow S)}$ for the production of a scalar meson with mass m_S reads:

$$\begin{aligned} A_{\alpha\beta}^{(\gamma\gamma^* \rightarrow S)} = & \left[g_{\alpha\beta} + \frac{4q_2^2}{(m_S^2 - q_2^2)^2} q_\alpha q_\beta - \frac{2}{m_S^2 - q_2^2} (q_{2\alpha} q_\beta + q_\alpha q_{2\beta}) \right] \\ & \times F_t(0, q_2^2, m_S^2) + \left[\frac{4q_2^2}{(m_S^2 - q_2^2)^2} q_\alpha q_\beta - \frac{2}{m_S^2 - q_2^2} q_\alpha q_{2\beta} \right] F_\ell(0, q_2^2, m_S^2). \end{aligned} \quad (7.38)$$

Here we have used $2(qq_2) = m_S^2 - q_2^2$. To make the term in front of $q_\alpha q_\beta$ non-singular at $m_S^2 \rightarrow q_2^2$, it is necessary that

$$[F_t(0, q_2^2, m_S^2) + F_\ell(0, q_2^2, m_S^2)]_{m_S^2 \rightarrow q_2^2} \sim (m_S^2 - q_2^2)^2. \quad (7.39)$$

This requirement is sufficient for the cancellation of the kinematical singularity in front of $q_\alpha q_{2\beta}$. However, to remove the kinematical singularity in the term $q_{2\alpha} q_\beta$, the following condition for $F_t(0, q_2^2, m_S^2)$ should be fulfilled:

$$F_t(0, q_2^2, m_S^2) \sim (m_S^2 - q_2^2) \quad \text{at} \quad (m_S^2 - q_2^2) \rightarrow 0. \quad (7.40)$$

The constraint (7.39) is in fact the requirement imposed on $F_\ell(0, q_2^2, m_S^2)$, but the $F_\ell(0, q_2^2, m_S^2)$ itself, as was noted above, does not participate in the definition of the decay partial width of the process $\gamma\gamma^* \rightarrow S$.

The second constraint given by (7.40) for $F_t(0, q_2^2, m_S^2)$ is the basic one for decay physics — in quantum mechanics it is known as Siegert's theorem [5].

Constraints (7.39), (7.40) are a source of other ambiguities in the presentation of the transition amplitude. One may extract the factor $(m_S^2 - q_2^2)$ from form factors, $F_t(0, q_2^2, m_S^2) = \frac{1}{2}(m_S^2 - q_2^2)f_t(0, q_2^2, m_S^2)$ and $F_\ell(0, q_2^2, m_S^2) = \frac{1}{2}(m_S^2 - q_2^2)f_\ell(0, q_2^2, m_S^2)$, and work with redefined form factors, $f_t(0, q_2^2, m_S^2)$ and $f_\ell(0, q_2^2, m_S^2)$, and spin operators. In this case, if one starts with the operator (7.35), the transition amplitude can be written as

$$A_{\alpha\beta}^{(\gamma\gamma^*\rightarrow S)} = [(qq_2)g_{\alpha\beta} - q_\alpha q_{2\beta}] f_{\gamma\gamma^*\rightarrow S}(0, q_2^2, m_S^2). \quad (7.41)$$

Let us remind once more that $(qq_2) = (m_S^2 - q_2^2)/2$.

All forms of representation of the transition amplitude (Eqs. (7.29), (7.33), (7.36) or (7.41)) are, in principle, equivalent to each other if the constraint requirements are fulfilled. We prefer to work with Eqs. (7.33) or (7.36) because within this choice calculations with composite particle amplitudes are more transparent.

7.3 Reaction $e^+e^- \rightarrow \gamma^* \rightarrow \gamma\pi\pi$

Using the basic reaction $e^+e^- \rightarrow \gamma^* \rightarrow \phi \rightarrow \gamma(\pi\pi)_{S\text{-wave}}$ and the subprocesses $\phi \rightarrow \gamma(\pi\pi)_{S\text{-wave}}$ and $\phi \rightarrow \gamma f_0$, we demonstrate in this section a way to handle the corresponding amplitudes in terms of the developed operator expansion technique. The interest in the consideration of this example is dictated by a number of studies of this reaction, see [6] and references therein.

Further, we consider the decay $\phi(1020) \rightarrow \gamma\pi\pi$ in the non-relativistic quark model approximation, perform the calculation of the form factor $\phi(1020) \rightarrow \gamma f_0^{(bare)}(700)$ and apply the K -matrix technique to the transition $f_0^{(bare)}(700) \rightarrow \pi\pi$.

7.3.1 Analytical structure of amplitudes in the reactions

$$e^+e^- \rightarrow \gamma^* \rightarrow \phi \rightarrow \gamma(\pi\pi)_S, \phi \rightarrow \gamma f_0 \text{ and} \\ \phi \rightarrow \gamma(\pi\pi)_S$$

Let us start with the general formula for the transition amplitude $e^+e^- \rightarrow \gamma\pi\pi$ assuming that the e^+e^- system is in the $1^{--}(V)$ state, the $\pi\pi$ system in the $I = 0, 0^{++}(S)$ state and the outgoing photon is real.

The amplitude of the reaction $V(e^+e^-) \rightarrow \gamma S(\pi\pi)$ reads:

$$A_{\mu\alpha}^{(V \rightarrow \gamma S)}(s_V, s_S, q^2=0) = \left(g_{\mu\alpha} - \frac{2q_\mu P_{V\alpha}}{s_V - s_S} \right) A_{V \rightarrow \gamma S}(s_V, s_S, q^2=0). \quad (7.42)$$

The indices μ and α refer, correspondingly, to the initial vector state $V(e^+e^-)$ (total momentum P_V and $P_V^2 = s_V$) and the outgoing photon (momentum q and $q^2 = 0$). We have $(P_V - q)^2 = s_S$ and $(P_V q) = (s_V - s_S)/2$. We use here the spin operator of Eq. (7.34), $\tilde{S}_{\alpha\mu}(q, P_V)$, with obvious renotations. Remind that $\tilde{S}_{\alpha\mu}(q, P_V)q_\alpha = 0$ and $P_{V\mu}\tilde{S}_{\alpha\mu}(q, P_V) = 0$.

The requirement of analyticity (the absence of the pole at $s_V = s_S$) leads to the condition (see (7.40)):

$$\left[A_{V \rightarrow \gamma S}(s_V, s_S, 0) \right]_{s_V \rightarrow s_S} \sim (s_V - s_S) \quad (7.43)$$

which is the threshold theorem for the transition amplitude $V(e^+e^-) \rightarrow \gamma S(\pi\pi)$.

Let us emphasise once more that the form of the spin operator in (7.42) is not unique: alternatively, one can write the spin factor as a metric tensor $g_{\mu\alpha}^{\perp\perp}$ which works in the space orthogonal to P_V and q , see (7.30). For the reaction $V(e^+e^-) \rightarrow \gamma S(\pi\pi)$ this means a replacement in (7.42):

$$\begin{aligned} \left(g_{\mu\alpha} - \frac{2q_\mu P_{V\alpha}}{s_V - s_S} \right) &\longrightarrow g_{\alpha\mu}^{\perp\perp}(q, P_V) = \\ &= \left(g_{\alpha\mu} + \frac{m_V^2}{(qP_V)^2} q_\alpha q_\mu - \frac{1}{(qP_V)} (q_\alpha P_{V\mu} + P_{V\alpha} q_\mu) \right). \end{aligned} \quad (7.44)$$

Ambiguities in the choice of the spin operator for the process $V(e^+e^-) \rightarrow \gamma S(\pi\pi)$ are due to the existence of the nilpotent operator in the case of emission of the real photon.

7.3.1.1 *The amplitude for the transition $\gamma^* \rightarrow \gamma(\pi\pi)_S$ and poles corresponding to subprocesses $\phi \rightarrow \gamma f_0$ and $\phi \rightarrow \gamma(\pi\pi)_S$*

Here we fix our attention on the amplitude of the reaction with hadrons, $\gamma^* \rightarrow \gamma(\pi\pi)_S$ which includes poles responsible for the subprocesses $\phi \rightarrow \gamma f_0$ and $\phi \rightarrow \gamma\pi\pi$. The amplitudes of the subprocesses are determined by corresponding residues of the pole terms in the basic amplitude $\gamma^* \rightarrow \gamma(\pi\pi)_S$.

For $\gamma^* \rightarrow \gamma(\pi\pi)_S$ the amplitude is written as follows:

$$\begin{aligned}
 & A_{\mu\alpha}^{\gamma^* \rightarrow \gamma(\pi\pi)_S}(s_V, s_S, 0) \\
 &= \left(g_{\mu\alpha} - \frac{2q_\mu P_{V\alpha}}{s_V - s_S} \right) \left[G_{\gamma^* \rightarrow \phi} \frac{A_{\phi \rightarrow \gamma f_0}(M_\phi^2, M_{f_0}^2, 0)}{(s_V - M_\phi^2)(s_S - M_{f_0}^2)} g_{f_0 \rightarrow \pi\pi} \right. \\
 & \quad \left. + G_{\gamma^* \rightarrow \phi} \frac{B_\phi(M_\phi^2, s_S, 0)}{s_V - M_\phi^2} + \frac{B_{f_0}(s_V, M_{f_0}^2, 0)}{s_S - M_{f_0}^2} g_{f_0 \rightarrow \pi\pi} + B_0(s_V, s_S, 0) \right].
 \end{aligned} \tag{7.45}$$

To avoid a change in the notation, we put $q_{\gamma^*} \equiv P_V$; the indices μ and α refer to γ^* and the outgoing photon, respectively.

The amplitude $A_{\mu\alpha}^{\gamma^* \rightarrow \gamma(\pi\pi)_S}(s_V, s_S, 0)$ contains the double-pole term ($\sim 1/(s_V - M_\phi^2)(s_S - M_{f_0}^2)$) and terms with one pole ($\sim 1/(s_V - M_\phi^2)$) and ($\sim 1/(s_S - M_{f_0}^2)$) where M_ϕ^2 and $M_{f_0}^2$ are complex masses squared; the numerators are determined as residues, so we put for them $s_V = M_\phi^2$ and $s_S = M_{f_0}^2$. In the Breit-Wigner approximation the complex masses are written as $M_\phi^2 = m_\phi^2 - im_\phi \Gamma_\phi$ and $M_{f_0}^2 = m_{f_0}^2 - im_{f_0} \Gamma_{f_0}$. The background term $B_0(s_V, s_S, 0)$ does not contain poles.

Different terms in the right-hand side of (7.45) are shown in Fig. 7.7: the double-pole term corresponds to Fig. 7.7a, the terms with poles ($\sim 1/(s_V - M_\phi^2)$) and ($\sim 1/(s_S - M_{f_0}^2)$) are given in Figs. 7.7b and 7.7c, respectively, and the last term in Eq. (7.45) is shown in Fig. 7.7d.

The analyticity requirement for the amplitude (7.45) is

$$\begin{aligned}
 & \left[G_{\gamma^* \rightarrow \phi} \frac{A_{\phi \rightarrow \gamma f_0}(M_\phi^2, M_{f_0}^2, 0)}{(s_V - M_\phi^2)(s_S - M_{f_0}^2)} g_{f_0 \rightarrow \pi\pi} + G_{\gamma^* \rightarrow \phi} \frac{B_\phi(M_\phi^2, s_S, 0)}{s_V - M_\phi^2} \right. \\
 & \quad \left. + \frac{B_{f_0}(s_V, M_{f_0}^2, 0)}{s_S - M_{f_0}^2} g_{f_0 \rightarrow \pi\pi} + B_0(s_V, s_S, 0) \right]_{s_V \rightarrow s_S} \sim (s_V - s_S). \tag{7.46}
 \end{aligned}$$

By this constraint we cancel the kinematic singularity on the first (physical) sheets of the complex variables s_V and s_S . Actually we do not know if this constraint works on unphysical sheets. But for small widths it looks reasonable to use (7.46) on the second sheet too. This expresses the hypothesis according to which the analytical continuation of the equality

$$\begin{aligned}
 & \left[G_{\gamma^* \rightarrow \phi} \frac{A_{\phi \rightarrow \gamma f_0}(M_\phi^2, M_{f_0}^2, 0)}{(s_V - M_\phi^2)(s_V - M_{f_0}^2)} g_{f_0 \rightarrow \pi\pi} + G_{\gamma^* \rightarrow \phi} \frac{B_\phi(M_\phi^2, s_V, 0)}{s_V - M_\phi^2} \right. \\
 & \quad \left. + \frac{B_{f_0}(s_V, M_{f_0}^2, 0)}{s_V - M_{f_0}^2} g_{f_0 \rightarrow \pi\pi} + B_0(s_V, s_V, 0) \right] = 0. \tag{7.47}
 \end{aligned}$$

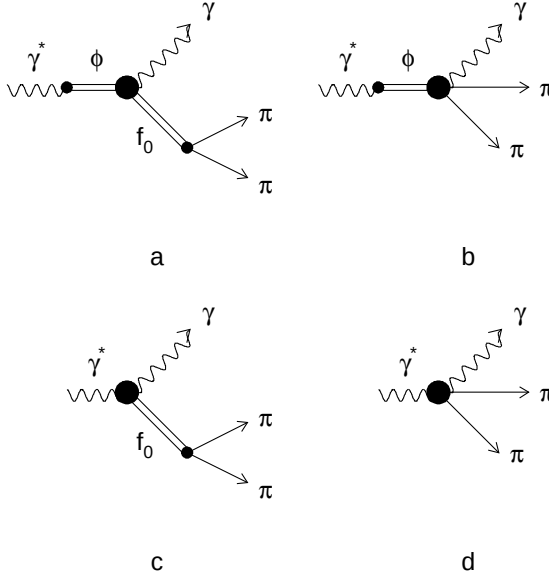


Fig. 7.7 The $e^+e^- \rightarrow \gamma^* \rightarrow \gamma\pi\pi$ process: residues in the γ^* and $(\pi\pi)_S$ channels determine the $\phi \rightarrow \gamma f_0$ amplitude.

is valid on the second sheet of s_V .

Owing to the pole terms in (7.47), this hypothesis leads to two additional constraints:

$$\left[G_{\gamma^* \rightarrow \phi} \frac{A_{\phi \rightarrow \gamma f_0}(M_\phi^2, M_{f_0}^2, 0)}{M_\phi^2 - M_{f_0}^2} g_{f_0 \rightarrow \pi\pi} + G_{\gamma^* \rightarrow \phi} B_\phi(M_\phi^2, M_\phi^2, 0) \right] = 0.$$

$$\left[G_{\gamma^* \rightarrow \phi} \frac{A_{\phi \rightarrow \gamma f_0}(M_\phi^2, M_{f_0}^2, 0)}{M_\phi^2 - M_{f_0}^2} g_{f_0 \rightarrow \pi\pi} + B_{f_0}(M_{f_0}^2, M_{f_0}^2, 0) g_{f_0 \rightarrow \pi\pi} \right] = 0.$$
(7.48)

The consideration presented in this section is an idealistic one: in reality we have no narrow f_0 mesons decaying into the $\pi\pi$ channel. The comparatively narrow resonance $f_0(980)$ is coupled with the $\pi\pi$ and $K\bar{K}$ channels, it is located near the $K\bar{K}$ threshold and is characterised by two poles.

7.3.1.2 Example of idealistic description of $\phi(1020) \rightarrow \gamma\pi\pi$

Nevertheless, to make clear our plan of further calculations, let us consider, as a first step, the ideal case: $f_0(980)$ is a standard Breit–Wigner reso-

nance, while the $K\bar{K}$ channel is strongly suppressed in the region under consideration and may be neglected. Moreover, since the width of $\phi(1020)$ is small ($\Gamma_\phi \simeq 4.5$ MeV), we consider the $\phi(1020)$ as a stable particle.

In this case the amplitude $\phi \rightarrow \gamma(\pi\pi)_S$ is determined by the residue of the pole term $1/(s_V - M_\phi^2)$ in (7.45). Supposing $\phi(1020)$ to be a stable particle, we put $M_\phi^2 = m_\phi^2 - im_\phi\Gamma_\phi \simeq m_\phi^2$. In this approximation we have:

$$A_{\mu\alpha}^{(\phi \rightarrow \gamma\pi\pi)}(m_\phi^2, s_S, 0) = \left(g_{\mu\alpha} - \frac{2q_\mu P_{\phi\alpha}}{m_\phi^2 - s_S} \right) \times \left[\frac{A_{\phi \rightarrow \gamma f_0}(m_\phi^2, M_{f_0}^2, 0)}{s_S - m_{f_0}^2 + i\Gamma_{f_0}m_{f_0}} g_{f_0 \rightarrow \pi\pi} + B_\phi(m_\phi^2, s_S, 0) \right], \quad (7.49)$$

where $m_\phi = 1020$ MeV.

The analyticity requirement in this case can be written as

$$\left[\frac{A_{\phi \rightarrow \gamma f_0}(m_\phi^2, M_{f_0}^2, 0)}{s_S - m_{f_0}^2 + i\Gamma_{f_0}m_{f_0}} g_{f_0 \rightarrow \pi\pi} + B_\phi(m_\phi^2, s_S, 0) \right] \sim (s_S - m_\phi^2). \quad (7.50)$$

Another requirement is related to the final state interactions of pions:

$$\begin{aligned} & \left(\frac{A_{\phi \rightarrow \gamma f_0}(m_\phi^2, M_{f_0}^2, 0)}{s_S - m_{f_0}^2 + i\Gamma_{f_0}m_{f_0}} g_{f_0 \rightarrow \pi\pi} + B_\phi(m_\phi^2, s_S, 0) \right) = \\ & = \left| \frac{A_{\phi \rightarrow \gamma f_0}(m_\phi^2, M_{f_0}^2, 0)}{s_S - m_{f_0}^2 + i\Gamma_{f_0}m_{f_0}} g_{f_0 \rightarrow \pi\pi} + B_\phi(m_\phi^2, s_S, 0) \right| \exp(i\delta_0^0(s_S)) \end{aligned} \quad (7.51)$$

The factor $\exp(i\delta_0^0(s_S))$, where $\delta_0^0(s_S)$ is the $\pi\pi$ scattering phase shift, appears in (7.51) owing to the pion rescatterings.

7.3.1.3 Description of the reaction $\phi(1020) \rightarrow \gamma(\pi\pi)_S$

The process $e^+e^- \rightarrow \gamma\pi\pi$ is determined by a number of subprocesses such as bremsstrahlung of photons by incoming electrons and outgoing pions, intermediate state transitions $\gamma^* \rightarrow \gamma + V'$, and so on. The discussion of all these subprocesses may be found in [7] (though in this paper the $f_0(980)$ is described as a standard one-pole resonance). Here we concentrate on the reaction $\gamma^* \rightarrow \phi(1020) \rightarrow \gamma(\pi\pi)_S$ considering the $f_0(980)$ in a realistic approach, *i.e.* taking into account the nearby threshold singularity at $s_S = 4m_{K\bar{K}}^2$.

As was noted above, the vector meson $\phi(1020)$ has a small decay width, $\Gamma_{\phi(1020)} \simeq 4.5$ MeV, and therefore it looks reasonable to treat $\phi(1020)$ as a stable particle. As to $f_0(980)$, the picture is not so well defined. In the PDG

compilation [6] the $f_0(980)$ width is given in the interval $40 \leq \Gamma_{f_0(980)} \leq 100$ MeV, and the width uncertainty is due not to the inaccuracy of the data (the experimental data are rather good) but to the vague definition of the width. The definition of the $f_0(980)$ width is aggravated by the $K\bar{K}$ threshold singularity that leads to the existence of two, not one, poles (this point was discussed in Chapter 3). Nevertheless, in the majority of analyses the width is determined by using the standard Breit–Wigner denominator, $1/(s_S - m_{f_0}^2 + i\Gamma_{f_0}m_{f_0})$, or its simple generalisation [8]:

$$\frac{1}{s_S - m_{f_0}^2 + i\Gamma_{f_0}m_{f_0}} \longrightarrow \frac{1}{s_S - m_{f_0}^2 + ig_{\pi\pi}^2\sqrt{s_S - 4m_{\pi\pi}^2} + ig_{K\bar{K}}^2\sqrt{s_S - 4m_{K\bar{K}}^2}}, \quad (7.52)$$

(here $s_S > 4m_{K\bar{K}}^2$). A more appropriate way for the description of the $f_0(980)$ is the application of the K -matrix approach (see, for example, [9, 10, 11] and references therein).

According to the K -matrix analyses [2, 11, 12], there are two poles in the $(IJ^{PC} = 00^{++})$ -wave at $s \sim 1.0$ GeV², namely, at $M^I \simeq 1.020 - i0.040$ GeV and $M^{II} \simeq 0.960 - i0.200$ GeV which are located on different complex- M sheets related to the $K\bar{K}$ -threshold (this was discussed in Chapters 2 and 3).

A significant trait of the K -matrix analysis is that it gives also, along with the characteristics of real resonances, the positions of levels before the onset of the decay channels, *i.e.* it determines the bare states. In addition, the K -matrix analysis allows us to observe the transformation of bare states into real resonances. In Chapter 3 we saw such a transformation of the 00^{++} -amplitude poles by switching off the decays $f_0 \rightarrow \pi\pi, K\bar{K}, \eta\eta, \eta\eta', \pi\pi\pi\pi$. After switching off the decay channels, the $f_0(980)$ turns into a stable state, approximately 300 MeV lower: $f_0(980) \longrightarrow f_0^{\text{bare}}(700 \pm 100)$.

The K -matrix amplitude of the 00^{++} -wave reconstructed in [2] gives us the possibility to trace the evolution of the transition form factor $\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(700 \pm 100)$ during the transformation of the bare state $f_0^{\text{bare}}(700 \pm 100)$ into the $f_0(980)$ resonance.

Using the diagrammatic language, one can say that the evolution of the form factor $F_{\phi \rightarrow \gamma f_0}^{(\text{bare})}$ occurs due to the processes shown in Fig. 7.8a: the ϕ -meson goes into $f_0^{\text{bare}}(n)$, with the emission of the photon, then $f_0^{\text{bare}}(n)$ decays into mesons $f_0^{\text{bare}}(n) \rightarrow h_a h_a = \pi\pi, K\bar{K}, \eta\eta, \eta\eta', \pi\pi\pi\pi$. The decay yields may rescatter thus coming to the final states.

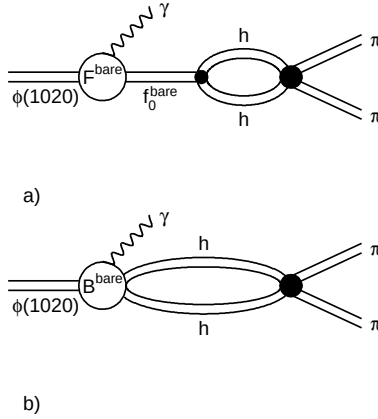


Fig. 7.8 Diagram for the $\phi(1020) \rightarrow \gamma\pi\pi$ transition, with the final state hadronic interaction taken into account, in the K -matrix approach (the right-hand side block $hh \rightarrow \pi\pi$): a) intermediate state production of f_0^{bare} and b) the background term.

With the use of the K -matrix technique, the amplitude $\phi(1020) \rightarrow \gamma\pi\pi$ is given by equation (7.49) with the following replacement (see Fig. 7.8):

$$\begin{aligned}
 & \left(g_{\mu\alpha} - \frac{2q_\mu P_{\phi\alpha}}{m_\phi^2 - s_S} \right) \left[\frac{A_{\phi \rightarrow \gamma f_0}(m_\phi^2, M_{f_0}^2, 0)}{s_S - m_{f_0}^2 + i\Gamma_{f_0} m_{f_0}} g_{f_0 \rightarrow \pi\pi} + B_\phi(m_\phi^2, s_S, 0) \right] \\
 & \longrightarrow \left(g_{\mu\alpha} - \frac{2q_\mu P_{\phi\alpha}}{m_\phi^2 - s_S} \right) \sum_a \left(\sum_n \frac{F_{\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(n)}^{(\text{bare})} g_a^{\text{bare}}(n)}{M_n^2 - s_S} + B_a(s_S) \right) \\
 & \times \left(\frac{1}{1 - i\hat{\rho}(s_S)\hat{K}(s_S)} \right)_{a,\pi\pi} = A_{\mu\alpha}^{(\phi \rightarrow \gamma\pi\pi)}(m_\phi^2, s_S, 0). \quad (7.53)
 \end{aligned}$$

Here the elements $K_{ab}(s_S)$, which correspond to meson rescatterings, contain the poles related to bare states:

$$K_{ab}(s) = \sum_n \frac{g_a^{\text{bare}}(n) g_b^{\text{bare}}(n)}{M_n^2 - s_S} + f_{ab}(s_S), \quad (7.54)$$

M_n is the mass of the bare state, and $g_a^{\text{bare}}(n)$ is the coupling for the transition $f_0^{\text{bare}}(n) \rightarrow a$ with $a = \pi\pi, K\bar{K}, \eta\eta, \eta\eta', \pi\pi\pi\pi$.

So, the factor $(1 - i\hat{\rho}(s)\hat{K}(s))^{-1}$ takes into account the rescatterings of the formed mesons. Recall that $\hat{\rho}(s_S)$ is the diagonal matrix of the phase spaces for hadronic states (for example, for the $\pi\pi$ system it reads: $\rho_{\pi\pi}(s_S) = \sqrt{(s_S - 4m_\pi^2)/s_S}$). The functions $B_a(s_S)$ and $f_{ab}(s)$ describe

background contributions, they are smooth in the right-hand side half-plane, at $\text{Re } s_S > 0$ (for details see Chapter 3).

The threshold condition now reads at $s_S \rightarrow m_\phi^2$:

$$\sum_a \left(\sum_n \frac{F_{\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(n)}^{\text{bare}} g_a^{\text{bare}}(n)}{M_n^2 - s_S} + B_a(s_S) \right) \left(\frac{1}{1 - i\hat{\rho}(s_S)\hat{K}(s_S)} \right)_{a,\pi\pi} \sim m_\phi^2 - s_S. \quad (7.55)$$

Since in the K -matrix approach the final state interaction is taken into account explicitly, the fitting procedure of the reaction $\phi \rightarrow \gamma\pi\pi$ should be performed with the threshold constraint (7.55) only. In the next section we give a more detailed consideration of the reaction $\phi \rightarrow \gamma\pi\pi$ in terms of the K -matrix.

7.3.2 Decay $\phi(1020) \rightarrow \gamma\pi\pi$: Non-relativistic quark model calculation of the form factor $\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(700)$ and the K -matrix consideration of the transition $f_0^{(\text{bare})}(700) \rightarrow \pi\pi$

It was emphasised above that the K -matrix analysis of meson spectra [2, 11, 13] and meson systematics [12, 14] indicates the quark–antiquark origin of $f_0(980)$. However, there exist widely discussed hypotheses where $f_0(980)$ is interpreted as a four-quark state [15], a $K\bar{K}$ molecule [16] or a vacuum scalar [17].

The radiative and weak decays involving $f_0(980)$ may give decisive arguments for understanding the nature of $f_0(980)$. In this way, as a first step, we consider the reaction $\phi(1020) \rightarrow \gamma f_0(980)$ in terms of the non-relativistic quark model, assuming $f_0(980)$ to be dominantly a $q\bar{q}$ state. The non-relativistic quark model is a good approach for the description of the lowest $q\bar{q}$ states of pseudoscalar and vector nonets, so one may hope that the lowest scalar $q\bar{q}$ states are also described with a reasonable accuracy. The choice of the non-relativistic approach for the analysis of the reaction $\phi(1020) \rightarrow \gamma f_0(980)$ was motivated by the fact that in its framework we can take into account not only the additive quark model processes (the emission of the photon by a constituent quark) but also those beyond it, using the dipole formula (the photon emission by the charge-exchange current gives us such an example). The dipole formula for the radiative transition of a vector state to a scalar one, $V \rightarrow \gamma S$, was applied before to the calculation of reactions with heavy quarks, see [18,

19] and references therein. Still, a straightforward application of the dipole formula to the reaction $\phi(1020) \rightarrow \gamma f_0(980)$ is hardly possible, for the $f_0(980)$ is certainly not a stable particle: this resonance is characterised by two poles laying on two different sheets of the complex M -plane, at $M^I = 1020 - i40$ MeV and $M^{II} = 960 - i200$ MeV. It should be emphasised that both these poles are important for the description of $f_0(980)$. Because of this, we use below the following method: we calculate the radiative transition to a stable bare f_0 state – this is $f_0^{\text{bare}}(700 \pm 100)$ and its parameters were obtained in the K -matrix analysis, see Chapter 3. This way we find the description of the process $\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(700 \pm 100)$; further, we switch on the hadronic decays and determine the transition $\phi(1020) \rightarrow \gamma \pi \pi$. The residue in the pole of this amplitude is the radiative transition amplitude $\phi(1020) \rightarrow \gamma f_0(980)$. This procedure gives us a successful description of the data for $\phi(1020) \rightarrow \gamma \pi \pi$ and $\phi(1020) \rightarrow \gamma f_0(980)$ if we assume that $f_0(980)$ is dominated by the quark–antiquark state.

We calculate the transition $\phi \rightarrow \gamma f_0^{\text{bare}}$ making use of two hypotheses:

- (i) The photon is emitted only by constituent quarks manifesting the dominance of the additive quark model.
- (ii) In the second version we suppose that the charge-exchange current provides a significant contribution to the transition $\phi \rightarrow \gamma f_0^{\text{bare}}$; then the corresponding form factor should be described by the dipole formula.

The matter is that the $f_0^{\text{bare}}(700)$ -mesons is a mixture of the $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$ and $s\bar{s}$ components. Such a multichannel structure of the $f_0^{\text{bare}}(700)$ may lead to the existence of the t -channel charge-exchange currents responsible for the transition $n\bar{n} \rightarrow s\bar{s}$.

7.3.2.1 The $V \rightarrow \gamma S$ process in the framework of the non-relativistic quark model

In the framework of the non-relativistic quark model we consider here the $V \rightarrow \gamma S$ transition for both cases: when charge-current exchange forces are absent or existing.

(i) Wave functions for vector and scalar composite particles.

The $q\bar{q}$ wave functions of vector (V) and scalar (S) particles are defined as follows:

$$\Psi_{V\mu}(\mathbf{k}) = \sigma_\mu \psi_V(k^2), \quad \Psi_S(\mathbf{k}) = (\boldsymbol{\sigma} \cdot \mathbf{k}) \psi_S(k^2), \quad (7.56)$$

where, using Pauli matrices, the spin factors are singled out. The parts dependent on the relative momentum squared are related to the vertices in

the following way:

$$\psi_V(k^2) = \frac{\sqrt{m}}{2} \frac{G_V(k^2)}{k^2 + m\varepsilon_V}, \quad \psi_S(k^2) = \frac{1}{2\sqrt{m}} \frac{G_S(k^2)}{k^2 + m\varepsilon_V}. \quad (7.57)$$

Here m is the quark mass, ε is the binding energy of the composite system: $\varepsilon_V = 2m - m_V$ and $\varepsilon_S = 2m - m_S$, where m_V and m_S are the masses of the bound states. The normalisation condition for the wave functions reads

$$\begin{aligned} \int \frac{d^3k}{(2\pi)^3} \text{Sp}_2 [\Psi_S^+(\mathbf{k})\Psi_S(\mathbf{k})] &= \int \frac{d^3k}{(2\pi)^3} \psi_S^2(k^2) \text{Sp}_2[(\boldsymbol{\sigma} \cdot \mathbf{k})(\boldsymbol{\sigma} \cdot \mathbf{k})] = 1, \\ \int \frac{d^3k}{(2\pi)^3} \text{Sp}_2 [\Psi_{V\mu}^+(\mathbf{k})\Psi_{V\mu'}(\mathbf{k})] &= \int \frac{d^3k}{(2\pi)^3} \psi_V^2(k^2) \text{Sp}_2[\sigma_\mu\sigma_{\mu'}] = \delta_{\mu\mu'}. \end{aligned} \quad (7.58)$$

(ii) Amplitude in the additive quark model.

In terms of the wave functions (7.56) the transition amplitude is written as follows:

$$\begin{aligned} \epsilon_\mu^{(V)} \epsilon_\alpha^{(\gamma)} A_{\mu\alpha}^{V \rightarrow \gamma S} &= e Z_{V \rightarrow \gamma S} \epsilon_\mu^{(V)} \epsilon_\alpha^{(\gamma)} F_{\mu\alpha}^{V \rightarrow \gamma S}, \\ F_{\mu\alpha}^{V \rightarrow \gamma S} &= \int \frac{d^3k}{(2\pi)^3} \text{Sp}_2 [\Psi_S^+(\mathbf{k}) 4k_\alpha \Psi_{V\mu}(\mathbf{k})]. \end{aligned} \quad (7.59)$$

Here $\epsilon_\mu^{(V)}$ and $\epsilon_\alpha^{(\gamma)}$ are polarisation vectors for V and γ : $\epsilon_\mu^{(V)} p_{V\mu} = 0$ and $\epsilon_\alpha^{(\gamma)} q_\alpha = 0$. The charge factor $Z_{V \rightarrow \gamma S}$ being different for different reactions is specified below. The expression for the transition amplitude (7.59) can be simplified after the substitution in the integrand

$$\text{Sp}_2[\sigma_\mu(\boldsymbol{\sigma} \cdot \mathbf{k})] k_\alpha \rightarrow \frac{2}{3} k^2 g_{\mu\alpha}^{\perp\perp}, \quad (7.60)$$

where, remind, $g_{\mu\alpha}^{\perp\perp}$ is the metric tensor in the space orthogonal to the momenta of the vector particle p_V and the photon q . The substitution (7.60) results in

$$\begin{aligned} A_{\mu\alpha}^{V \rightarrow \gamma S} &= e g_{\mu\alpha}^{\perp\perp} A_{V \rightarrow \gamma S}, \\ A_{V \rightarrow \gamma S} &= Z_{V \rightarrow \gamma S} \int_0^\infty \frac{dk^2}{\pi} \psi_S(k^2) \psi_V(k^2) \frac{2}{3\pi} k^3. \end{aligned} \quad (7.61)$$

The amplitudes $A_{\mu\alpha}^{V \rightarrow \gamma S}$ and $A_{V \rightarrow \gamma S}$ in the form (7.61) were used in [20, 21] where the relativistic and non-relativistic treatments of the decay amplitude $\phi(1020) \rightarrow \gamma f_0(980)$ were discussed.

However, for further discussion it would be suitable not to deal with equation (7.61) but to use the form factor $F_{\mu\alpha}^{V\rightarrow\gamma S}$ (7.59) rewritten in the coordinate representation. One has

$$\Psi_{V\mu}(\mathbf{k}) = \int d^3r e^{i\mathbf{k}\cdot\mathbf{r}} \Psi_{V\mu}(\mathbf{r}), \quad \Psi_S(\mathbf{k}) = \int d^3r e^{i\mathbf{k}\cdot\mathbf{r}} \Psi_S(\mathbf{r}). \quad (7.62)$$

Then the form factor $F_{\mu\alpha}^{V\rightarrow\gamma S}$ can be represented as follows:

$$F_{\mu\alpha}^{V\rightarrow\gamma S} = \int d^3r \text{Sp}_2 [\Psi_S^+(\mathbf{r}) 4k_\alpha \Psi_{V\mu}(\mathbf{r})], \quad (7.63)$$

where k_α is the operator $k_\alpha = -i\nabla_\alpha$. This operator can be written as the commutator of r_α and the kinetic energy $T = -\nabla^2/m$:

$$2i m(T r_\alpha - r_\alpha T) = 4(-i\nabla_\alpha). \quad (7.64)$$

Let us consider the case when the quark-quark interaction is rather simple, say, it is given by the relative interquark distance with the potential $U(r)$. For vector and scalar composite systems we use also an additional simplifying assumption: vector and scalar mesons consist of quarks of the same flavour ($q\bar{q}$). If so, we have the following Hamiltonian for ($q\bar{q}$)-states:

$$H = -\frac{\nabla^2}{m} + U(r), \quad (7.65)$$

and can rewrite (7.64) as

$$2i m(H r_\alpha - r_\alpha H) = 4(-i\nabla_\alpha). \quad (7.66)$$

After substituting the commutator in (7.63), the transition form factor for the reaction $V \rightarrow \gamma S$ reads

$$F_{\mu\alpha}^{V\rightarrow\gamma S} = \int d^3r \text{Sp}_2 [\Psi_S^+(r) r_\alpha \Psi_{V\mu}(r)] 2i m(\varepsilon_V - \varepsilon_S). \quad (7.67)$$

Here we have used that $(H + \varepsilon_V)\Psi_V = 0$ and $(H + \varepsilon_S)\Psi_S = 0$.

The factor $\varepsilon_V - \varepsilon_S$ in the right-hand side (7.67) is a manifestation of the threshold theorem: at $(\varepsilon_V - \varepsilon_S) = (m_S - m_V) \rightarrow 0$ the form factor $F_{\mu\alpha}^{V\rightarrow\gamma S}$ turns to zero. Actually, in the additive quark model the amplitude of the $V \rightarrow \gamma S$ transition cannot be zero if V and S are basic states with a radial quantum number $n = 1$: in this case the wave functions $\psi_V(k^2)$ and $\psi_S(k^2)$ do not change sign, and the right-hand side (7.61) does not equal zero. To resolve this contradiction, let us consider as an example the wave functions $\psi_V(k^2)$ and $\psi_S(k^2)$ in an exponential form.

(iii) Basic vector and scalar $q\bar{q}$ states: an example of the exponential approach to wave functions.

We parametrise the ground-state wave functions of scalar and vector particles as follows:

$$\begin{aligned}\Psi_{V_\mu}(r) &= \sigma_\mu \psi_V(r^2), & \psi_V(r^2) &= \frac{1}{2^{5/4} \pi^{3/4} b_V^{3/4}} \exp \left[-\frac{r^2}{4b_V} \right], \\ \Psi_S(r) &= (\boldsymbol{\sigma} \cdot \mathbf{r}) \psi_S(r^2), & \psi_S(r^2) &= \frac{i}{2^{5/4} \pi^{3/4} b_S^{5/4} \sqrt{3}} \exp \left[-\frac{r^2}{4b_S} \right].\end{aligned}\quad (7.68)$$

The wave functions with $n = 1$ have no nodes; the numerical factors take into account the normalisation conditions

$$\int d^3r \text{Sp}_2 [\Psi_S^+(\mathbf{r}) \Psi_S(\mathbf{r})] = 1, \quad \int d^3r \text{Sp}_2 [\Psi_{V_\mu}^+(r) \Psi_{V_{\mu'}}(r)] = \delta_{\mu\mu'}. \quad (7.69)$$

With exponential wave functions the matrix element for $V \rightarrow \gamma S$ given by the additive quark model diagram, equation (7.63), is equal to

$$\epsilon_\mu^{(V)} \epsilon_\alpha^{(\gamma)} F_{\mu\alpha}^{V \rightarrow \gamma S}(\text{additive}) = (\epsilon^{(V)} \epsilon^{(\gamma)}) \frac{2^{7/2}}{\sqrt{3}} \frac{b_V^{3/4} b_S^{5/4}}{(b_V + b_S)^{5/2}}. \quad (7.70)$$

The formula for $F_{\mu\alpha}^{V \rightarrow \gamma S}$ written in the frame of the dipole emission, see (7.67), reads

$$\epsilon_\mu^{(V)} \epsilon_\alpha^{(\gamma)} F_{\mu\alpha}^{V \rightarrow \gamma S}(\text{dipole}) = (\epsilon^{(V)} \epsilon^{(\gamma)}) \frac{2^{7/2}}{\sqrt{3}} \frac{b_V^{7/4} b_S^{5/4}}{(b_V + b_S)^{5/2}} m(m_V - m_S). \quad (7.71)$$

In the considered case (one-flavour quarks with a Hamiltonian given by (7.65)) the equations (7.70) and (7.71) coincide, $F_{\mu\alpha}^{V \rightarrow \gamma S}(\text{additive}) = F_{\mu\alpha}^{V \rightarrow \gamma S}(\text{dipole})$, therefore

$$m(m_V - m_S) = b_V^{-1}, \quad (7.72)$$

which means that the factor $(\varepsilon_S - \varepsilon_V)$ in the right-hand side (7.67) is related to the difference between the V and S levels and is defined by b_V only. In this way, the form factor $F_{\mu\alpha}^{V \rightarrow \gamma S}$ turns to zero only when b_V (or b_S) tends to infinity.

The considered example does not mean that the threshold theorem for the reaction $V \rightarrow \gamma S$ does not work, it tells us only that we should interpret and use it carefully.

7.3.2.2 *Quantum mechanical consideration of the reaction*
 $\phi \rightarrow \gamma f_0$ *with the simplifying assumption of*
 ϕ *and* f_0 *being stable particles*

We have considered above the model for the reaction $V \rightarrow \gamma S$, when V and S are formed by quarks of the same flavour (one-channel model for V and S). The one-channel approach for $\phi(1020)$ (the dominance of the $s\bar{s}$ component) looks acceptable, though for f_0 mesons it is definitely not so: scalar–isoscalar states are multicomponent ones.

The existence of several components in the f_0 -mesons changes the picture of the $\phi \rightarrow \gamma f_0$ decays: equations (7.63) and (7.67) for the $\phi \rightarrow \gamma f_0$ decay turn out to be non-equivalent because of a possible photon emission by the t -channel exchange currents.

As a next step, we consider in detail a simple model for ϕ and f_0 : the ϕ meson is treated as an $s\bar{s}$ -system, with no admixture of either the non-strange quarkonium, $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$, or the gluonium (gg), while the f_0 meson is a mixture of $s\bar{s}$ and gg . Despite its simplicity, this model can be used as a guide for the rough study of the reaction $\phi(1020) \rightarrow \gamma f_0(980)$. Indeed, $\phi(1020)$ is almost a pure $s\bar{s}$ state, the admixture of the $n\bar{n}$ component in $\phi(1020)$ is small, $\leq 5\%$. Concerning $f_0(980)$, the K -matrix fit to the data gives the following constraints for the $s\bar{s}$, $n\bar{n}$ and gg -components in $f_0(980)$ [2, 12]: $50\% \lesssim W_{s\bar{s}}[f_0(980)] < 100\%$, $0 \lesssim W_{n\bar{n}}[f_0(980)] < 50\%$, $0 \lesssim W_{gg}[f_0(980)] < 25\%$. Also, the $f_0(980)$ may contain a long-range $K\bar{K}$ component, on the level of 10–20%. Therefore, these estimates permit the version when the probability of the $n\bar{n}$ component is small, and $f_0(980)$ is a mixture of $s\bar{s}$ and gg only.

Bearing in mind this estimate, we consider such a two-component model for ϕ and f_0 , though supposing for the sake of simplicity that these particles are stable with respect to hadronic decays. Note that it is not difficult to generalise the two-component model for f_0 to the three-component one ($f_0 \rightarrow n\bar{n}, s\bar{s}, gg$): the corresponding formulae are also given in this section.

(i) Two-component model ($s\bar{s}$, gg) for f_0 and ϕ .

Let us now present the model where f_0 has only two components: the strange quarkonium ($s\bar{s}$ in the P wave) and the gluonium (gg in the S wave). The spin structure of the $s\bar{s}$ wave function is given in section 7.2: it contains the factor $(\boldsymbol{\sigma} \cdot \mathbf{r})$ in the coordinate representation. For the gg system we have the spin operator δ_{ab} or, in terms of polarisation vectors, the convolution $(\boldsymbol{\epsilon}_1^{(g)} \boldsymbol{\epsilon}_2^{(g)})$. We consider a simple interaction, when the potential does not depend on spin variables — in this case one may forget about

the vector structure of gg working as if the gluon component consisted of spinless particles. Concerning ϕ , it is considered as a pure $s\bar{s}$ state in the S wave, with the wave function spin factor $\sim \sigma_\mu$, see section 7.2. So, the wave functions of f_0 and ϕ mesons are written as follows:

$$\begin{aligned}\hat{\Psi}_{f_0}(\mathbf{r}) &= \begin{pmatrix} \Psi_{f_0(s\bar{s})}(\mathbf{r}) \\ \Psi_{f_0(gg)}(r) \end{pmatrix} = \begin{pmatrix} (\boldsymbol{\sigma} \cdot \mathbf{r})\psi_{f_0(s\bar{s})}(r) \\ \psi_{f_0(gg)}(r) \end{pmatrix}, \\ \hat{\Psi}_{\phi\mu}(\mathbf{r}) &= \begin{pmatrix} \Psi_{\phi(s\bar{s})\mu}(\mathbf{r}) \\ \Psi_{\phi(gg)\mu}(r) \end{pmatrix} = \begin{pmatrix} \sigma_\mu\psi_{\phi(s\bar{s})}(r) \\ 0 \end{pmatrix}.\end{aligned}\quad (7.73)$$

The normalisation condition is given by (7.69), with the obvious replacement: $\Psi_S \rightarrow \hat{\Psi}_{f_0}$ and $\Psi_{V\mu} \rightarrow \hat{\Psi}_{\phi\mu}$.

The Schrödinger equation for the two-component states, $s\bar{s}$ and gg , reads

$$\begin{vmatrix} k^2/m + U_{s\bar{s} \rightarrow s\bar{s}}(r) & U_{s\bar{s} \rightarrow gg}(\mathbf{r}) \\ U_{s\bar{s} \rightarrow gg}^+(\mathbf{r}) & k^2/m_g + U_{gg \rightarrow gg}(r) \end{vmatrix} \begin{pmatrix} \Psi_{s\bar{s}}(\mathbf{r}) \\ \Psi_{gg}(\mathbf{r}) \end{pmatrix} = E \begin{pmatrix} \Psi_{s\bar{s}}(\mathbf{r}) \\ \Psi_{gg}(\mathbf{r}) \end{pmatrix}.\quad (7.74)$$

Further, we denote the Hamiltonian in the left-hand side of (7.74) as $\hat{H}_0$.

We put the gg component in ϕ to be zero. This means that the potential $U_{s\bar{s} \rightarrow gg}(\mathbf{r})$ satisfies the following constraints:

$$\langle 0^+ s\bar{s} | U_{s\bar{s} \rightarrow gg}(\mathbf{r}) | 0^+ gg \rangle \neq 0, \quad \langle 1^- s\bar{s} | U_{s\bar{s} \rightarrow gg}(\mathbf{r}) | 1^- gg \rangle = 0. \quad (7.75)$$

These constraints do not look surprising for mesons in the region 1.0–1.5 GeV because the scalar glueball is located just in this mass region, while the vector one has a considerably higher mass, ~ 2.5 GeV [22].

(ii) Dipole emission of the photon in $\phi \rightarrow \gamma f_0$ decay.

To describe the interaction of a composite system with the electromagnetic field, we should consider the full Hamiltonian which reads:

$$\hat{H}(0) = \begin{vmatrix} (k_1^2 + k_2^2)/2m + U_{s\bar{s} \rightarrow s\bar{s}}(\mathbf{r}_1 - \mathbf{r}_2) & \hat{U}_{s\bar{s} \rightarrow gg}(\mathbf{r}_1 - \mathbf{r}_2) \\ \hat{U}_{s\bar{s} \rightarrow gg}(\mathbf{r}_1 - \mathbf{r}_2) & (k_1^2 + k_2^2)/2m_g + U_{gg \rightarrow gg}(\mathbf{r}_1 - \mathbf{r}_2) \end{vmatrix}.\quad (7.76)$$

The coordinates ($\mathbf{r}_a$) and the momenta ($\mathbf{k}_a = -i\nabla_a$) of the constituents are related to the characteristics of the centre-of-mass system of ($\mathbf{R}, \mathbf{P}$) and relative motion ($\mathbf{r}, \mathbf{k}$) as follows:

$$\mathbf{r}_1 = \frac{1}{2}\mathbf{r} + \mathbf{R}, \quad \mathbf{r}_2 = -\frac{1}{2}\mathbf{r} + \mathbf{R}, \quad \mathbf{k}_1 = \mathbf{k} + \frac{1}{2}\mathbf{P}, \quad \mathbf{k}_2 = -\mathbf{k} + \frac{1}{2}\mathbf{P}. \quad (7.77)$$

The electromagnetic interaction is included into our consideration by substituting in (7.76)

$$\mathbf{k}_1^2 \rightarrow (\mathbf{k}_1 - e_1 \mathbf{A}(r_1))^2, \quad \mathbf{k}_2^2 \rightarrow (\mathbf{k}_2 - e_2 \mathbf{A}(r_2))^2, \quad (7.78)$$

$$\hat{U}_{s\bar{s} \rightarrow gg}(\mathbf{r}_1 - \mathbf{r}_2) \rightarrow \hat{U}_{s\bar{s} \rightarrow gg}(\mathbf{r}_1 - \mathbf{r}_2) \exp \left[ie_1 \int_{-\infty}^{\mathbf{r}_1} d\mathbf{r}'_\alpha A_\alpha(r') + ie_2 \int_{-\infty}^{\mathbf{r}_2} d\mathbf{r}'_\alpha A_\alpha(r') \right],$$

with $e_1 = -e_2 = e_s$. After that we obtain the gauge-invariant Hamiltonian $\hat{H}(\mathbf{A})$. Indeed, it is invariant under the transformation:

$$\hat{H}(\mathbf{A}) = \hat{\chi}^+ \hat{H}(\mathbf{A} + \nabla\chi) \hat{\chi}, \quad (7.79)$$

where the following substitution is made:

$$\mathbf{A}(r_a) \rightarrow \mathbf{A}(r_a) + \nabla\chi(r_a), \quad (7.80)$$

with the matrix $\hat{\chi}$ which is written as:

$$\hat{\chi} = \begin{vmatrix} \exp[ie_s\chi(r_1) - ie_s\chi(r_2)] & , & 0 \\ 0 & , & 1 \end{vmatrix}. \quad (7.81)$$

For the transition $\phi \rightarrow \gamma f_0$, keeping the terms proportional to the s -quark charge, e_s , we have the following operator for the dipole emission:

$$\hat{d}_\alpha = \begin{vmatrix} 2(k_{1\alpha} - k_{2\alpha}) & , & i(r_{1\alpha} - r_{1\alpha})\hat{U}_{s\bar{s} \rightarrow gg}(\mathbf{r}_1 - \mathbf{r}_2) \\ -i(r_{1\alpha} - r_{1\alpha})\hat{U}_{s\bar{s} \rightarrow gg}(\mathbf{r}_1 - \mathbf{r}_2) & , & 0 \end{vmatrix}. \quad (7.82)$$

We should emphasise that here we consider a particular example of interaction. There exist, of course, other mechanisms of the photon emission which, being beyond the additive quark model, lead us to the dipole formula for the $V \rightarrow \gamma S$ transition; an example is provided by the **(LS)**-interaction in the quark-antiquark component [18, 19]. The short-range **(LS)**-interaction in the $q\bar{q}$ systems was discussed in [23, 24] as a source of the nonet splitting. Actually the point-like **(LS)**-interaction gives us (v/c) -corrections to the non-relativistic approach. In the relativistic quark model approaches based on the Bethe-Salpeter equation the gluon-exchange forces result in a similar nonet splitting as for the **(LS)**-interaction; see, for example, [25].

For the transition $V \rightarrow \gamma S$, where we keep the terms proportional to the charge e , we have the following operator for the dipole emission:

$$\hat{d}_\alpha = \begin{vmatrix} 2k_\alpha & , & ir_\alpha U_{s\bar{s} \rightarrow gg}(\mathbf{r}) \\ -ir_\alpha U_{s\bar{s} \rightarrow gg}^+(\mathbf{r}) & , & 0 \end{vmatrix}. \quad (7.83)$$

The transition form factor is given by a formula similar to (7.63) for the one-channel case, it reads

$$F_{\mu\alpha}^{\phi \rightarrow \gamma f_0} = \int d^3r \text{Sp}_2 \left[\hat{\Psi}_{f_0}^+(\mathbf{r}) 2\hat{d}_\alpha \hat{\Psi}_{\phi\mu}(\mathbf{r}) \right]. \quad (7.84)$$

Drawing explicitly the two-component wave functions, one can rewrite equation (7.84) as follows:

$$F_{\mu\alpha}^{\phi \rightarrow \gamma f_0} = \int d^3r \text{Sp}_2 \left[\Psi_{f_0(s\bar{s})}^+(\mathbf{r}) 4k_\alpha \Psi_{\phi(s\bar{s})\mu}(\mathbf{r}) \right] \\ + \int d^3r \text{Sp}_2 \left[\Psi_{f_0(gg)}^+(\mathbf{r}) (-ir_\alpha U_{gg \rightarrow s\bar{s}}(\mathbf{r})) \Psi_{\phi(s\bar{s})\mu}(\mathbf{r}) \right]. \quad (7.85)$$

The first term in the right-hand side (7.85), with the operator $4k_\alpha$, is responsible for the interaction of a photon with a constituent quark. This is the additive quark model contribution, while the term $(-ir_\alpha U_{gg \rightarrow s\bar{s}}(\mathbf{r}))$ describes the interaction of the photon with the charge flowing through the t -channel – this term describes the photon interaction with the fermion exchange current.

Let us return to Eq. (7.84) and rewrite it in a form similar to (7.67). One can see that

$$i m \left(\hat{H}_0 \hat{r}_\alpha - \hat{r}_\alpha \hat{H}_0 \right) = \hat{d}_\alpha, \quad (7.86)$$

where $\hat{H}_0$ is the Hamiltonian for composite systems written in the left-hand side of (7.74), and the operator $\hat{r}_\alpha$ is determined as

$$\hat{r}_\alpha = \begin{pmatrix} r_\alpha & 0 \\ 0 & 0 \end{pmatrix}. \quad (7.87)$$

Substituting equation (7.86) into (7.84), we have for the dipole emission of a photon:

$$F_{\mu\alpha}^{\phi \rightarrow \gamma f_0} = \int d^3r \text{Sp}_2 \left[(\boldsymbol{\sigma} \cdot \mathbf{r}) \psi_{f_0(s\bar{s})}(r) r_\alpha \sigma_\mu \psi_{\phi(s\bar{s})}(r) \right] 2i m (\varepsilon_\phi - \varepsilon_{f_0}). \quad (7.88)$$

This formula is similar to (7.67) for the one-channel model.

(iii) Partial width of the $\phi \rightarrow \gamma f_0$ decay.

The partial width of the decay $\phi \rightarrow \gamma f_0$ in the case when ϕ is a pure $s\bar{s}$ state is determined by the following formula:

$$m_\phi \Gamma_{\phi \rightarrow \gamma f_0} = \frac{1}{6} \alpha \frac{m_\phi^2 - m_{f_0}^2}{m_\phi^2} \left| A_{\phi \rightarrow \gamma f_0(s\bar{s})} \right|^2, \quad (7.89)$$

with $\alpha = 1/137$ and the $A_{\phi \rightarrow \gamma f_0(s\bar{s})}$ given by (7.61), with obvious substitutions $V \rightarrow \phi$, $S \rightarrow f_0$ and $Z_{V \rightarrow \gamma S} \rightarrow Z_{\phi \rightarrow \gamma f_0} = -2/3$.

(iv) Three-component model ($s\bar{s}$, $n\bar{n}$, gg) for f_0 and ϕ .

The above formula can be easily generalised for the case when f_0 is a three-component system ($s\bar{s}$, $n\bar{n}$, gg) and ϕ is a two-component one ($s\bar{s}$,

$n\bar{n}$), while gg is supposed to be negligibly small in ϕ . We have two transition form factors:

$$\begin{aligned} F_{\mu\alpha}^{\phi \rightarrow \gamma f_0(s\bar{s})} &= \int d^3r \text{Sp}_2 [(\boldsymbol{\sigma}\mathbf{r})\psi_{f_0(s\bar{s})}(r)r_\alpha\sigma_\mu\psi_{\phi(s\bar{s})}(r)] \ 2i \ m(\varepsilon_\phi - \varepsilon_{f_0}) , \\ F_{\mu\alpha}^{\phi \rightarrow \gamma f_0(n\bar{n})} &= \int d^3r \text{Sp}_2 [(\boldsymbol{\sigma}\mathbf{r})\psi_{f_0(n\bar{n})}(r)r_\alpha\sigma_\mu\psi_{\phi(n\bar{n})}(r)] \ 2i \ m(\varepsilon_\phi - \varepsilon_{f_0}). \end{aligned} \quad (7.90)$$

The partial width reads

$$m_\phi \Gamma_{\phi \rightarrow \gamma f_0} = \frac{1}{6} \alpha \frac{m_\phi^2 - m_{f_0}^2}{m_\phi^2} |A_{\phi \rightarrow \gamma f_0(s\bar{s})} + A_{\phi \rightarrow \gamma f_0(n\bar{n})}|^2, \quad (7.91)$$

with $A_{\phi \rightarrow \gamma f_0}$ defined by (7.61). The charge factors, which were separated in (7.59), are equal to $Z_{\phi \rightarrow \gamma f_0}^{(s\bar{s})} = -2/3$, $Z_{\phi \rightarrow \gamma f_0}^{(n\bar{n})} = 1/3$; the combinatorial factor 2 is related to two diagrams with photon emission by a quark and an antiquark, see [20, 21] for more details.

7.3.2.3 *K-matrix calculation of the decay amplitude*

$$\phi(1020) \rightarrow \gamma f_0(980)$$

As was discussed above, we treat $\phi(1020)$ as a stable particle. The pole structure of the $f_0(980)$ is more complicated: the $K\bar{K}$ threshold singularity leads to the existence of two poles, see Fig. 7.9. By switching off the decay $f_0(980) \rightarrow K\bar{K}$, both poles begin to move to one another, and they coincide after switching off the $K\bar{K}$ channel completely. Usually, when one discusses the $f_0(980)$, the resonance is characterised by the closest pole on the second sheet, $M^I = 1020 - i40$ MeV. However, when we are interested in how far from each other $\phi(1020)$ and $f_0(980)$ are, we should not forget about the second pole on the third sheet, $M^{II} = 960 - i200$ MeV. Keeping in mind the existence of two poles, one should accept that the $f_0(980)$ resonance can hardly be represented as a stable particle.

The pole residues in the $\pi\pi$ channel of the amplitude $\phi(1020) \rightarrow \gamma\pi\pi$ provides us with two transition amplitudes $\phi(1020) \rightarrow \gamma f_0^N(980)$, with $N = I, II$ (recall that the resonance poles are contained in the factor $[1 - i\hat{\rho}(s)\hat{K}(s)]^{-1}$). Near the pole which we study, the amplitude (7.53) for the $\phi(1020) \rightarrow \gamma\pi\pi$ transition is written as:

$$\begin{aligned} \sum_a \left(\sum_n \frac{F_{\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(n)}^{\text{(bare)}} g_a^{\text{bare}}(n)}{M_n^2 - s_S} + B_a(s_S) \right) \left[\frac{1}{1 - i\hat{\rho}(s_S)\hat{K}(s_S)} \right]_{a,\pi\pi} \\ \simeq \frac{A_{\phi(1020) \rightarrow \gamma f_0(980)}^N}{M_{f_0(980)}^{N2} - s_S} g_{f_0(980) \rightarrow \pi\pi}^N + \text{smooth contributions} . \end{aligned} \quad (7.92)$$

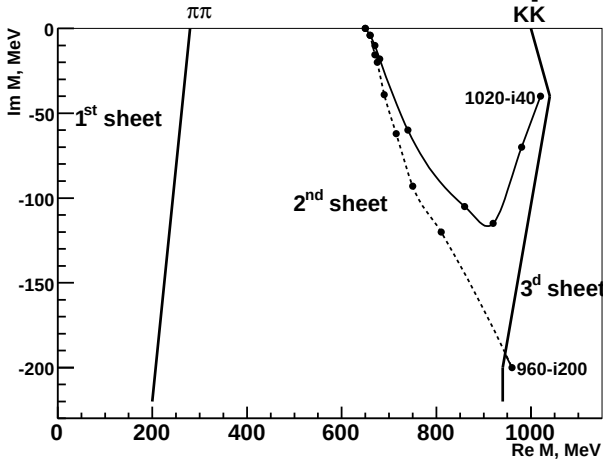


Fig. 7.9 The complex- M plane (we denote $M = \sqrt{s_S}$) and the location of the poles in the vicinity of $f_0(980)$; the cuts related to the $\pi\pi$ and $K\bar{K}$ thresholds are shown as thick solid lines. The trajectories of the pole motion corresponding to a uniform onset of the decay channels are shown for the $f_0(980)$: the solid lines give the trajectories on the visible parts of the second and third sheets, the dotted line is the trajectory of the second pole on the non-visible part of the third sheet.

Remind that M_n is the mass of bare state, while $M_{f_0}^N(980)$ is the complex-valued resonance mass: $M_{f_0(980)}^I \rightarrow M^I \simeq 1020 - i40$ MeV for the first pole, and $M_{f_0(980)}^{II} \rightarrow M^{II} \simeq 960 - i200$ MeV for the second one. The transition amplitudes $A_{\phi(1020) \rightarrow \gamma f_0(980)}^I$ and $A_{\phi(1020) \rightarrow \gamma f_0(980)}^{II}$ are different for different poles. The couplings $g_{f_0(980) \rightarrow \pi\pi}^I$ and $g_{f_0(980) \rightarrow \pi\pi}^{II}$ are different as well.

We see that the radiative transition $\phi(1020) \rightarrow \gamma f_0(980)$ is determined by two amplitudes, $A_{\phi(1020) \rightarrow \gamma f_0(M^I)} \equiv A_{\phi(1020) \rightarrow \gamma f_0(980)}^I$ and $A_{\phi(1020) \rightarrow \gamma f_0(M^{II})} \equiv A_{\phi(1020) \rightarrow \gamma f_0(980)}^{II}$, and just these amplitudes are the subjects of our interest in the investigation of $\phi(1020) \rightarrow \gamma f_0(980)$.

The amplitudes $A_{\phi(1020) \rightarrow \gamma f_0(980)}^I$, $A_{\phi(1020) \rightarrow \gamma f_0(980)}^{II}$ are contributions of different bare states:

$$\begin{aligned}
 A_{\phi(1020) \rightarrow \gamma f_0(980)}^I &= \sum_n \zeta_n^{(I)}[f_0(980)] F_{\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(n)}^{(\text{bare})}, \\
 A_{\phi(1020) \rightarrow \gamma f_0(980)}^{II} &= \sum_n \zeta_n^{(II)}[f_0(980)] F_{\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(n)}^{(\text{bare})}. \quad (7.93)
 \end{aligned}$$

To calculate the constants $\zeta_n[f_0(m_R)]$, we use the K -matrix solution for the 00^{++} wave amplitude denoted in Chapter 2 as II-2 (see also [2]). In

this solution there are five bare states $f_0^{\text{bare}}(n)$ in the mass interval 290–1950 MeV: four of them are members of the $q\bar{q}$ nonets ($1^3P_0q\bar{q}$ and $2^3P_0q\bar{q}$) and the fifth state is the glueball. Namely:

$$\begin{aligned} 1^3P_0q\bar{q} : & f_0^{\text{bare}}(700 \pm 100), \quad f_0^{\text{bare}}(1220 \pm 30), \\ 2^3P_0q\bar{q} : & f_0^{\text{bare}}(1230 \pm 40), \quad f_0^{\text{bare}}(1800 \pm 40), \\ \text{glueball} : & f_0^{\text{bare}}(1580 \pm 50). \end{aligned} \quad (7.94)$$

For the first pole of $f_0(980)$, $M^I = 1020 - i40$ MeV, we have:

$$\begin{aligned} \zeta_{700}^{(I)}[f_0(980)] &= 0.62 \exp(-i144^\circ), \quad \zeta_{1220}^{(I)}[f_0(980)] = 0.37 \exp(-i41^\circ), \\ \zeta_{1230}^{(I)}[f_0(980)] &= 0.19 \exp(i1^\circ), \quad \zeta_{1800}^{(I)}[f_0(980)] = 0.02 \exp(-i112^\circ), \\ \zeta_{1580}^{(I)}[f_0(980)] &= 0.02 \exp(i5^\circ). \end{aligned} \quad (7.95)$$

An interesting fact is that the phases of constants $\zeta_{700}^{(I)}[f_0(980)]$ and $\zeta_{1220}^{(I)}[f_0(980)]$ have a relative shift close to 90° . This means that the contributions of $f_0^{\text{bare}}(700 \pm 100)$ and $f_0^{\text{bare}}(1220 \pm 30)$ (both are members of the basic $1^3P_0q\bar{q}$ nonet) practically do not interfere in the calculation of the probability for the decay $\phi(1020) \rightarrow \gamma f_0^{(I)}(980)$.

Actually, one may neglect the contributions of the bare states $f_0^{\text{bare}}(1230)$, $f_0^{\text{bare}}(1800)$, $f_0^{\text{bare}}(1580)$ into the amplitude $\phi(1020) \rightarrow \gamma f_0^{(I)}(980)$, because the form factors for the production of radial excited states ($n \geq 2$) are noticeably suppressed $\left| F_{\phi(1020) \rightarrow \gamma f_0(2^3P_0q\bar{q})}^{(\text{bare})} \right| \ll \left| F_{\phi(1020) \rightarrow \gamma f_0(1^3P_0q\bar{q})}^{(\text{bare})} \right|$ (this point is discussed below, see also [21]).

For the second pole, which is located on the third sheet at $M^{\text{II}} = 960 - i200$ MeV, we have:

$$\begin{aligned} \zeta_{700}^{(\text{II})}[f_0(980)] &= 1.00 \exp(i6^\circ), \quad \zeta_{1220}^{(\text{II})}[f_0(980)] = 0.33 \exp(i113^\circ), \\ \zeta_{1230}^{(\text{II})}[f_0(980)] &= 0.32 \exp(i148^\circ), \quad \zeta_{1800}^{(\text{II})}[f_0(980)] = 0.08 \exp(i4^\circ), \\ \zeta_{1580}^{(\text{II})}[f_0(980)] &= 0.04 \exp(i98^\circ). \end{aligned} \quad (7.96)$$

Here, as before, the transitions $\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(1230)$, $\gamma f_0^{\text{bare}}(1580)$, $\gamma f_0^{\text{bare}}(1800)$ are negligibly small.

The bare states $f_0^{\text{bare}}(700)$ and $f_0^{\text{bare}}(1220)$ are mixtures of the $n\bar{n}$ and $s\bar{s}$ components, $n\bar{n} \cos \varphi + s\bar{s} \sin \varphi$, with the mixing angles φ $[f_0^{\text{bare}}(700)] = -70^\circ \pm 10^\circ$ and φ $[f_0^{\text{bare}}(1220)] = 20^\circ \pm 10^\circ$ (see Chapter 3). Assuming $\phi(1020)$ to be a pure $s\bar{s}$ state, the transition amplitude for $\phi(1020) \rightarrow \gamma f_0(980)$ is written as

$$\begin{aligned} A_{\phi(1020) \rightarrow \gamma f_0(980)}^N &\simeq \zeta_{700}^{(\text{N})}[f_0(980)] \sin \varphi [f_0^{\text{bare}}(700)] F_{\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(700)}^{(\text{bare})} \\ &+ \zeta_{1220}^{(\text{N})}[f_0(980)] \sin \varphi [f_0^{\text{bare}}(1220)] F_{\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(1220)}^{(\text{bare})}. \end{aligned} \quad (7.97)$$

One can see that the factor $\zeta_{1220}[f_0(980)] \sin \varphi [f_0^{\text{bare}}(1220)]$ is numerically small, and may be neglected. Then for two poles we have:

$$\begin{aligned} s_S &= (M^I)^2 : \quad A_{\phi(1020) \rightarrow \gamma f_0(980)}^I \simeq (0.58 \pm 0.04) F_{\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(700)}^{(\text{bare})}, \\ s_S &= (M^{II})^2 : \quad A_{\phi(1020) \rightarrow \gamma f_0(980)}^{II} \simeq (0.92 \pm 0.06) F_{\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(700)}^{(\text{bare})} \quad (7.98) \end{aligned}$$

We see that the $A_{\phi(1020) \rightarrow \gamma f_0(980)}^{II}$ amplitude practically does not change its value in the course of the evolution from bare state to resonance, while the decrease of $A_{\phi(1020) \rightarrow \gamma f_0(980)}^I$ is significant.

7.3.2.4 Comparison to data

Comparing the above-written formulae to experimental data, we have parametrised the wave functions of the $q\bar{q}$ states in an exponent-type form, see (7.68). For $\phi(1020)$, we accept its mean radius squared to be close to the pion radius, $R_{\phi(1020)}^2 \simeq R_\pi^2$ (both states are members of the same 36-plet). This value of the mean radius squared for $\phi(1020)$ fixes the wave function by $b_\phi = 10 \text{ GeV}^{-2}$.

For $f_0^{\text{bare}}(700)$, we change the value b_{f_0} in the interval $5 \text{ GeV}^{-2} \leq b_{f_0}^{(\text{bare})} \leq 15 \text{ GeV}^{-2}$ that corresponds to the interval $(0.5\text{--}1.5)R_\pi^2$ for the mean radius squared of $f_0^{\text{bare}}(700)$.

We have the following data for the branching ratios [26, 27]:

$$\begin{aligned} \text{BR}[\phi(1020) \rightarrow \gamma f_0(980)] &= (4.47 \pm 0.21) \times 10^{-4}, \\ \text{BR}[\phi(1020) \rightarrow \gamma f_0(980)] &= (2.90 \pm 0.21 \pm 1.54) \times 10^{-4}; \quad (7.99) \end{aligned}$$

the PDG group gives $\text{BR}[\phi(1020) \rightarrow \gamma f_0(980)] = (4.40 \pm 0.21) \times 10^{-4}$ [6]. For the extraction of the branching ratios (7.99) simplified formulae were used, describing $f_0(980)$ as a Breit–Wigner resonance. Nevertheless, we estimate below the experimental amplitude $A_{\phi \rightarrow \gamma f_0}^{(\text{exp})}$ on the basis of the PDG fit value.

We have for the radiative decay width:

$$\begin{aligned} m_\phi \Gamma_{\phi \rightarrow \gamma f_0} &= \frac{1}{6} \alpha \frac{m_\phi^2 - m_{f_0}^2}{m_\phi^2} |A_{\phi \rightarrow \gamma f_0}|^2, \\ \Gamma_{\phi \rightarrow \gamma f_0} &= \text{BR}[\phi(1020) \rightarrow \gamma f_0(980)] \Gamma_{\text{tot}}[\phi(1020)]. \quad (7.100) \end{aligned}$$

Using experimental values for $\text{BR}[\phi(1020) \rightarrow \gamma f_0(980)]$ and equation (7.100), we write the decay amplitude:

$$A_{\phi(1020) \rightarrow \gamma f_0(980)}^{(\text{exp})} = 0.137 \pm 0.014 \text{ GeV}. \quad (7.101)$$

Here $\alpha = 1/137$, $m_\phi = 1.02$ GeV and $m_{f_0} = 0.975$ GeV (the mass reported in [26, 27] for the measured $\gamma f_0(980)$ signal) and $\Gamma_{\text{tot}}[\phi(1020)] = 4.26 \pm 0.05$ MeV [6].

The right-hand side of (7.101) should be compared with $A_{\phi(1020) \rightarrow \gamma f_0(980)}^I$ (the residue in the pole near the physical region, Eq. (7.100)); we have:

$$A_{\phi(1020) \rightarrow \gamma f_0(980)}^{I(\text{calc})}(\text{dipole}) \simeq (0.58 \pm 0.04) \sqrt{W_{q\bar{q}}[f_0^{\text{bare}}(700)]} Z_{\phi \rightarrow \gamma f_0}^{(s\bar{s})} \times \frac{2^{7/2}}{\sqrt{3}} \frac{b_\phi^{7/4} b_{f_0}^{5/4}}{(b_\phi + b_{f_0})^{5/2}} m_s [m_\phi - (0.7 \pm 0.1) \text{ GeV}]. \quad (7.102)$$

In (7.102) the factor (0.58 ± 0.04) takes into account the change of the transition amplitude caused by the final-state hadron interaction, see (7.98). The probability to find the quark-antiquark component in the bare state $f_0^{\text{bare}}(700)$ is denoted as $W_{q\bar{q}}[f_0^{\text{bare}}(700)]$: one can guess that it is of the order of $(80 - 90)\%$, or even more. The mass of the strange constituent quark is equal to $m_s \simeq 0.5$ GeV.

The comparison of data (7.101) with the calculated amplitude (7.102) at $b_\phi = 10 \text{ GeV}^{-2}$ and $5 < b_{f_0} < 15 \text{ GeV}^{-2}$ is shown in Fig. 7.10. We see that the amplitude (7.102) is in agreement with the data, when $M_{f_0^{(\text{bare})}}$ is inside the error bars given by the K -matrix analysis (see Chapter 3 and [2]): $M_{f_0^{(\text{bare})}} = 0.7 \pm 0.1 \text{ GeV}$.

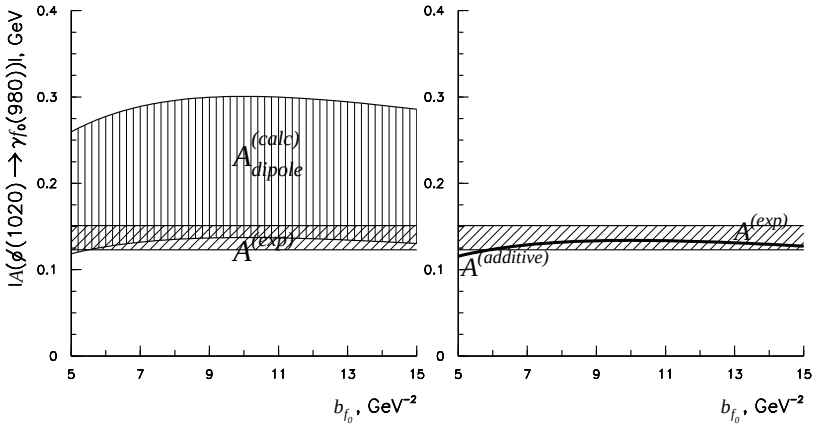


Fig. 7.10 The experimental amplitude $A^{(\text{exp})}$ versus the calculated one in the non-relativistic quark model: a) dipole amplitude, $A^{(\text{dipole})}$, and b) additive quark model amplitude, $A^{(\text{additive})}$.

7.3.2.5 The additive quark model, does it work?

If the contributions of the charge-exchange currents are small, the additive quark model should give for the process $\phi(1020) \rightarrow \gamma f_0^{\text{bare}}(700 \pm 100)$ the same result as the dipole formula. The comparison of the dipole formula (7.71) with that for the triangle diagram contribution (additive quark model, equation (7.70)) tells us that both formulae lead to the same result if

$$m_s[m_\phi - M_{f_0^{\text{bare}}}] = b_\phi^{-1}. \quad (7.103)$$

At $m_s = 0.5 \text{ GeV}$ and $b_\phi = 10 \text{ GeV}^{-2}$ the equality (7.103) is almost fulfilled, when $M_{f_0^{\text{bare}}} \simeq 0.8 \text{ GeV}$ (remind once more that the K -matrix fit [2] gives us $M_{f_0^{\text{bare}}} = 0.7 \pm 0.1 \text{ GeV}$). If $\phi(1020)$, being a $s\bar{s}$ system, is more compact than the non-strange members of the 36-plet (*i.e.* if $b_\phi < 10 \text{ GeV}^{-2}$) the condition (7.103) requires a smaller value for $M_{f_0^{\text{bare}}}$. For example, for $b_\phi = 7 \text{ GeV}^{-2}$ one has $M_{f_0^{\text{bare}}} \simeq 0.7 \text{ GeV}$.

It means that using $F_{\mu\alpha}^{\phi \rightarrow \gamma f_0}(\text{additive})$, equation (7.70), for the calculation of $A_{\phi(1020) \rightarrow \gamma f_0(980)}^{I(\text{calc})}$, we should get an agreement with the experimental data. Indeed, we have:

$$A_{\phi(1020) \rightarrow \gamma f_0(980)}^{I(\text{calc})}(\text{additive}) \simeq (0.58 \pm 0.04) \sqrt{W_{s\bar{s}}[f_0^{\text{bare}}(700)]} Z_{\phi \rightarrow \gamma f_0}^{(s\bar{s})} \times \frac{2^{7/2}}{\sqrt{3}} \frac{b_\phi^{3/4} b_{f_0}^{5/4}}{(b_\phi + b_{f_0})^{5/2}}. \quad (7.104)$$

To be illustrative, in Fig. 7.10 we demonstrate $A_{\phi(1020) \rightarrow \gamma f_0(980)}^{I(\text{calc})}(\text{additive})$ versus $A_{\phi(1020) \rightarrow \gamma f_0(980)}^{(\text{exp})}$: there is a good agreement with the data.

We think that the coincidence of the dipole formula with the additive model calculations is the result of either the gluonic nature of the t -channel forces or the gluonium dominance in the quark mixing $s\bar{s} \rightarrow \text{gluonium} \rightarrow n\bar{n}$.

Miniconclusion

A correct determination of the origin of $f_0(980)$ is a key for understanding the status of the light σ and the classification of heavier mesons $f_0(1300)$, $f_0(1500)$, $f_0(1750)$ and the broad state $f_0(1200\text{--}1600)$.

It is seen that experimental data on the reaction $\phi(1020) \rightarrow \gamma f_0(980)$ do not contradict the suggestion about the dominance of the quark-antiquark component in $f_0(980)$.

However, one would come to the opposite conclusion assuming naïvely that the $f_0(980)$ may be a stable particle and applying the dipole formula directly to the decay $\phi(1020) \rightarrow \gamma f_0(980)$.

7.3.3 Form factors in the additive quark model and confinement

The Feynman diagram technique may be an appropriate starting point for the calculation of amplitudes in the framework of the quark model. But in the Feynman technique the requirement of the quark confinement was not imposed directly. Here we consider form factors in the framework of the additive quark model and, going to the non-relativistic limit, we show how to impose the requirement of confinement.

As an example, we consider form factors of the radiative decays $V \rightarrow \gamma P$ and $V \rightarrow \gamma S$, written in terms of Feynman triangle diagrams and then, going to the non-relativistic approximation, we transform them to diagrams of the additive quark model with the confinement constraints.

In the additive quark model the radiative decay is a three-stage process: the transition $V \rightarrow q\bar{q}$, photon emission by one of the quarks and the fusion of quarks into a final meson (S or P), see Fig. 7.11a. The considered processes, $V \rightarrow \gamma S$ and $V \rightarrow \gamma P$, are transitions of both electric and magnetic types. So, it is convenient, depending on the studied reaction, to write the quark-photon vertex (γ_α) in two equivalent forms: $\gamma_\alpha \leftrightarrow (k_{1\alpha} + k'_{1\alpha})/2m + \sigma_{\alpha\beta}q_\beta/2m$ where m is the quark mass, $\sigma_{\alpha\beta} = (\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)/2$, for the notations of momenta see Fig. 7.11a. Such a representation of the vertex is equivalent to the expression with the use of γ_α , it simplifies the calculations related to the transformation to the non-relativistic limit.

In the calculations we work with amplitudes written as $\epsilon_\alpha^{(\gamma)}\epsilon_\mu^{(V)}A_{\mu\alpha}^{V \rightarrow \gamma S/P}$ taking into account the requirements $\epsilon_\alpha^{(\gamma)}q_\alpha = 0$ and $\epsilon_\mu^{(V)}p_\mu = 0$. Hence, the calculated amplitudes obey the constraints $q_\alpha A_{\mu\alpha}^{V \rightarrow \gamma S/P} = 0$ and $p_\mu A_{\mu\alpha}^{V \rightarrow \gamma S/P} = 0$.

The Feynman integral for the diagram of Fig. 7.11a reads:

$$\epsilon_\alpha^{(\gamma)}\epsilon_\mu^{(V)} \int \frac{d^4k}{i(2\pi)^4} \quad (7.105)$$

$$\times g_V \frac{(-)\text{Sp} \left[\hat{G}_\mu^{(V)}(\hat{k}_1 + m)\Gamma_\alpha(\hat{k}'_1 + m)\hat{G}^{(S/P)}(-\hat{k}_2 + m) \right]}{(m^2 - k_1^2 - i0)(m^2 - k_1'^2 - i0)(m^2 - k_2^2 - i0)} g_{S/P},$$

where for the vertices $V \rightarrow q\bar{q}$, $S \rightarrow q\bar{q}$, $P \rightarrow q\bar{q}$ we write $\hat{G}^{(V)} = \gamma_\mu$, $\hat{G}^{(S)} = I$, $\hat{G}^{(P)} = \gamma_5$ and for the photon-quark interaction: $\Gamma_\alpha = (k_{1\alpha} + k'_{1\alpha})/2m + \sigma_{\alpha\beta}q_\beta/2m$. The vertices $V \rightarrow q\bar{q}$, $q\bar{q} \rightarrow S$ and $q\bar{q} \rightarrow P$ are denoted as g_V , g_S and g_P .

Let us emphasise that, writing the triangle diagram of Fig. 7.11a in the form (7.105), we work with a non-confined quark: this diagram con-

tains threshold singularities at $p^2 = 4m^2$ and $p'^2 = 4m^2$ which reflect the possibility for quarks to fly out at $p^2 > 4m^2$ and $p'^2 > 4m^2$. Below, introducing $q\bar{q}$ wave functions, we demonstrate the method of keeping quarks in the confinement trap which works for both approaches: the non-relativistic expansion and the spectral integral approximation.

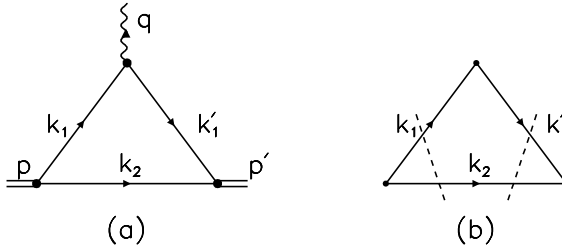


Fig. 7.11 Diagram for the transition form factor in the additive quark model (a) and corresponding cuts in its double spectral integral representation (b).

7.3.3.1 Triangle diagrams in non-relativistic approximation

A suitable transformation procedure for getting a non-relativistic expression is to introduce in (7.105) two-component spinors for the quark and antiquark, φ_j and χ_j . This is realised by substituting

$$\begin{aligned} (\hat{k}_1 + m) &\rightarrow \sum_{j=1,2} \psi^j(k_1) \bar{\psi}^j(k_1), & (\hat{k}'_1 + m) &\rightarrow \sum_{j=1,2} \psi^j(k'_1) \bar{\psi}^j(k'_1), \\ (\hat{k}_2 - m) &\rightarrow \sum_{j=3,4} \psi^j(-k_2) \bar{\psi}^j(-k_2), \end{aligned} \quad (7.106)$$

with

$$\psi^j(k) = \begin{pmatrix} \sqrt{k_0 + m} \varphi_j \\ \frac{\boldsymbol{\sigma} \mathbf{k}}{\sqrt{k_0 + m}} \varphi_j \end{pmatrix}, \quad \psi^j(-k) = i \begin{pmatrix} \frac{\boldsymbol{\sigma} \mathbf{k}}{\sqrt{k_0 + m}} \chi^j \\ \sqrt{k_0 + m} \chi^j \end{pmatrix}, \quad (7.107)$$

leading to the two-dimensional trace in the integrand (7.105).

We turn now to the non-relativistic approximation in the vector-particle rest frame. Denoting the four-momentum of the vector particle as $p = (p_0, \mathbf{p}_\perp, p_z)$, we have in this frame: $p = (2m - \varepsilon_V, \mathbf{0}, 0)$, where ε_V is the binding energy of the vector particle which is supposed to be small as compared to the quark mass, $\varepsilon_V \ll m$. Let the photon fly along the z -axis, then $q = (q_z, \mathbf{0}, q_z)$, and the polarisation vector of the photon lays in

the (x, y) -plane. The four-momentum of a scalar (pseudoscalar) particle is equal to $p' = (2m - \varepsilon_V - q_z, \mathbf{0}, -q_z) \simeq \left(2m - \varepsilon + \frac{q_z^2}{2m}, \mathbf{0}, -q_z\right)$. Here ε is the binding energy of a scalar (pseudoscalar) particle, which is also small compared to the mass of the constituent $\varepsilon \ll m$.

(i) The reaction $V \rightarrow \gamma S$.

The transition to the non-relativistic approximation in the numerator of the integrand (7.105) provides the following formula for the reaction $V \rightarrow \gamma S$:

$$- \text{Sp}_2 [2m\sigma_\mu (\mathbf{k}_2 - \mathbf{k}'_1)\boldsymbol{\sigma}] (k_{1\alpha} + k'_{1\alpha}) . \quad (7.108)$$

The notation Sp_2 stands for the trace of two-dimensional matrices. In the transition to the non-relativism, the following terms are kept in (7.108), being of the leading order:

- 1) in $q\gamma q$ -vertex: $\bar{\psi}(k_1)[(k_{1\alpha} + k'_{1\alpha})]/2m\psi(k'_1) \rightarrow \varphi^+(1)(k_{1\alpha} + k'_{1\alpha})\varphi(1')$,
- 2) in the $V \rightarrow q\bar{q}$ vertex: $\bar{\psi}(-k_2)\gamma_\mu\psi(k_1) \rightarrow \chi^+(2)2m\sigma_\mu\varphi(1)$,
- 3) in the $q\bar{q} \rightarrow S$ vertex: $\bar{\psi}(k'_1)\psi(-k_2) \rightarrow \varphi^+(1')\boldsymbol{\sigma}(\mathbf{k}_2 - \mathbf{k}'_1)\chi(2)$.

For the non-relativistic case the constituent propagators should be replaced in a standard way: $(m^2 - k^2 - i0)^{-1} \rightarrow (-2mE + \mathbf{k}^2 - i0)^{-1}$, with $E = k_0 - m$ and $m^2 - k_0^2 \simeq -2mE$.

Then the amplitude of Fig. 7.11a for the transition $V \rightarrow \gamma S$ reads:

$$\begin{aligned} & \epsilon_\mu^{(V)} \epsilon_\alpha^{(\gamma)} \int \frac{dE d^3k}{i(2\pi)^4} \\ & \times g_V \frac{-\text{Sp}_2[2m\sigma_\mu \cdot (\mathbf{k}_2 - \mathbf{k}'_1)\boldsymbol{\sigma}](k_{1\alpha} + k'_{1\alpha})}{(-2mE_1 + \mathbf{k}_1^2 - i0)(-2mE'_1 + \mathbf{k}'_1{}^2 - i0)(-2mE_2 + \mathbf{k}_2^2 - i0)} g_S . \end{aligned} \quad (7.109)$$

Further, we denote $E_2 \equiv E$, $\mathbf{k}_2 \equiv \mathbf{k}$. With these notations one should include the energy-momentum conservation laws: $E_1 = -\varepsilon_V - E$, $\mathbf{k}_1 = -\mathbf{k}$, $E'_1 = -\varepsilon_V - E - q_z$, $\mathbf{k}'_1 = -\mathbf{k} - \mathbf{q}$, and integrate over E that is equivalent to the substitution in (7.109): $2m(-2mE + \mathbf{k}^2 - i0)^{-1} \rightarrow 2\pi i\delta(E - \mathbf{k}^2/2m)$. By fixing $E = \mathbf{k}^2/2m$, we can evaluate the order of value of the momenta entering (7.109). We have

$$q_z \simeq \varepsilon_V - \varepsilon \ll |\mathbf{k}| , \quad (7.110)$$

because $\mathbf{k}^2 \sim 2m\varepsilon \sim 2m\varepsilon_V$ and $q_z^2/2m$ is the value of the next-to-leading order. Thus, within the non-relativistic approximation, the amplitude for the transition $V \rightarrow \gamma S$ reads:

$$\epsilon_\mu^{(V)} \epsilon_\alpha^{(\gamma)} \int \frac{d^3k}{(2\pi)^3} \psi_V(k) \psi_S(k) \frac{(-4)}{2m} \text{Sp}_2[2m\sigma_\mu \cdot 2(\mathbf{k}\boldsymbol{\sigma})](-2k_\alpha), \quad (7.111)$$

where the requirement (7.110) is duely taken into account.

In (7.111) an important step is made for the further use of this equation in the quark model: we rewrite (7.111) in terms of the wave functions for vector and scalar particles:

$$\psi_V(k) = \frac{g_V}{4(m\varepsilon_V + \mathbf{k}^2)} , \quad \psi_S(k) = \frac{g}{4(m\varepsilon + \mathbf{k}^2)} . \quad (7.112)$$

We return to this point below.

Let us now recall once more that the polarisation vector $\epsilon_\mu^{(V)}$ does not contain the time-like component and the polarisation vector of the photon belongs to the (x, y) -plane. Accounting for $\text{Sp}_2[\sigma_\mu \sigma_\beta] = 2\delta_{\mu\beta}$, where $\delta_{\mu\beta}$ is the three-dimensional Kronecker symbol, and substituting in the integrand $k_\mu k_\alpha \rightarrow \delta_{\mu\alpha} \mathbf{k}^2/3$, we have the final expression:

$$A_{V \rightarrow \gamma S} = \left(\epsilon^{(V)} \epsilon^{(\gamma)} \right) \int_0^\infty \frac{dk^2}{\pi} \psi_V(k) \psi_S(k) \frac{8}{3\pi} k^3 . \quad (7.113)$$

Here we redenoted $\mathbf{k}^2 \rightarrow k^2$.

(ii) The reaction $V \rightarrow \gamma P$.

In the reaction $V \rightarrow \gamma P$ the non-relativistic spin factor (the numerator of the integrand of (7.105) has the form:

$$(-) \text{Sp}_2[2m\sigma_\mu \cdot i\varepsilon_{\alpha\beta\gamma} q_\beta \sigma_\gamma \cdot 2m] = -i8m^2 \varepsilon_{\mu\alpha\beta} q_\beta , \quad (7.114)$$

where $\varepsilon_{\alpha\beta\gamma}$ is a three-dimensional antisymmetric tensor. As a result, we have:

$$\begin{aligned} A_{V \rightarrow \gamma P} &= -i\varepsilon_{\mu\alpha\nu_1\nu_2} \epsilon_\mu^{(V)} \epsilon_\alpha^{(\gamma)} q_{\nu_1} p_{\nu_2} F_{V \rightarrow \gamma P}(q^2) \\ &= -i\varepsilon_{\mu\alpha\beta} \epsilon_\mu^{(V)} \epsilon_\alpha^{(\gamma)} q_\beta \int_0^\infty \frac{dk^2}{\pi} \psi_V(k) \psi_P(k) \frac{4km}{\pi} . \end{aligned} \quad (7.115)$$

(iii) Normalisation of wave functions.

The normalisation condition for the wave function $\psi_V(k)$, $\psi_S(k)$ and $\psi_P(k)$ can be formulated as a requirement for the charge form factor at $q^2 = 0$, namely, $F_{charge}(0) = 1$. Consider as an example the charge form factor of a scalar particle. It is defined by the triangle diagram of the Fig. 7.11a type. Using the same calculation technique which resulted in formula (7.111), we obtain:

$$F_{charge}^{(S)}(0) = \int \frac{d^3k}{(2\pi)^3} \psi_S^2(k) \frac{2}{m} \text{Sp}_2[2(\mathbf{k}\boldsymbol{\sigma}) \cdot 2(\mathbf{k}\boldsymbol{\sigma})] \frac{1}{2} . \quad (7.116)$$

In the same way as for equation (7.111), the factor $2/m$ arises due to the integration over E and the definition of the wave function $\psi_S(k)$; the

vertex $S \rightarrow q\bar{q}$ is equal to $2(\mathbf{k}\boldsymbol{\sigma})$, and the factor $1/2$ appeared because of the substitution $k_{1\alpha} + k'_{1\alpha} \rightarrow (p_{1\alpha} + p'_{1\alpha})/2$. For $\alpha = 0$ this corresponds to the interaction with the Coulomb field, we have $k_{10} = k'_{10} \simeq m$ and $p_{10} = p'_{10} \simeq 2m$. The condition $F_{charge}^{(S)}(0) = 1$ gives us:

$$\int_0^\infty \frac{dk^2}{\pi} \psi_S^2(k) \frac{2k^3}{\pi m} = 1. \quad (7.117)$$

The normalisation for the pseudoscalar composite particle is the same as for the vector one. We have:

$$\int_0^\infty \frac{dk^2}{\pi} \psi_P^2(k) \frac{2km}{\pi} = \int_0^\infty \frac{dk^2}{\pi} \psi_V^2(k) \frac{2km}{\pi} = 1. \quad (7.118)$$

Miniconclusion

Starting from Feynman triangle diagram integrals, we obtained for the transitions $V \rightarrow \gamma S$ and $V \rightarrow \gamma P$ the formulae of the non-relativistic additive quark model. We show that, after a correct transition to the non-relativistic approximation, these decay amplitudes for the emission of a real photon ($q^2 \rightarrow 0$) are determined by the convolution of wave functions, with no additional energy dependence like that in [28]. This is a natural consequence of the Lorentz-invariant structure of the transition amplitudes: as we show in the next section, it is a common property independent of whether we use relativistic or non-relativistic representations of the amplitude.

7.3.3.2 Requirement for quark confinement

The direct application of the Feynman technique to quark diagrams leads to a problem with confinement: an intermediate state quark is able to move alone at large distances that is reflected in the quark threshold singularities. Indeed, the Feynman amplitude of the triangle diagram of Fig. 7.11 contains the quark threshold singularities at $M_{\text{meson}}^2 = 4m^2$. Such singularities exist in both incoming and outgoing meson channels due to the integrand poles $(m\varepsilon_V + \mathbf{k}^2)^{-1}$ and $(m\varepsilon + \mathbf{k}^2)^{-1}$. However, rewriting the transition amplitudes with the use of wave functions (here $-\psi_V(k)$, $\psi_S(k)$ and $\psi_P(k)$) we open a way to eliminate these singularities. For example, the final formulae for transitions $V \rightarrow \gamma S$ and $V \rightarrow \gamma P$, (7.113) and (7.115), operate with the confined quarks if we use exponential wave functions.

Miniconclusion

In (7.112) we introduce $q\bar{q}$ wave functions for mesons and, after that, we rewrite all amplitudes in terms of $\psi_V(k)$ and $\psi_S(k)$, thus hiding the pole factors $(m\varepsilon_V + \mathbf{k}^2)^{-1}$ and $(m\varepsilon_S + \mathbf{k}^2)^{-1}$ in the integrand of (7.113). If we write the wave functions with these pole factors, the threshold singularities exist, and we work with non-confined quarks. But if we use the wave functions of the type discussed in (7.68) (without pole factors that correspond to $V(r) \rightarrow \infty$ at $r \rightarrow \infty$), we deal with confined quarks.

The spectral integral approach, being an ingenious generalisation of quantum mechanics, allows one to work with wave functions both containing or not containing pole singularities, *i.e.* to work with non-confined and confined constituents.

7.4 Spectral Integral Technique in the Additive Quark Model: Transition Amplitudes and Partial Widths of the Decays $(q\bar{q})_{in} \rightarrow \gamma + V(q\bar{q})$

The spectral integration technique is in some important points similar to the description of processes used in quantum mechanics. In both cases time-ordered processes are considered, the intermediate state particles are on the mass-shell, both methods operate with energy non-conservation diagrams. Moreover, the introduction of the quark confinement constraints in the calculated amplitudes is performed in both approaches in an analogous way. To underline the common ideas of the spectral integral technique and that applied in quantum mechanics, we present here the calculation of the same processes which were considered above in the non-relativistic approach.

In this section, we calculate the radiative decays of the quark–antiquark composite systems, $(q\bar{q})_{in}$, with $J^{PC} = 0^{++}, 0^{-+}, 2^{++}, 1^{++}$ when the radiative decays are realised through the additive quark model transitions $(q\bar{q})_{in} \rightarrow \gamma + V(q\bar{q})_{out}$, see Fig. 7.11. The method is based on the spectral integration over the masses of composite particles, it was briefly discussed for simple examples (scalar mesons and scalar or pseudoscalar constituents) in Chapter 3 (section 3.3). The method gives us relativistic and gauge invariant amplitudes. The obtained transition amplitudes (form factors) are determined by the quark wave functions of the composite systems $(q\bar{q})_{in}$ and $(q\bar{q})_{out}$.

The consideration of triangle diagrams in terms of the spectral integral over the mass of a composite particle, or an interacting system, has a long

history. Triangle diagrams appear at the rescattering of the three-particle systems, and the energy dependence of corresponding amplitudes (on either the total energy or the energy of two particles) were studied rather long ago, though in non-relativistic approximation, in the dispersion relation technique applied to the analysis of the threshold singularities (see [29] and references therein). The relativistic approximation was used for the extraction of logarithmic singularities of the triangle diagram, see Chapter 4 as well as [30, 31]. Relativistic dispersion relation equations for three-particle interacting systems were given in [32, 33]. The double dispersion relation representation of the triangle diagram without accounting for the spin structure was written in [34].

In the consideration of radiative decays of the spin particles, one of the most important point is a correct construction of gauge invariant spin operators allowing us to perform the expansion of the decay amplitude (written in terms of external variables) and to give the double discontinuity of the spectral integral (written in terms of the composite particle constituents). Such a procedure has been realised for the deuteron in [35, 36] and, correspondingly, for the elastic scattering and photodisintegration amplitude. A generalisation of the method for composite quark systems has been performed in [20, 37, 38].

There are two basic points which should be accounted for the form factor processes shown in Fig. 7.11 considered in terms of the spectral integration technique:

- (i) The amplitude of the process $(q\bar{q})_{in} \rightarrow \gamma(q\bar{q})_{out}$ should be expanded in a series over a full set of spin operators, and this expansion should be done in a uniform way for both internal quark and external boson states. The spin operators should be orthogonal, and the spectral integrals are to be written for the amplitudes related to this set of orthogonal operators.
- (ii) It should be taken into account that in the processes with real photons (with the photon four-momentum $q^2 \rightarrow 0$) nilpotent spin operators appear, their norm being equal to zero [4]. Because of that, even if the representation of the spin factors of the amplitudes for the same processes may nominally be different, this does not affect the calculation result for partial widths.

As was noted above, we explain the main points of the spectral integration considering the same transitions as in the previous section: $q\bar{q}(0^{-+}) \rightarrow \gamma + q\bar{q}(1^{--})$ and $q\bar{q}(0^{++}) \rightarrow \gamma + q\bar{q}(1^{--})$. In terms of the spectral integral technique, these reactions were studied in papers [20, 37, 38].

As the next step, we apply the method to the transitions $q\bar{q}(2^{++}) \rightarrow \gamma + q\bar{q}(1^{--})$ and $q\bar{q}(1^{++}) \rightarrow \gamma + q\bar{q}(1^{--})$ (see also [39]). Let us emphasise that the cases $q\bar{q}(2^{++}) \rightarrow \gamma + q\bar{q}(1^{--})$ and $q\bar{q}(1^{++}) \rightarrow \gamma + q\bar{q}(1^{--})$ are rather general and can be used as a pattern for the consideration of the spectral integral representation of the amplitudes $(q\bar{q})_{in} \rightarrow \gamma + (q\bar{q})_{out}$ for the $q\bar{q}$ states with arbitrary spins.

7.4.1 Radiative transitions $P \rightarrow \gamma V$ and $S \rightarrow \gamma V$

In its main part, this section is an introductory one: we remind here notations and collect properties of the spectral integrals presented in the previous sections.

7.4.1.1 The decay of the pseudoscalar meson $P \rightarrow \gamma V$

First, we consider the transition $P \rightarrow \gamma^* V$ for the virtual photon, see Fig. 7.11a. We write the spin operator for both initial mesons in the triangle diagram and the quark intermediate states in the triangle diagram discontinuity with the cuttings shown in Fig. 7.11b. Then we extract the invariant part of the discontinuity, calculate the double dispersion integral for the form factor amplitude and present it for the emission of a real photon ($q^2 \rightarrow 0$).

(i) Amplitude for the decay $P \rightarrow \gamma V$.

Let us remind that the decay amplitude $P \rightarrow \gamma^* V$ is written as a product of a spin-dependent multiplier and an invariant form factor:

$$\begin{aligned} A_{P \rightarrow \gamma^* V} &= \epsilon_{\alpha}^{(\gamma^*)} \epsilon_{\beta}^{(V)} A_{\alpha\beta}^{(P \rightarrow \gamma^* V)}, \\ A_{\alpha\beta}^{(P \rightarrow \gamma^* V)} &= e \varepsilon_{\alpha\beta\mu\nu} q_{\mu}^{\perp} p_{\nu} F_{P \rightarrow \gamma V}(q^2). \end{aligned} \quad (7.119)$$

In (7.119) the electron charge is singled out, and $\varepsilon_{\alpha\beta\mu\nu}$ is a totally anti-symmetric tensor. This expression can be used for virtual and real photon emissions. The spin operator reads:

$$S_{\alpha\beta}^{(P \rightarrow \gamma V)}(p, q) = \varepsilon_{\alpha\beta qp}. \quad (7.120)$$

(ii) Partial widths for $P \rightarrow \gamma V$ and $V \rightarrow \gamma P$.

The partial width for the decay with the emission of a real photon $P \rightarrow \gamma V$ is equal to:

$$M_P \Gamma_{P \rightarrow \gamma V} = \int d\Phi_2(p; q, p') \left| \sum_{\alpha\beta} A_{\alpha\beta}^{(P \rightarrow \gamma V)} \right|^2 = \alpha \frac{M_P^2 - M_V^2}{8M_P^2} |F_{P \rightarrow \gamma V}(0)|^2, \quad (7.121)$$

where

$$d\Phi_2(p; q, p') = \frac{1}{2} \frac{d^3 q}{(2\pi)^3 2q_0} \frac{d^3 p'}{(2\pi)^3 2p'_0} (2\pi)^4 \delta^{(4)}(p - q - p'). \quad (7.122)$$

The summation is carried out over the photon and vector meson polarisations; in the final expression $\alpha = e^2/4\pi = 1/137$. The same form factor gives the partial width for the decay $V \rightarrow \gamma P$:

$$M_V \Gamma_{V \rightarrow \gamma P} = \frac{1}{3} \int d\Phi_2(p; q, p') \left| \sum_{\alpha\beta} A_{\alpha\beta}^{(V \rightarrow \gamma P)} \right|^2 = \alpha \frac{M_V^2 - M_P^2}{24M_V^2} |F_{P \rightarrow \gamma V}(0)|^2. \quad (7.123)$$

(iii) Double spectral integral representation of the triangle diagram for the $P \rightarrow \gamma V$ transition.

To derive the double spectral integral for the form factor $F_{P \rightarrow \gamma^* V}(q^2)$, one needs to calculate the double discontinuity of the triangle diagram of Fig. 7.11b, where the cuttings are shown by dotted lines. In the dispersion representation the invariant energy in the intermediate state differs from those of the initial and final states. Because of that, in the double discontinuity $P \neq p$ and $P' \neq p'$. Following [35, 36, 38], the requirements are imposed on the momenta in the diagram of Fig. 7.11b :

$$(k_1 + k_2)^2 = P^2 > 4m^2, \quad (k'_1 + k_2)^2 = P'^2 > 4m^2 \quad (7.124)$$

at the fixed photon momentum squared $(P' - P)^2 = (k'_1 - k_1)^2 = q^2$. In the spirit of the dispersion relation representation, we denote $P^2 = s$, $P'^2 = s'$.

Calculating the double discontinuity starting with the Feynman diagram, the propagators should be substituted by the residues in the poles. This is equivalent to the substitution as follows: $(m^2 - k_i^2)^{-1} \rightarrow \delta(m^2 - k_i^2)$. Then the double discontinuity of the amplitude $A_{\alpha\beta}^{(P \rightarrow \gamma^* V)}$ becomes proportional to the three factors:

$$\begin{aligned} disc_s disc_{s'} A_{\alpha\beta}^{(P \rightarrow \gamma^* V(L))} &\sim Z_{P \rightarrow \gamma V} g_P(s) g_{V(L)}(s') \quad (7.125) \\ &\times d\Phi_2(P; k_1, k_2) d\Phi_2(P'; k'_1, k'_2) (2\pi)^3 2k_{20} \delta^3(\mathbf{k}'_2 - \mathbf{k}_2) \\ &\times \text{Sp} \left[i\gamma_5 (\hat{k}_1 + m) \gamma_\alpha^\perp \gamma^* (\hat{k}'_1 + m) \hat{G}_\beta^{(1,L,1)}(k')(m - \hat{k}_2) \right]. \end{aligned}$$

The first factor in the right-hand side of (7.126) includes the quark charge factor $Z_{P \rightarrow \gamma V}$ (for the one-flavour states $Z_{P \rightarrow \gamma V} = e_q$) and the transition vertices $P \rightarrow q\bar{q}$ and $V \rightarrow q\bar{q}$ which are denoted as $g_P(s)$ and $g_{V(L)}(s')$ (the transition $V \rightarrow q\bar{q}$ is characterised by two angular momenta $L = 0, 2$).

The second factor includes the space volumes of the two-particle states: $d\Phi_2(P; k_1, k_2)$ and $d\Phi_2(P'; k'_1, k'_2)$ that correspond to two cuts in the diagram of Fig. 7.11b (the space volume is determined in (7.122)). The factor $(2\pi)^3 2k_{20} \delta^3(\mathbf{k}'_2 - \mathbf{k}_2)$ takes into account the fact that one quark line is cut twice.

The third factor in (7.126) is the trace coming from the summation over the quark spin states. Since the transition $V \rightarrow q\bar{q}$ may be of two types (with $L = 0$ or $L = 2$), we have the following versions for spin factors $\hat{G}_\beta^{(S,L,J)}(k')$:

$$\hat{G}_\beta^{(1,0,1)}(k') = \gamma_\beta^\perp V, \quad \hat{G}_\beta^{(1,2,1)}(k') = \sqrt{2} \gamma_\beta X_{\beta'\beta}^{(2)}(k'). \quad (7.126)$$

For quarks of equal masses, we have $k' = (k'_1 - k_2)/2$ and $k' \perp P' = k'_1 + k_2$.

The whole vertex $\hat{G}_\beta^V(k')$ of the vector state is the sum of two components with $L = 0$ and $L = 2$:

$$\hat{G}_\beta^V(k') = \hat{G}_\beta^{(1,0,1)}(k') g_{V(L=0)}(s') + \hat{G}_\beta^{(1,2,1)}(k') g_{V(L=2)}(s'). \quad (7.127)$$

Correspondingly, the whole form factor is the sum of two components too:

$$F_{P \rightarrow \gamma^* V}(q^2) = F_{P \rightarrow \gamma^* V(0)}(q^2) + F_{P \rightarrow \gamma^* V(2)}(q^2). \quad (7.128)$$

So, in the double discontinuity we have two traces for two different transitions: $P \rightarrow \gamma^* V(L=0)$ and $P \rightarrow \gamma^* V(L=2)$:

$$\begin{aligned} \text{Sp}_{\alpha\beta}^{(P \rightarrow \gamma^* V(0))} &= -\text{Sp}[\hat{G}_\beta^{(1,0,1)}(k')(\hat{k}'_1 + m) \gamma_\alpha^\perp \gamma^*(\hat{k}_1 + m) i\gamma_5(-\hat{k}_2 + m)], \\ \text{Sp}_{\alpha\beta}^{(P \rightarrow \gamma^* V(2))} &= -\text{Sp}[\hat{G}_\beta^{(1,2,1)}(k')(\hat{k}'_1 + m) \gamma_\alpha^\perp \gamma^*(\hat{k}_1 + m) i\gamma_5(-\hat{k}_2 + m)]. \end{aligned} \quad (7.129)$$

To calculate the invariant form factor $F_{P \rightarrow \gamma V(L)}(q^2)$, we should extract from (7.129) the spin factor analogous to $S_{\alpha\beta}^{(P \rightarrow \gamma V)}(q, p)$ given by (7.120).

For the $q\bar{q}$ quark states, this operator reads:

$$S_{\alpha\beta}^{(P \rightarrow \gamma V)}(\tilde{q}, P') = \varepsilon_{\alpha\beta\tilde{q}P'}, \quad (7.130)$$

where $\tilde{q} = P' - P$, while $P' = k'_1 + k_2$ and $P = k_1 + k_2$. Thus we have:

$$\text{Sp}_{\alpha\beta}^{(P \rightarrow \gamma^* V(L))} = S_{\alpha\beta}^{(P \rightarrow \gamma V)}(\tilde{q}, P') S_{P \rightarrow \gamma^* V(L)}(s, s', q^2); \quad (7.131)$$

here

$$S_{P \rightarrow \gamma^* V(L)}(s, s', q^2) = \frac{S_{\alpha\beta}^{(P \rightarrow \gamma^* V(L))} S_{\alpha\beta}^{(P \rightarrow \gamma V)}(\tilde{q}, P')}{S_{\alpha'\beta'}^{(P \rightarrow \gamma V)}(\tilde{q}, P') S_{\alpha'\beta'}^{(P \rightarrow \gamma V)}(\tilde{q}, P')}. \quad (7.132)$$

As a result, we obtain:

$$S_{P \rightarrow \gamma^* V(0)}(s, s', q^2) = 4m ,$$

$$S_{P \rightarrow \gamma^* V(2)}(s, s', q^2) = \frac{m}{\sqrt{2}} \left[(2m^2 + s) - \frac{6ss'q^2}{\lambda(s, s', q^2)} \right] , \quad (7.133)$$

with

$$\lambda(s, s', q^2) = (s - s')^2 - 2q^2(s + s') + q^4. \quad (7.134)$$

The double discontinuity of the amplitude (7.126) is equal to

$$\begin{aligned} & disc_s disc_{s'} A_{\alpha\beta}^{(P \rightarrow \gamma^* V(L))} \\ &= S_{\alpha\beta}^{(P \rightarrow \gamma^* V(L))}(\tilde{q}, P') disc_s disc_{s'} F_{P \rightarrow \gamma^* V(L)}(s, s', q^2) , \end{aligned} \quad (7.135)$$

where

$$\begin{aligned} & disc_s disc_{s'} F_{P \rightarrow \gamma^* V(L)}(q^2) = Z_{P \rightarrow \gamma V} g_P(s) g_{V(L)}(s') d\Phi_2(P; k_1, k_2) \\ & \times d\Phi_2(P'; k'_1, k'_2) (2\pi)^3 2k_{20} \delta^3(\mathbf{k}'_2 - \mathbf{k}_2) S_{P \rightarrow \gamma^* V(L)}(s, s', q^2) . \end{aligned} \quad (7.136)$$

It defines the form factor in terms of the double dispersion integral as follows:

$$F_{P \rightarrow \gamma^* V(L)}(q^2) = \int_{4m^2}^{\infty} \frac{ds}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{disc_s disc_{s'} F_{P \rightarrow \gamma^* V(L)}(s, s', q^2)}{(s - M_P^2)(s' - M_V^2)}. \quad (7.137)$$

We have written the expression for $F_{P \rightarrow \gamma^* V(L)}(q^2)$ without subtraction terms, assuming that the convergence of (7.137) is guaranteed by the vertices $g_P(s)$ and $g_{V(L)}(s')$.

Further, we define the wave functions for the pseudoscalar and vector $q\bar{q}$ systems:

$$\psi_P(s) = \frac{g_P(s)}{s - M_P^2}, \quad \psi_{V(L)}(s) = \frac{g_{V(L)}(s)}{s - M_V^2}, \quad L = 0, 2. \quad (7.138)$$

After integrating over the momenta one can, in accordance with (7.136), represent (7.137) in the following form:

$$\begin{aligned} F_{P \rightarrow \gamma^* V(L)}(q^2) &= Z_{P \rightarrow \gamma V} \int_{4m^2}^{\infty} \frac{ds ds'}{16\pi^2} \psi_P(s) \psi_{V(L)}(s') \\ &\times \frac{\Theta(-ss'q^2 - m^2\lambda(s, s', q^2))}{\sqrt{\lambda(s, s', q^2)}} S_{P \rightarrow \gamma^* V(L)}(s, s', q^2), \end{aligned} \quad (7.139)$$

where $\Theta(X)$ equals $\Theta(X) = 1$ at $X \geq 0$ and $\Theta(X) = 0$ at $X < 0$.

To calculate the integral at small q^2 , we make the substitution similar to that which was made in section 3.3 (Chapter 3): $s = \Sigma + zQ/2$, $s' = \Sigma - zQ/2$, $q^2 = -Q^2$, thus representing the form factor as follows:

$$\begin{aligned}
 F_{P \rightarrow \gamma V(L)}(0) &= F_{P \rightarrow \gamma^* V(L)}(-Q^2 \rightarrow 0) \\
 &= Z_{P \rightarrow \gamma V} \int_{4m^2}^{\infty} \frac{d\Sigma}{\pi} \psi_P(\Sigma) \psi_{V(L)}(\Sigma) \int_{-b}^{+b} \frac{dz}{\pi} \frac{\bar{S}_{P \rightarrow \gamma^* V(L)}(\Sigma, z, -Q^2)}{16\sqrt{\Lambda(\Sigma, z, Q^2)}} , \\
 b &= \sqrt{\Sigma\left(\frac{\Sigma}{m^2} - 4\right)}, \quad \Lambda(\Sigma, z, Q^2) = (z^2 + 4\Sigma)Q^2 . \quad (7.140)
 \end{aligned}$$

After integrating over z and substituting $\Sigma \rightarrow s$, the form factors read:

$$\begin{aligned}
 F_{P \rightarrow \gamma V(0)}(0) &= Z_{P \rightarrow \gamma V} m \int_{4m^2}^{\infty} \frac{ds}{4\pi^2} \psi_P(s) \psi_{V(0)}(s) \ln \frac{s + \sqrt{s(s - 4m^2)}}{s - \sqrt{s(s - 4m^2)}} , \\
 F_{P \rightarrow \gamma V(2)}(0) &= Z_{P \rightarrow \gamma V} m \int_{4m^2}^{\infty} \frac{ds}{4\pi^2} \psi_P(s) \psi_{V(2)}(s) \quad (7.141) \\
 &\quad \times \left[(2m^2 + s) \ln \frac{\sqrt{s} + \sqrt{s - 4m^2}}{\sqrt{s} - \sqrt{s - 4m^2}} - 3\sqrt{s(s - 4m^2)} \right] .
 \end{aligned}$$

The whole form factor (7.128) is a sum of the form factors with $L = 0, 2$.

7.4.1.2 Decay of the scalar meson $S \rightarrow \gamma V$

The process $S \rightarrow \gamma V$ gives us a more complicated example than that considered above — in this reaction we face the problem of the nilpotent spin operators. But recent experiments provide us with data for reactions with the emission of a real photon. Because of that, we consider here a case which can give us the limit $q^2 \rightarrow 0$ easily: the case of the transversely polarised photon.

Following our considerations presented in the previous section, we repeat briefly the main steps in the calculation of the quark triangle diagram of Fig. 7.11 modifying them to the case of the scalar meson decay $S \rightarrow \gamma V$.

(i) Spin operator decomposition of the quark states in the triangle diagram for the transversely polarised photon.

As was explained above, in the $q\bar{q}$ systems there are two possibilities to construct vector mesons — with angular momenta $L = 0$ and $L = 2$. For the transitions $V \rightarrow q\bar{q}(L)$ we apply the vertices introduced in (7.126): $\hat{G}_{\beta}^{(1,0,1)}$

and $\hat{G}_\beta^{(1,2,1)}$. For the transition $S \rightarrow q\bar{q}(L)$ we use the spin operator mI , where I is the unit matrix. The traces for two processes with the different vector-meson wave functions ($L = 0, 2$) are written as:

$$\begin{aligned} \text{Sp}_{\alpha\beta}^{(S \rightarrow \gamma_\perp^* V(0))} &= -\text{Sp}[\hat{G}_\beta^{(1,0,1)}(\hat{k}'_1 + m)\gamma_\alpha^\perp \gamma^*(\hat{k}_1 + m)mI(-\hat{k}_2 + m)] , \\ \text{Sp}_{\alpha\beta}^{(S \rightarrow \gamma_\perp^* V(2))} &= -\text{Sp}[\hat{G}_\beta^{(1,2,1)}(\hat{k}'_1 + m)\gamma_\alpha^\perp \gamma^*(\hat{k}_1 + m)mI(-\hat{k}_2 + m)] . \end{aligned} \quad (7.142)$$

Calculating the invariant form factor for the transversely polarised photon (we denote it as $F_{S \rightarrow \gamma_\perp V(L)}(q^2)$), one should extract from (7.142) the corresponding spin factor. For the quark states this operator reads:

$$S_{\alpha\beta}^{(S \rightarrow \gamma_\perp V)}(\tilde{q}, P') = g_{\alpha\beta}^{\perp\perp}(\tilde{q}, P') . \quad (7.143)$$

Recall that $P' = k'_1 + k_2$ and $\tilde{q} = P - P' = k_1 - k'_1$. We have:

$$\text{Sp}_{\alpha\beta}^{(S \rightarrow \gamma_\perp^* V(L))} = S_{\alpha\beta}^{(S \rightarrow \gamma_\perp V)}(\tilde{q}, P') S_{S \rightarrow \gamma_\perp^* V(L)}(s, s', q^2) , \quad (7.144)$$

where

$$S_{S \rightarrow \gamma_\perp^* V(L)}(s, s', q^2) = \frac{\text{Sp}_{\alpha\beta}^{(S \rightarrow \gamma_\perp^* V(L))} S_{\alpha\beta}^{(S \rightarrow \gamma_\perp V)}(\tilde{q}, P')}{S_{\alpha'\beta'}^{(S \rightarrow \gamma_\perp V)}(\tilde{q}, P') S_{\alpha'\beta'}^{(S \rightarrow \gamma_\perp^* V)}(\tilde{q}, P')} . \quad (7.145)$$

The spin factors $S_{S \rightarrow \gamma_\perp^* V(L)}(s, s', q^2)$ at $L = 0, 2$ equal

$$\begin{aligned} S_{S \rightarrow \gamma_\perp^* V(0)}(s, s', q^2) &= -2m[(s - s' + q^2 + 4m^2) - \frac{4s'q^4}{\lambda(s, s', q^2)}] , \\ S_{S \rightarrow \gamma_\perp^* V(2)}(s, s', q^2) &= -\frac{m}{2\sqrt{2}}[4m^4 - 2m^2(3s + s' - q^2) + s(s - s' + q^2) \\ &\quad + \frac{2ss'q^2}{\lambda(s, s', q^2)}(16m^2 + 3q^2 - s - 3s')] , \end{aligned} \quad (7.146)$$

with $\lambda(s, s', q^2)$ given by (7.134).

(ii) Form factor amplitudes.

The form factor of the considered process takes the form:

$$\begin{aligned} F_{S \rightarrow \gamma_\perp^* V(L)}(q^2) &= Z_{S \rightarrow \gamma V} \int_{4m^2}^{\infty} \frac{ds ds'}{16\pi^2} \psi_S(s) \psi_{V(L)}(s') \\ &\quad \times \frac{\Theta(-ss'q^2 - m^2\lambda(s, s', q^2))}{\sqrt{\lambda(s, s', q^2)}} S_{S \rightarrow \gamma^* V(L)}(s, s', q^2) . \end{aligned} \quad (7.147)$$

To calculate the integral at $q^2 \rightarrow 0$, we make, similarly to the calculations of (7.140), the following substitution: $q^2 = -Q^2$, $s = \Sigma + zQ/2$, $s' = \Sigma - zQ/2$.

After the integration over z in the limit $Q^2 \rightarrow 0$ and substituting $\Sigma \rightarrow s$, we have:

$$\begin{aligned}
 F_{S \rightarrow \gamma V(0)}(0) &= Z_{S \rightarrow \gamma V} \frac{m}{4\pi} \int_{4m^2}^{\infty} \frac{ds}{\pi} \psi_S(s) \psi_{V(0)}(s) I_{S \rightarrow \gamma V}(s), \\
 F_{S \rightarrow \gamma V(2)}(0) &= Z_{S \rightarrow \gamma V} \frac{m}{2\pi} \int_{4m^2}^{\infty} \frac{ds}{\pi} \psi_S(s) \psi_{V(2)}(s) (-s + 4m^2) I_{S \rightarrow \gamma V}(s), \\
 I_{S \rightarrow \gamma V}(s) &= \sqrt{s(s - 4m^2)} - 2m^2 \ln \frac{\sqrt{s} + \sqrt{s - 4m^2}}{\sqrt{s} - \sqrt{s - 4m^2}}.
 \end{aligned} \tag{7.148}$$

The whole form factor is

$$F_{S \rightarrow \gamma V}(0) = F_{S \rightarrow \gamma V(0)}(0) + F_{S \rightarrow \gamma V(2)}(0). \tag{7.149}$$

(iii) Partial widths for the decay processes with the emission of real photons.

Similarly to the form factor calculations performed above, the partial width of the scalar meson decay $S \rightarrow \gamma V$ reads:

$$M_S \Gamma_{S \rightarrow \gamma V} = \int d\Phi_2(p; q, p') \left| \sum_{\alpha\beta} A_{\alpha\beta}^{(S \rightarrow \gamma V)} \right|^2 = \alpha \frac{M_S^2 - M_V^2}{2M_S^2} |F_{S \rightarrow \gamma V}(0)|^2. \tag{7.150}$$

Recall that in the final expression $\alpha = e^2/4\pi = 1/137$. Likewise, the partial width of the vector meson decay $V \rightarrow \gamma S$ is equal to:

$$M_V \Gamma_{V \rightarrow \gamma S} = \alpha \frac{M_V^2 - M_S^2}{6M_V^2} |F_{S \rightarrow \gamma V}(0)|^2. \tag{7.151}$$

7.4.1.3 Normalisation conditions for wave functions of $q\bar{q}$ states

It is convenient to write the normalisation conditions for P , S and V meson wave functions using the charge form factor of this meson:

$$F_{charge}(0) = 1. \tag{7.152}$$

The amplitude of the charge factor is defined by the diagram of Fig. 7.11, with $(q\bar{q})_{in} = (q\bar{q})_{out}$. For P and S mesons the amplitude takes the form:

$$A_\alpha(q) = e(p + p')_\alpha F_{charge}(q^2). \tag{7.153}$$

For the pion, the $F_{charge}(q^2)$ is calculated in Appendix 7.A. For vector and scalar particles the calculations are similar.

Considering the meson V , we take the amplitude averaged over the spins of the vector particle. At $q^2 = 0$, it can be written as

$$A_{\alpha;\mu\mu}^{(V)}(q^2 \rightarrow 0) = 3e(p + p')_{\alpha} F_{charge}^{(V)}(0). \quad (7.154)$$

The normalisation conditions based on the formula (7.153) for P and S mesons read:

$$\begin{aligned} 1 &= \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_P^2(s) 2s \sqrt{\frac{s-4m^2}{s}}, \\ 1 &= \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_S^2(s) 2m^2 (s-4m^2) \sqrt{\frac{s-4m^2}{s}}. \end{aligned} \quad (7.155)$$

For the vector mesons V the normalisation condition is:

$$\begin{aligned} 1 &= W_{00}[V] + W_{02}[V] + W_{22}[V], \\ W_{00}[V] &= \frac{1}{3} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{V(0)}^2(s) 4(s+2m^2) \sqrt{\frac{s-4m^2}{s}}, \\ W_{02}[V] &= \frac{\sqrt{2}}{3} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{V(0)}(s) \psi_{V(2)}(s) (s-4m^2)^2 \sqrt{\frac{s-4m^2}{s}}, \\ W_{22}[V] &= \frac{2}{3} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{V(2)}^2(s) \frac{(8m^2+s)(s-4m^2)^2}{16} \sqrt{\frac{s-4m^2}{s}}. \end{aligned} \quad (7.156)$$

For more details in calculating the charge form factors for the vector and scalar mesons see [20, 37].

7.4.2 Transitions $T(2^{++}) \rightarrow \gamma V$ and $A(1^{++}) \rightarrow \gamma V$

Making use of the decays of the mesons $T(2^{++})$ and $A(1^{++})$, in this section we calculate form factors in a way which can be easily generalised for particles with arbitrary spins.

As a first step, we consider, as before, the emission of transversely polarised photons, *i.e.* reactions $T(2^{++}) \rightarrow \gamma_{\perp}^* V$ and $A(1^{++}) \rightarrow \gamma_{\perp}^* V$. Then we give expressions for form factors and decay partial widths for the production of real photons.

7.4.2.1 Transition $T \rightarrow \gamma_\perp^* V$

To operate with the tensor meson, we use the polarisation tensor $\epsilon_{\mu\nu}(a)$ with five components $a = 1, \dots, 5$. This polarisation tensor, being symmetrical and traceless, obeys the completeness condition:

$$\sum_{a=1, \dots, 5} \epsilon_{\mu\nu}(a) \epsilon_{\mu'\nu'}^+(a) = \frac{1}{2} \left(g_{\mu\mu'}^\perp g_{\nu\nu'}^\perp + g_{\mu\nu}^\perp g_{\nu\mu'}^\perp - \frac{2}{3} g_{\mu\nu}^\perp g_{\mu'\nu'}^\perp \right) = O_{\mu\nu}^{\mu'\nu'}(\perp p),$$

$$\sum_{a=1, \dots, 5} \epsilon_{\mu\nu}(a) \epsilon_{\mu\nu}^+(a) = 5. \quad (7.157)$$

Here $O_{\mu\nu}^{\mu'\nu'}(\perp p)$ is a standard projection operator for a system with the angular momentum $L = 2$ and the momentum p which obeys the requirements: $O_{\mu\nu}^{\mu'\nu'}(\perp p) O_{\mu'\nu'}^{\mu''\nu''}(\perp p) = O_{\mu\nu}^{\mu''\nu''}(\perp p)$ and $O_{\mu\nu}^{\mu\nu}(\perp p) = 0$, see Chapter 3 and [1] for more details.

In terms of the polarisation tensor $\varepsilon_{\mu\nu}$ and the vectors $\epsilon_\alpha^{(\gamma_\perp^*)}$, $\epsilon_\beta^{(V)}$, one has five independent spin structures for the decay amplitudes with the emission of virtual photons ($q^2 \neq 0$) in different final state waves:

$$\begin{aligned} (1) \quad S\text{-wave} : & \quad \epsilon_{\mu\nu} \epsilon_\mu^{(\gamma_\perp^*)} \epsilon_\nu^{(V)}, \\ (2) \quad D\text{-wave} : & \quad \epsilon_{\mu\nu} X_{\mu\nu}^{(2)}(q^\perp) (\epsilon^{(\gamma_\perp^*)} \epsilon^{(V)}), \\ (3) \quad D\text{-wave} : & \quad \epsilon_{\mu\nu} X_{\nu\beta}^{(2)}(q^\perp) \epsilon_\mu^{(\gamma_\perp^*)} \epsilon_\beta^{(V)}, \\ (4) \quad D\text{-wave} : & \quad \epsilon_{\mu\nu} X_{\nu\alpha}^{(2)}(q^\perp) \epsilon_\alpha^{(\gamma_\perp^*)} \epsilon_\mu^{(V)}, \\ (5) \quad G\text{-wave} : & \quad \epsilon_{\mu\nu} X_{\mu\nu\alpha\beta}^{(4)}(q^\perp) \epsilon_\alpha^{(\gamma_\perp^*)} \epsilon_\beta^{(V)}. \end{aligned} \quad (7.158)$$

Consequently, we have five independent form factors which describe the transition $T(2^{++}) \rightarrow \gamma_\perp^* V$. But for the real photon ($q^2 = 0$) the number of independent form factors is reduced to three.

(i) Spin operators in the $T \rightarrow \gamma_\perp^* V$ reaction.

For the transversely polarised photon with $q^2 \neq 0$ we introduce the following spin operators corresponding to the spin structures given in (7.158):

$$\begin{aligned} S_{\mu\nu, \alpha\beta}^{(1)} &= O_{\mu\nu}^{\mu'\nu'}(\perp p) g_{\mu'\alpha}^\perp g_{\nu'\beta}^\perp, \\ S_{\mu\nu, \alpha\beta}^{(2)} &= -\frac{1}{q_\perp^2} X_{\mu\nu}^{(2)}(q^\perp) g_{\alpha\alpha'}^\perp g_{\beta\beta'}^\perp, \\ S_{\mu\nu, \alpha\beta}^{(3)} &= -\frac{1}{q_\perp^2} O_{\mu\nu}^{\mu'\nu'}(\perp p) X_{\nu'\beta'}^{(2)}(q^\perp) g_{\mu'\alpha}^\perp g_{\beta'\beta}^\perp, \\ S_{\mu\nu, \alpha\beta}^{(4)} &= -\frac{1}{q_\perp^2} O_{\mu\nu}^{\mu'\nu'}(\perp p) X_{\nu'\alpha'}^{(2)}(q^\perp) g_{\alpha'\alpha}^\perp g_{\beta'\beta}^\perp, \\ S_{\mu\nu, \alpha\beta}^{(5)} &= \frac{1}{q_\perp^4} X_{\mu\nu\alpha'\beta'}^{(4)}(q^\perp) g_{\alpha'\alpha}^\perp g_{\beta'\beta}^\perp. \end{aligned} \quad (7.159)$$

Recall that here $q_\alpha^\perp = g_{\alpha\alpha'}^\perp q_{\alpha'} = q_\alpha - p_\alpha(pq)/p^2$ and $g_{\alpha\alpha'}^\perp = g_{\alpha\alpha'} - p_\alpha p_{\alpha'}/p^2$. Let us remind the method of construction of these operators by considering the G -wave spin structure from (7.158): one should multiply the G -wave spin structure $\epsilon_{\mu'\nu'} X_{\mu'\nu'\alpha'\beta'}^{(4)}(q^\perp) \epsilon_{\alpha'}^{(\gamma_\perp^*)} \epsilon_{\beta'}^{(V)}$ by the polarisations $\epsilon_{\mu\nu}^+(a)$, $\epsilon_{\alpha}^{(\gamma_\perp^*+)}(b)$, $\epsilon_{\beta}^{(V)+}(c)$, and perform summations over a, b, c :

$$\sum_{a,b,c} \epsilon_{\mu\nu}^+(a) \epsilon_{\mu'\nu'}(a) X_{\mu'\nu'\alpha'\beta'}^{(4)}(q^\perp) \epsilon_{\alpha'}^{(\gamma_\perp^*)}(b) \epsilon_{\alpha}^{(\gamma_\perp^*+)}(b) \epsilon_{\beta'}^{(V)}(c) \epsilon_{\beta}^{(V)+}(c). \quad (7.160)$$

The operators (7.159) should be orthogonalised as follows:

$$\begin{aligned} S_{\mu\nu,\alpha\beta}^{\perp(1)}(p', q) &= S_{\mu\nu,\alpha\beta}^{(1)}, \\ S_{\mu\nu,\alpha\beta}^{\perp(2)}(p', q) &= S_{\mu\nu,\alpha\beta}^{(2)} - S_{\mu\nu,\alpha\beta}^{\perp(1)}(p', q) \frac{\left(S_{\mu'\nu',\alpha'\beta'}^{\perp(1)}(p', q) S_{\mu'\nu',\alpha'\beta'}^{(2)} \right)}{\left(S_{\mu'\nu',\alpha'\beta'}^{\perp(1)}(p', q) S_{\mu'\nu',\alpha'\beta'}^{\perp(1)}(p', q) \right)}, \\ S_{\mu\nu,\alpha\beta}^{\perp(3)}(p', q) &= S_{\mu\nu,\alpha\beta}^{(3)} - S_{\mu\nu,\alpha\beta}^{\perp(1)}(p', q) \frac{\left(S_{\mu'\nu',\alpha'\beta'}^{\perp(1)}(p', q) S_{\mu'\nu',\alpha'\beta'}^{(3)} \right)}{\left(S_{\mu'\nu',\alpha'\beta'}^{\perp(1)}(p', q) S_{\mu'\nu',\alpha'\beta'}^{\perp(1)}(p', q) \right)} \\ &\quad - S_{\mu\nu,\alpha\beta}^{\perp(2)}(p', q) \frac{\left(S_{\mu'\nu',\alpha'\beta'}^{\perp(2)}(p', q) S_{\mu'\nu',\alpha'\beta'}^{(3)} \right)}{\left(S_{\mu'\nu',\alpha'\beta'}^{\perp(2)}(p', q) S_{\mu'\nu',\alpha'\beta'}^{\perp(2)}(p', q) \right)}. \end{aligned} \quad (7.161)$$

Thus we construct three operators, $i = 1, 2, 3$. The operators $S_{\mu\nu,\alpha\beta}^{\perp(4)}(p', q)$ and $S_{\mu\nu,\alpha\beta}^{\perp(5)}(p', q)$ are nilpotent at $q^2 = 0$, so we do not present explicit expressions for them here but concentrate on the calculation of the amplitude for the emission of the real photon.

The orthogonalised operator norm which determines the decay partial width is defined as follows:

$$S_{\mu\nu,\alpha\beta}^{\perp(a)}(p', q) S_{\mu\nu,\alpha\beta}^{\perp(b)}(p', q) = z_{ab}^\perp (M_T^2, M_V^2, q^2). \quad (7.162)$$

At $q^2 = 0$ we have:

$$\begin{aligned} z_{11}^\perp(M_T^2, M_V^2, 0) &= \frac{3M_T^4 + 34M_T^2 M_V^2 + 3M_V^4}{12M_T^2 M_V^2}, \\ z_{22}^\perp(M_T^2, M_V^2, 0) &= 9 \frac{M_T^4 + 10M_T^2 M_V^2 + M_V^4}{3M_T^4 + 34M_T^2 M_V^2 + 3M_V^4}, \\ z_{33}^\perp(M_T^2, M_V^2, 0) &= \frac{9}{2} \frac{(M_T^2 + M_V^2)^2}{M_T^4 + 10M_T^2 M_V^2 + M_V^4}. \end{aligned} \quad (7.163)$$

(ii) Calculation of the transition amplitude $T(L) \rightarrow \gamma V(L')$ for the emission of the real photon.

So, the decay amplitude $T \rightarrow \gamma V$ is written using the operators (7.161) as follows:

$$\begin{aligned} A_{\mu\nu;\alpha\beta}^{T(L) \rightarrow \gamma V(L')} &= \sum_{i=1,2,3} S_{\mu\nu;\alpha\beta}^{\perp(i)}(p', q) F_{T \rightarrow \gamma V}^{(i)}(0) \\ &= \sum_{i=1,2,3} S_{\mu\nu;\alpha\beta}^{\perp(i)}(p', q) \sum_{L=1,3; L'=0,2} F_{T(L) \rightarrow \gamma V(L')}^{(i)}(0), \end{aligned} \quad (7.164)$$

where $F_{T \rightarrow \gamma V}^{(i)}(0) = \sum_{L=1,3; L'=0,2} F_{T(L) \rightarrow \gamma V(L')}^{(i)}(0)$ are the form factors at $q^2 = 0$. But for performing calculations, it is convenient to consider first the case $q^2 \neq 0$ and then put $q^2 \rightarrow 0$.

So, we write the double discontinuity related to the diagram of Fig. 7.11b at $q^2 = \tilde{q}^2 = (k_1 - k'_1)^2 \neq 0$ and expand over the spin operators the corresponding traces:

$$\begin{aligned} \text{Sp}_{\mu\nu,\alpha\beta}^{T(1) \rightarrow \gamma^* V(0)} &= -\text{Sp} \left[\hat{G}_{\beta}^{(1,0,1)}(k') (\hat{k}'_1 + m) \gamma_{\alpha}^{\perp} \gamma^* (\hat{k}_1 + m) \hat{G}_{\mu\nu}^{(1,1,2)}(k) (-\hat{k}_2 + m) \right], \\ \text{Sp}_{\mu\nu,\alpha\beta}^{T(1) \rightarrow \gamma^* V(2)} &= -\text{Sp} \left[\hat{G}_{\beta}^{(1,2,1)}(k') (\hat{k}'_1 + m) \gamma_{\alpha}^{\perp} \gamma^* (\hat{k}_1 + m) \hat{G}_{\mu\nu}^{(1,1,2)}(k) (-\hat{k}_2 + m) \right], \\ \text{Sp}_{\mu\nu,\alpha\beta}^{T(3) \rightarrow \gamma^* V(0)} &= -\text{Sp} \left[\hat{G}_{\beta}^{(1,0,1)}(k') (\hat{k}'_1 + m) \gamma_{\alpha}^{\perp} \gamma^* (\hat{k}_1 + m) \hat{G}_{\mu\nu}^{(1,3,2)}(k) (-\hat{k}_2 + m) \right], \\ \text{Sp}_{\mu\nu,\alpha\beta}^{T(3) \rightarrow \gamma^* V(2)} &= -\text{Sp} \left[\hat{G}_{\beta}^{(1,2,1)}(k') (\hat{k}'_1 + m) \gamma_{\alpha}^{\perp} \gamma^* (\hat{k}_1 + m) \hat{G}_{\mu\nu}^{(1,3,2)}(k) (-\hat{k}_2 + m) \right], \end{aligned} \quad (7.165)$$

where the vertices $\hat{G}_{\beta}^{(1,0,1)}(k')$ and $\hat{G}_{\beta}^{(1,2,1)}(k')$ for $L' = 0, 2$ are given in (7.126), and

$$\begin{aligned} \hat{G}_{\mu\nu}^{(1,1,2)}(k) &= \frac{3}{\sqrt{2}} \left[k_{\mu} \gamma_{\nu} + k_{\nu} \gamma_{\mu} - \frac{2}{3} g_{\mu\nu}^{\perp} \hat{k} \right], \\ \hat{G}_{\mu\nu}^{(1,3,2)}(k) &= \frac{5}{\sqrt{2}} \left[k_{\mu} k_{\nu} \hat{k} - \frac{1}{5} k^2 (g_{\mu\nu}^{\perp} \hat{k} + \gamma_{\mu} k_{\nu} + k_{\mu} \gamma_{\nu}) \right]. \end{aligned} \quad (7.166)$$

Remind that we have used here the notations $\gamma_{\alpha}^{\perp} \gamma^* = g_{\alpha\alpha'}^{\perp} \gamma_{\alpha'}^*$, $k = (k_1 - k_2)/2$, $k' = (k'_1 - k'_2)/2$.

The expansion (7.165) over the spin operators $S_{\mu\nu,\alpha\beta}^{\perp(i)}(P', \tilde{q})$ reads:

$$\text{Sp}_{\mu\nu,\alpha\beta}^{(T(L) \rightarrow \gamma^* V(L'))} = \sum_{i=1,2,3} S_{\mu\nu,\alpha\beta}^{\perp(i)}(P', \tilde{q}) S_{T(L) \rightarrow \gamma_{\perp}^* V(L')}^{(i)}(s, s', q^2), \quad (7.167)$$

Let us emphasise that in (7.167) the spin operators depend on the intermediate-state quark variables, P' and $\tilde{q}$. The invariant spin factors

are determined by convolutions:

$$S_{T(L) \rightarrow \gamma_{\perp}^* V(L')}^{\perp(i)}(s, s', q^2) = \frac{Sp_{\mu\nu, \alpha\beta}^{(T(L) \rightarrow \gamma_{\perp}^* V(L'))} S_{\mu\nu, \alpha\beta}^{\perp(i)}(P', \tilde{q})}{S_{\mu'\nu', \alpha'\beta'}^{\perp(i)}(P', \tilde{q}) S_{\mu'\nu', \alpha'\beta'}^{\perp(i)}(P', \tilde{q})}, \quad (7.168)$$

where $i = 1, 2, 3$. The invariant spin factors determine the form factors in a standard way:

$$F_{T(L) \rightarrow \gamma_{\perp}^* V(L')}^{(i)}(q^2) = Z_{T \rightarrow \gamma V} \int_{4m^2}^{\infty} \frac{ds ds'}{16\pi^2} \psi_{T(L)}(s) \psi_{V(L')}(s') \quad (7.169)$$

$$\times \frac{\Theta(-ss'q^2 - m^2\lambda(s, s', q^2))}{\sqrt{\lambda(s, s', q^2)}} S_{T(L) \rightarrow \gamma_{\perp}^* V(L')}^{\perp(i)}(s, s', q^2).$$

To calculate the integral at $q^2 \rightarrow 0$, we make, as before (see (7.140)), the following substitution: $q^2 = -Q^2$, $s = \Sigma + zQ/2$, $s' = \Sigma - zQ/2$ and perform the integration over z . We have:

$$F_{T(L) \rightarrow \gamma V(L')}^{(i)}(0) = Z_{T \rightarrow \gamma V} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{T(L)}(s) \psi_{V(L')}(s) J_{T(L) \rightarrow \gamma V(L')}^{(i)}(s). \quad (7.170)$$

Here

$$J_{T(1) \rightarrow \gamma V(0)}^{(1)}(s) = -\frac{\sqrt{3}}{5}(8m^2 + 3s)I_{T \rightarrow \gamma V}^{(1)}(s), \quad (7.171)$$

$$J_{T(1) \rightarrow \gamma V(0)}^{(2)}(s) = \frac{2}{3}J_{T(1) \rightarrow \gamma V(0)}^{(3)}(s) = -\frac{2}{3\sqrt{3}}I_{T \rightarrow \gamma V}^{(2)}(s),$$

$$J_{T(1) \rightarrow \gamma V(2)}^{(1)}(s) = -\frac{\sqrt{6}}{40}(16m^2 - 3s)(4m^2 - s)I_{T \rightarrow \gamma V}^{(1)}(s),$$

$$J_{T(1) \rightarrow \gamma V(2)}^{(2)}(s) = \frac{2}{3}J_{T(1) \rightarrow \gamma V(2)}^{(3)}(s) = -\frac{\sqrt{2}}{12\sqrt{3}}(8m^2 + s)I_{T \rightarrow \gamma V}^{(2)}(s),$$

$$J_{T(3) \rightarrow \gamma V(0)}^{(1)}(s) = -\frac{3\sqrt{2}}{20}(4m^2 - s)^2I_{T \rightarrow \gamma V}^{(1)}(s),$$

$$J_{T(3) \rightarrow \gamma V(0)}^{(2)}(s) = \frac{2}{3}J_{T(3) \rightarrow \gamma V(0)}^{(3)}(s) = -\frac{\sqrt{2}}{18}(6m^2 + s)I_{T \rightarrow \gamma V}^{(2)}(s),$$

$$J_{T(3) \rightarrow \gamma V(2)}^{(1)}(s) = -\frac{3}{80}(4m^2 - s)^2(8m^2 + s)I_{T \rightarrow \gamma V}^{(1)}(s),$$

$$J_{T(3) \rightarrow \gamma V(2)}^{(2)}(s) = \frac{2}{3}J_{T(3) \rightarrow \gamma V(2)}^{(3)}(s) = -\frac{1}{72}(16m^2 - 3s)(4m^2 - s)I_{T \rightarrow \gamma V}^{(2)}(s),$$

and

$$I_{T \rightarrow \gamma V}^{(1)}(s) = 2m^2 \ln \frac{\sqrt{s} + \sqrt{s - 4m^2}}{\sqrt{s} - \sqrt{s - 4m^2}} - \sqrt{s(s - 4m^2)}, \quad (7.172)$$

$$I_{T \rightarrow \gamma V}^{(2)}(s) = m^2(m^2 + s) \ln \frac{\sqrt{s} + \sqrt{s - 4m^2}}{\sqrt{s} - \sqrt{s - 4m^2}} - \frac{1}{12} \sqrt{s(s - 4m^2)}(s + 26m^2).$$

(iii) Normalisation of tensor meson wave function and partial widths.

The normalisation condition for the wave functions of tensor mesons reads:

$$1 = W_{11}[T] + W_{13}[T] + W_{33}[T], \quad (7.173)$$

$$W_{11}[T] = \frac{1}{5} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{T(1)}^2(s) \frac{1}{2} (8m^2 + 3s)(s - 4m^2) \sqrt{\frac{s - 4m^2}{s}},$$

$$W_{13}[T] = \frac{1}{5} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{T(1)}(s) \psi_{T(3)}(s) \frac{\sqrt{3}}{2\sqrt{2}} (s - 4m^2)^3 \sqrt{\frac{s - 4m^2}{s}},$$

$$W_{33}[T] = \frac{1}{5} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{T(3)}^2(s) \frac{1}{16} (6m^2 + s)(s - 4m^2)^3 \sqrt{\frac{s - 4m^2}{s}}.$$

The partial width of the $T \rightarrow \gamma V$ decay is equal to:

$$\begin{aligned} m_T \Gamma_{T \rightarrow \gamma V} &= e^2 \int d\Phi_2(p; q, p') \frac{1}{5} \sum_{\mu\nu, \alpha\beta} \left| A_{\mu\nu, \alpha\beta} \right|^2 = \frac{\alpha}{20} \frac{m_T^2 - m_V^2}{m_T^2} \\ &\times \left[z_{11}^\perp(M_T^2, M_V^2, 0) \left(F_{T \rightarrow \gamma V}^{(1)}(0) \right)^2 + z_{22}^\perp(M_T^2, M_V^2, 0) \left(F_{T \rightarrow \gamma V}^{(2)}(0) \right)^2 \right. \\ &\left. + z_{33}^\perp(M_T^2, M_V^2, 0) \left(F_{T \rightarrow \gamma V}^{(3)}(0) \right)^2 \right]. \end{aligned} \quad (7.174)$$

The same block of form factors determines the partial width for $V \rightarrow \gamma T$:

$$\begin{aligned} m_V \Gamma_{V \rightarrow \gamma T} &= e^2 \int d\Phi_2(p; q, p') \frac{1}{3} \sum_{\mu\nu, \alpha\beta} \left| A_{\mu\nu, \alpha\beta} \right|^2 = \frac{\alpha}{12} \frac{m_V^2 - m_T^2}{m_V^2} \\ &\times \left[z_{11}^\perp(M_T^2, M_V^2, 0) \left(F_{T \rightarrow \gamma V}^{(1)}(0) \right)^2 + z_{22}^\perp(M_T^2, M_V^2, 0) \left(F_{T \rightarrow \gamma V}^{(2)}(0) \right)^2 \right. \\ &\left. + z_{33}^\perp(M_T^2, M_V^2, 0) \left(F_{T \rightarrow \gamma V}^{(3)}(0) \right)^2 \right]. \end{aligned} \quad (7.175)$$

Let us emphasise that the factors $z_{aa}^\perp(M_T^2, M_V^2, 0)$ are symmetrical with respect to the $T \leftrightarrow V$ permutation: $z_{aa}^\perp(M_T^2, M_V^2, 0) = z_{aa}^\perp(M_V^2, M_T^2, 0)$.

7.4.2.2 Transition $A \rightarrow \gamma V$

For the reaction $A(1^{++}) \rightarrow \gamma^* V(1^{--})$ one has three partial states: the S -wave state and two D -wave states. Generally, we have three spin structures,

but only two of them survive in the case of a transversely polarised photon $\gamma_{\perp}^*$:

$$\begin{aligned} S_{\mu,\alpha\beta}^{(1)}(p, q) &= g_{\alpha\alpha'}^{\perp\perp} g_{\beta\beta'}^{\perp V} \varepsilon_{\mu\alpha'\beta'p} , \\ S_{\mu,\alpha\beta}^{(2)}(p, q) &= -\frac{1}{q_{\perp}^2} q_{\beta'}^{\perp} g_{\mu\mu'}^{\perp} g_{\alpha\alpha'}^{\perp\perp} g_{\beta\beta'}^{\perp V} \varepsilon_{\mu'\alpha'q^{\perp}p} = -\frac{1}{q_{\perp}^2} q_{\beta'}^{\perp} g_{\beta\beta'}^{\perp V} \varepsilon_{\mu\alpha q^{\perp}p} , \\ S_{\mu,\alpha\beta}^{(3)}(p, q) &= -\frac{1}{q_{\perp}^2} q_{\alpha'}^{\perp} g_{\mu\mu'}^{\perp} g_{\alpha\alpha'}^{\perp\perp} g_{\beta\beta'}^{\perp V} \varepsilon_{\mu'\beta'q^{\perp}p} = 0 . \end{aligned} \quad (7.176)$$

Here, as previously, p is the momentum of the decaying particle, q is that of the outgoing photon, and we use the abridged form $\varepsilon_{\mu\alpha\beta\xi} p_{\xi} \equiv \varepsilon_{\mu\alpha\beta p}$. The vanishing of $S_{\mu,\alpha\beta}^{(3)}(p, q)$ is due to the equality $q_{\xi}^{\perp} g_{\alpha\xi}^{\perp\perp} = 0$.

(i) Spin operators and decay amplitude.

The operators $S_{\mu,\alpha\beta}^{(i)}(p, q)$ should be orthogonalised:

$$\begin{aligned} S_{\mu,\alpha\beta}^{\perp(1)}(p, q) &\equiv S_{\mu,\alpha\beta}^{(1)}(p, q) , \\ S_{\mu,\alpha\beta}^{\perp(2)}(p, q) &= S_{\mu,\alpha\beta}^{(2)}(p, q) - S_{\mu,\alpha\beta}^{\perp(1)}(p, q) \frac{\left(S_{\mu',\alpha'\beta'}^{\perp(1)}(p, q) S_{\mu',\alpha'\beta'}^{(2)}(p, q) \right)}{\left(S_{\mu',\alpha'\beta'}^{\perp(1)}(p, q) S_{\mu',\alpha'\beta'}^{\perp(1)}(p, q) \right)} . \end{aligned} \quad (7.177)$$

We determine the convolutions

$$S_{\mu,\alpha\beta}^{\perp(a)}(p, q) S_{\mu,\alpha\beta}^{\perp(b)}(p, q) \equiv z_{ab}^{\perp} (M_A^2, M_V^2, q^2) . \quad (7.178)$$

At $q^2 = 0$ (see Appendix 6.C for details), they are

$$\begin{aligned} z_{11}^{\perp} (M_A^2, M_V^2, 0) &= -\frac{M_A^4 + 6M_A^2 M_V^2 + M_V^4}{2M_V^2} , \\ z_{22}^{\perp} (M_A^2, M_V^2, 0) &= -\frac{2M_A^2 (M_A^2 + M_V^2)^2}{M_A^4 + 6M_A^2 M_V^2 + M_V^4} . \end{aligned} \quad (7.179)$$

The transition amplitude $A \rightarrow \gamma V$ reads:

$$A_{\mu,\alpha\beta}^{(A \rightarrow \gamma V)} = \sum_{i=1,2} S_{\mu,\alpha\beta}^{\perp(i)}(p, q) F_{A \rightarrow \gamma V}^{(i)}(0) , \quad (7.180)$$

being determined by two form factors $F_{A \rightarrow \gamma V}^{(i)}(0)$ ($i = 1, 2$).

(ii) Calculation of the quark triangle diagram for the emission of the real photon.

The vector state has two components, so the diagram of Fig. 7.11b for the processes $A \rightarrow \gamma^* V(L)$ ($L = 0, 2$) is determined by the following traces:

$$\begin{aligned} \text{Sp}_{\mu,\alpha\beta}^{(A \rightarrow \gamma^* V(0))} &= -\text{Sp} \left[\hat{G}_{\beta}^{(1,0,1)}(k') (\hat{k}'_1 + m) \gamma_{\alpha}^{\perp \gamma^*} (\hat{k}_1 + m) A_{\mu}(k) (-\hat{k}_2 + m) \right] , \\ \text{Sp}_{\mu,\alpha\beta}^{(A \rightarrow \gamma^* V(2))} &= -\text{Sp} \left[\hat{G}_{\beta}^{(1,2,1)}(k') (\hat{k}'_1 + m) \gamma_{\alpha}^{\perp \gamma^*} (\hat{k}_1 + m) A_{\mu}(k) (-\hat{k}_2 + m) \right] , \end{aligned} \quad (7.181)$$

where the vertices $\hat{G}^{(1,0,1)}$, $\hat{G}^{(1,2,1)}$ refer to the vector state (see (7.126)). The spin vertex for the transition $A \rightarrow q\bar{q}$ reads:

$$A_\mu(k) = \sqrt{\frac{2}{3s}} i \varepsilon_{\mu k \gamma P} , \quad (7.182)$$

and, as previously, $k = (k_1 - k_2)/2$, $P = k_1 + k_2$.

To calculate the invariant form factor, we should expand (7.181) into a series with respect to the spin operators $S_{\mu,\alpha\beta}^\perp(P, \tilde{q})$ (recall that $\tilde{q} = P - P'$ and $\tilde{q}^2 = q^2$) and perform calculations for $F_{A \rightarrow \gamma_\perp^* V(L)}^{(i)}(q^2)$ in a way developed above. After performing these calculations, we obtain in the limit $q^2 \rightarrow 0$:

$$\begin{aligned} F_{A \rightarrow \gamma V(L)}^{(i)}(0) &= Z_{A \rightarrow \gamma V} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} (s) \psi_A(s) \psi_{V(L)}(s) J_{A \rightarrow \gamma V(L)}^{(i)}(s) , \\ J_{A \rightarrow \gamma V(0)}^{(1)}(s) &= -\sqrt{\frac{3}{2}} I_{A \rightarrow \gamma V}(s), \quad J_{A \rightarrow \gamma V(2)}^{(2)}(s) = \frac{\sqrt{3}}{8} (4m^2 - s) I_{A \rightarrow \gamma V}(s), \\ I_{A \rightarrow \gamma V}(s) &= \sqrt{s} \left(2m^2 \ln \frac{\sqrt{s} + \sqrt{s - 4m^2}}{\sqrt{s} - \sqrt{s - 4m^2}} - \sqrt{s(s - 4m^2)} \right). \end{aligned} \quad (7.183)$$

The whole form factor equals

$$F_{A \rightarrow \gamma V}^{(i)}(0) = \sum_{L=0,2} F_{A \rightarrow \gamma V(L)}^{(i)}(0) . \quad (7.184)$$

(iii) Wave function normalisation condition and partial widths.

The normalisation condition for the 1^{++} meson wave function reads:

$$1 = \frac{1}{2} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_A^2(s) s(s - 4m^2) \sqrt{\frac{s - 4m^2}{s}}. \quad (7.185)$$

The partial width of the decay $A \rightarrow \gamma V$ is

$$\begin{aligned} m_A \Gamma_{A \rightarrow \gamma V} &= e^2 \int d\Phi_2(p; q, p') \frac{1}{3} \sum_{\mu,\alpha\beta} \left| A_{\mu,\alpha\beta} \right|^2 = \frac{\alpha}{12} \frac{m_A^2 - m_V^2}{m_A^2} \\ &\times \left[z_{11}^\perp(M_A^2, M_V^2, 0) \left(F^{(1)}(0) \right)^2 + z_{22}^\perp(M_A^2, M_V^2, 0) \left(F^{(2)}(0) \right)^2 \right]. \end{aligned} \quad (7.186)$$

For the partial width of the decay $V \rightarrow \gamma A$ one has:

$$\begin{aligned} m_V \Gamma_{V \rightarrow \gamma A} &= \frac{\alpha}{12} \frac{m_V^2 - m_A^2}{m_V^2} \left[z_{11}^\perp(M_V^2, M_A^2, 0) \left(F^{(1)}(0) \right)^2 \right. \\ &\quad \left. + z_{22}^\perp(M_V^2, M_A^2, 0) \left(F^{(2)}(0) \right)^2 \right]. \end{aligned} \quad (7.187)$$

Let us emphasise that $z_{aa}^\perp(M_V^2, M_A^2, 0) \neq z_{aa}^\perp(M_A^2, M_V^2, 0)$.

Miniconclusion

Actually, the tensor meson decay is a pattern for an amplitude, where the parity of the initial meson coincides with the parity of the final state. For this case we construct the spin factors as convolutions of polarisation and angular momentum functions $X_{\mu_1 \dots \mu_L}^{(L)}(k^\perp)$, see equation (7.158) for the tensor meson. With the completeness condition for the vector and tensor polarisations, we construct gauge invariant spin operators (7.159) for the tensor mesons. The orthogonalisation of these operators for the case of the real photon emission allows us to single out the operators with non-zero norm and the nilpotent operators. These operators are used in the expansion of the amplitude in a series with respect to external particles (equation (7.164)), as well as for the quark states when we consider the triangle diagram discontinuity (Eqs. (7.167) and (7.168)). The spectral integrals are written for the invariant form factors, which are the coefficients in front of the orthogonalised operators.

As was noted above, the spectral integral expressions for the form factors have many common features with those in quantum mechanics. Let us emphasise once more that the confinement in the spectral integral representation, as in quantum mechanics, is the underlying property of the $q\bar{q}$ wave functions of mesons. Namely, the singular behaviour of the interaction at large distances results in a type of wave functions forbidding the quarks to leave the confinement trap. In terms of analytical properties, this means that the wave functions of the confined quarks have no poles at $s = M_{\text{meson}}^2$.

7.5 Determination of the Quark–Antiquark Component of the Photon Wave Function for u, d, s -Quarks

The establishing of the quark–gluon content of mesons and subsequent systematisation provide the basis for strong interaction physics. The radiative decay is a powerful tool for the qualitative evaluation of the quark–antiquark components of mesons. An important role in this line of investigation plays the study of the two-photon transitions such as $meson \rightarrow \gamma\gamma$ and, more generally, $meson \rightarrow \gamma^*\gamma^*$. Within the additive quark model the corresponding diagrams are shown in Fig. 7.12.

Experimental data accumulated by the collaborations L3 [40, 41], ARGUS [42], CELLO [43], TRC/2 γ [44], CLEO [45], Mark II [46], Crystal

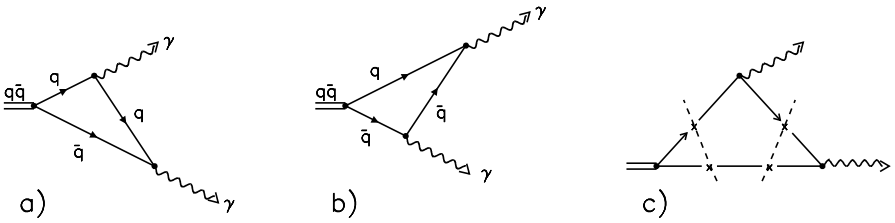


Fig. 7.12 Diagrams for the two-photon decay of a $q\bar{q}$ state with the emission of a photon in the intermediate state by a quark (a) and an antiquark (b). Figure (c) demonstrates the cuttings of the diagram (a) in the double spectral integral.

Ball [47], and others make it obvious that the calculation of the processes $meson \rightarrow \gamma^* \gamma^*$ is up to date. To make this reaction informative concerning the meson quark–gluon content, one needs a reliably determined initial and final state interactions of quarks, *i.e.* their wave functions, see Figs. 7.13, 7.14.

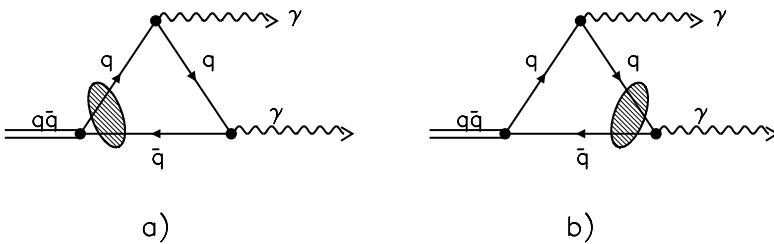


Fig. 7.13 Diagrams for the two-photon decay of a $q\bar{q}$ state: quark interaction in the initial (a) and the final state (b).

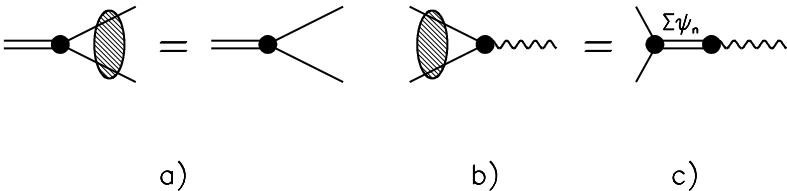


Fig. 7.14 Inclusion of the initial quark interaction into meson wave function (a); rewritten final state interaction in terms of the vector dominance model (b and c).

Conventionally, one may consider two pieces of the photon wave function: the soft and hard components. The hard component is related to the point-like vertex $\gamma \rightarrow q\bar{q}$, it is responsible for the production of a quark–antiquark pair at high photon virtuality. In the case of the e^+e^- system, at high energies the ratio of the cross sections $R = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ is determined by the hard component of the photon wave function, while the soft component is responsible for the production of low-energy quark–antiquark vector states such as ρ^0 , ω , $\phi(1020)$ and their excitations.

The first evaluation of the photon wave function in terms of the spectral integral technique was carried out for the transitions $\gamma^* \rightarrow u\bar{u}, d\bar{d}, s\bar{s}$ in [38] (on the basis of data of the CLEO Collaboration [45] on the Q^2 -dependent transition form factors $\pi^0 \rightarrow \gamma\gamma^*$, $\eta \rightarrow \gamma\gamma^*$, and $\eta' \rightarrow \gamma\gamma^*$). As the next step, in [48] the information on the processes $e^+e^- \rightarrow V$ was added that made it possible to determine the wave function $\gamma^* \rightarrow u\bar{u}, d\bar{d}, s\bar{s}$ more precisely.

The photon wave function depends on the invariant energy squared of the $q\bar{q}$ system:

$$\psi_{\gamma^*(Q^2) \rightarrow q\bar{q}}(s) = \frac{G_{\gamma \rightarrow q\bar{q}}(s)}{s + Q^2}, \quad (7.188)$$

here $G_{\gamma \rightarrow q\bar{q}}(s)$ is the vertex for the transition of a photon into a $q\bar{q}$ state, and $(s + Q^2)^{-1}$ presents the wave function denominator ($q^2 = -Q^2$).

Schematically, the vertex function $G_{\gamma \rightarrow q\bar{q}}(s)$ may be represented as

$$\sum_a C_a e^{-b_a s} + \Theta(s - s_0), \quad (7.189)$$

where the first terms stand for the soft component which is due to the transition of a photon to vector mesons $\gamma \rightarrow V \rightarrow q\bar{q}$, see Figs. 7.14b,c, while the second one describes the point-like interaction in the hard domain, see Fig. 7.14a, (here the step-function $\Theta(s - s_0) = 0$ at $s < s_0$ and $\Theta(s - s_0) = 1$ at $s \geq s_0$; we extract the quark charge from our photon wave function). The basic characteristics of the soft component of $G_{\gamma \rightarrow q\bar{q}}(s)$ are the threshold value of the vertex, $\sum C_a \exp(-4m^2 b_a)$, and the rate of its decrease with energy given by the slopes b_a . The hard component of the vertex is characterised by the value of s_0 , which is the quark energy squared when the point-like interaction becomes dominant.

In [38] the photon wave function has been found with the assumption that the quark relative-momentum dependence is the same for all quark vertices. The hypothesis of the vertex universality for u and d quarks,

$$G_{\gamma \rightarrow u\bar{u}}(s) = G_{\gamma \rightarrow d\bar{d}}(s) \equiv G_{\gamma}(s), \quad (7.190)$$

looks rather trustworthy because of the degeneracy of the ρ and ω states, though the similarity in the k -dependence for the non-strange and strange quarks (which results from the SU(6)-symmetry) may be not precise.

Our strategy in the determination of the parameters for the photon wave function for non-strange and strange quarks is as follows (see also [48]). As the first step, we present the formulae for the transition form factors $\pi^0, \eta, \eta' \rightarrow \gamma(Q_1^2)\gamma(Q_2^2)$ (the charge form factor of the pseudoscalar meson, which determines the meson wave function, is calculated in the way discussed above. Then we consider the e^+e^- annihilation processes: the partial decay widths $\omega, \rho^0, \phi \rightarrow e^+e^-$ and the ratio $R(E_{e^+e^-}) = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ at $1 \leq E_{e^+e^-} \leq 3.7$ GeV. Thus, fitting to data, we obtain the photon wave function $\gamma \rightarrow q\bar{q}$ for the light quarks.

7.5.1 Transition form factors $\pi^0, \eta, \eta' \rightarrow \gamma^*(Q_1^2)\gamma^*(Q_2^2)$

Using the same technique as for the $\text{meson} \rightarrow \gamma^*(Q^2)V$ amplitude, we can write the formulae for the transition form factors of the pseudoscalar mesons $\pi^0, \eta, \eta' \rightarrow \gamma^*(Q_1^2)\gamma^*(Q_2^2)$. The corresponding diagrams are shown in Fig. 7.12.

The general structure of the amplitude for these processes is as follows:

$$A_{\mu\nu}^{(P \rightarrow \gamma^* \gamma^*)}(Q_1^2, Q_2^2) = e^2 \epsilon_{\mu\nu\alpha\beta} q_\alpha p_\beta F_{(\pi, \eta, \eta') \rightarrow \gamma^* \gamma^*}(Q_1^2, Q_2^2), \quad (7.191)$$

where $q = (q_1 - q_2)/2$ and $p = q_1 + q_2$ (recall that $q_i^2 = -Q_i^2$).

Let us make use, first, of the light-cone variables $(x, \mathbf{k}_\perp)$; in terms of these variables the expression for the transition form factor $\pi^0 \rightarrow \gamma^*(Q_1^2)\gamma^*(Q_2^2)$, being determined by two diagrams of Fig. 7.12a and Fig. 7.12b, reads:

$$F_{\pi \rightarrow \gamma^* \gamma^*}(Q_1^2, Q_2^2) = \zeta_{\pi \rightarrow \gamma\gamma} \frac{\sqrt{N_c}}{16\pi^3} \int_0^1 \frac{dx}{x(1-x)^2} \int d^2 k_\perp \Psi_\pi(s) \\ \times \left(S_{\pi \rightarrow \gamma^* \gamma^*}(s, s'_1, Q_1^2) \frac{G_{\gamma^*}(s'_1)}{s'_1 + Q_2^2} + S_{\pi \rightarrow \gamma^* \gamma^*}(s, s'_2, Q_2^2) \frac{G_{\gamma^*}(s'_2)}{s'_2 + Q_1^2} \right), \quad (7.192)$$

where

$$s = \frac{m^2 + k_\perp^2}{x(1-x)}, \quad s'_i = \frac{m^2 + (\mathbf{k}_\perp - x\mathbf{Q}_i)^2}{x(1-x)}, \quad (i = 1, 2). \quad (7.193)$$

For pseudoscalar states the spin factor depends only on the quark mass:

$$S_{\pi \rightarrow \gamma^* \gamma^*}(s, s'_i, Q_i^2) = 4m. \quad (7.194)$$

The charge factor for the decay $\pi^0 \rightarrow \gamma\gamma$ is equal to

$$\zeta_{\pi \rightarrow \gamma\gamma} = \frac{e_u^2 - e_d^2}{\sqrt{2}} = \frac{1}{3\sqrt{2}}. \quad (7.195)$$

The factor $\sqrt{N_c}$ in the right-hand side of (7.192) appears owing to the definition of the colour wave function for the photon which differs from that for the pion: in the pion wave function there is a factor $1/\sqrt{N_c}$ while in the photon wave function this factor is absent.

In terms of the spectral integrals over the (s, s') variables, the transition form factor for $\pi^0 \rightarrow \gamma^*(Q_1^2)\gamma^*(Q_2^2)$ reads:

$$\begin{aligned} F_{\pi \rightarrow \gamma^*\gamma^*}(Q_1^2, Q_2^2) &= \zeta_{\pi \rightarrow \gamma\gamma} \frac{\sqrt{N_c}}{16} \int_{4m^2}^{\infty} \frac{ds}{\pi} \frac{ds'}{\pi} \Psi_{\pi}(s) \times \\ &\times \left[\frac{\Theta(s'sQ_1^2 - m^2\lambda(s, s', -Q_1^2))}{\sqrt{\lambda(s, s', -Q_1^2)}} S_{\pi \rightarrow \gamma^*\gamma^*}(s, s', Q_1^2) \frac{G_{\gamma^*}(s')}{s' + Q_2^2} \right. \\ &\left. + \frac{\Theta(s'sQ_2^2 - m^2\lambda(s, s', -Q_2^2))}{\sqrt{\lambda(s, s', -Q_2^2)}} S_{\pi \rightarrow \gamma^*\gamma^*}(s, s', Q_2^2) \frac{G_{\gamma^*}(s')}{s' + Q_1^2} \right], \quad (7.196) \end{aligned}$$

where $\lambda(s, s', -Q_i^2)$ is determined in (7.134).

Similar expressions may be written for the transitions $\eta, \eta' \rightarrow \gamma^*(Q_1^2)\gamma^*(Q_2^2)$. One should bear in mind that, because of the presence of two quarkonium components in the η, η' -mesons, their flavour wave functions are mixtures of the two components as follows: $\eta = \sin\theta n\bar{n} - \cos\theta s\bar{s}$ and $\eta' = \cos\theta n\bar{n} + \sin\theta s\bar{s}$. Therefore, the transition form factors have two components as well:

$$\begin{aligned} F_{\eta \rightarrow \gamma\gamma}(s) &= \sin\theta F_{\eta/\eta'(n\bar{n}) \rightarrow \gamma\gamma}(s) - \cos\theta F_{\eta/\eta'(s\bar{s}) \rightarrow \gamma\gamma}(s), \\ F_{\eta' \rightarrow \gamma\gamma}(s) &= \cos\theta F_{\eta/\eta'(n\bar{n}) \rightarrow \gamma\gamma}(s) + \sin\theta F_{\eta/\eta'(s\bar{s}) \rightarrow \gamma\gamma}(s). \quad (7.197) \end{aligned}$$

The spin factors for non-strange components of η and η' are the same as those for the pion, see (7.194); a different quark mass is entering the strange component:

$$S_{\eta/\eta'(n\bar{n}) \rightarrow \gamma^*\gamma^*}(s, s', Q^2) = 4m, \quad S_{\eta/\eta'(s\bar{s}) \rightarrow \gamma^*\gamma^*}(s, s', Q^2) = 4m_s. \quad (7.198)$$

Charge factors for the $n\bar{n}$ and $s\bar{s}$ components are:

$$\zeta_{\eta/\eta'(n\bar{n}) \rightarrow \gamma\gamma} = \frac{5}{9\sqrt{2}}, \quad \zeta_{\eta/\eta'(s\bar{s}) \rightarrow \gamma\gamma} = \frac{1}{9}. \quad (7.199)$$

In the calculation of transition form factors for pseudoscalar mesons, the wave functions related to non-strange quarks in η and η' are assumed to be the same as for the pion:

$$\Psi_{\eta/\eta'(n\bar{n})}(s) = \Psi_{\pi}(s) = C_{\pi} \left[\exp(-b_{\pi}^{(1)}s) + \delta_{\pi} \exp(-b_{\pi}^{(2)}s) \right], \quad (7.200)$$

with the following pion wave function parameters (see Appendix 6.A): $C_\pi = 209.36 \text{ GeV}^{-2}$, $\delta_\pi = 0.01381$, $b_\pi^{(1)} = 3.57 \text{ GeV}^{-2}$, $b_\pi^{(2)} = 0.4 \text{ GeV}^{-2}$.

As to the strange components of the wave functions, they may be different, but we suppose a similar shape for $n\bar{n}$ and $s\bar{s}$. We write:

$$\Psi_{\eta/\eta'(s\bar{s})}(s) = C_{\eta/\eta'(s\bar{s})} \left[\exp(-b_{\eta/\eta'(s\bar{s})}^{(1)} s) + \delta_{\eta/\eta'(s\bar{s})} \exp(-b_{\eta/\eta'(s\bar{s})}^{(2)} s) \right] \quad (7.201)$$

with $C_{\eta/\eta'(s\bar{s})} = 528.78 \text{ GeV}^{-2}$, $\delta_{\eta/\eta'(s\bar{s})} = \delta_\pi$, $b_{\eta/\eta'(s\bar{s})}^{(1)} = b_\pi^{(1)}$, $b_{\eta/\eta'(s\bar{s})}^{(2)} = b_\pi^{(2)}$. The change of the normalisation parameter, $C_{\eta/\eta'(s\bar{s})}$, is due to a larger value of the strange quark mass.

Equations (7.200), (7.201) express the use of the SU(6)-symmetry relations for the wave functions of the lightest pseudoscalar mesons.

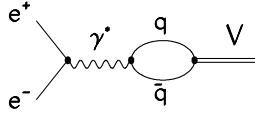


Fig. 7.15 Production of a vector $q\bar{q}$ state in the e^+e^- -annihilation.

7.5.2 e^+e^- -annihilation

The e^+e^- -annihilation processes provide us with additional information about the photon wave function:

(i) The partial width of the transitions $\omega, \rho^0, \phi \rightarrow e^+e^-$ is defined by the quark loop diagrams, which contain the product $G_{\gamma^*}(s)\Psi_V(s)$, where $\Psi_V(s)$ is the quark wave function of the vector meson ($V = \omega, \rho^0, \phi$). Supposing that the radial wave functions of ω, ρ^0, ϕ coincide with those of the lowest pseudoscalar mesons (this is a plausible assumption, for these mesons are members of the same lowest 36-plet), we can estimate $G_{\gamma^*}(s)$ and $G_{\gamma^*(s\bar{s})}(s)$ from the data on the $\omega, \rho^0, \phi \rightarrow e^+e^-$ decays.

(ii) The ratio $R(s) = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ below the open charm production ($\sqrt{s} \equiv E_{e^+e^-} < 3.7 \text{ GeV}$) is determined by hard components of the photon vertices $G_{\gamma^*}(s)$ and $G_{\gamma^*(s\bar{s})}(s)$ (transitions $\gamma^* \rightarrow u\bar{u}, d\bar{d}, s\bar{s}$), thus giving us the well-known quantity $R(s) = 2$ (small violations of $R(s) = 2$ come from corrections related to the gluon emission $\gamma^* \rightarrow q\bar{q}g$, see [49] and references therein). Hence, the deviation of the ratio from the value $R(s) = 2$ at decrease of $E_{e^+e^-}$ provides us with an information about the energies, when the hard components in $G_{\gamma^*}(s)$ and $G_{\gamma^*(s\bar{s})}(s)$ stop to work, while soft components start to play their role.

7.5.2.1 Partial decay widths $\omega, \rho^0, \phi \rightarrow e^+e^-$

Figure 7.15 is a diagrammatic representation of the reaction $V \rightarrow e^+e^-$: the virtual photon produces a $q\bar{q}$ pair, which turns into a vector meson.

The partial width of the vector meson is determined as follows:

$$m_V \Gamma_{V \rightarrow e^+e^-} = \pi \alpha^2 A_{e^+e^- \rightarrow V}^2 \frac{1}{m_V^4} \left(\frac{4}{3} m_V^2 + \frac{8}{3} m_e^2 \right) \sqrt{\frac{m_V^2 - 4m_e^2}{m_V^2}}, \quad (7.202)$$

where m_V is the vector meson mass, the factor $1/m_V^2$ is associated with the photon propagator, and $\alpha = e^2/(4\pi)$. In (7.202), the integration over the electron-positron phase space results in $\sqrt{(1 - 4m_e^2/m_V^2)}/(16\pi)$, while the averaging over vector meson polarisations and summing over electron-positron spins lead to $\text{Sp} \left[\gamma_\mu^\perp (\hat{k}_1 + m_e) \gamma_\mu'^\perp (-\hat{k}_2 + m_e) \right] = 4m_V^2 + 8m_e^2$. The amplitude $A_{V \rightarrow e^+e^-}$ is determined with the help of the quark-antiquark loop calculations, in the framework of the spectral integration technique. Thus we get for the decays $\omega, \rho^0 \rightarrow e^+e^-$:

$$\begin{aligned} A_{\omega, \rho^0 \rightarrow e^+e^-} &= Z_{\omega, \rho^0} \frac{\sqrt{N_c}}{16\pi} \int_{4m^2}^{\infty} \frac{ds}{\pi} G_{\gamma^*(q^2) \rightarrow q\bar{q}}(s) \Psi_{\omega, \rho}(s) \\ &\times \sqrt{\frac{s - 4m^2}{s}} \left(\frac{8}{3} m^2 + \frac{4}{3} s \right), \end{aligned} \quad (7.203)$$

where Z_{ω, ρ^0} is the quark charge factor for vector mesons: $Z_\omega = 1/(3\sqrt{2})$ and $Z_{\rho^0} = 1/\sqrt{2}$. We have a similar expression for the $\phi(1020) \rightarrow e^+e^-$ amplitude:

$$\begin{aligned} A_{\phi \rightarrow e^+e^-} &= Z_\phi \frac{\sqrt{N_c}}{16\pi} \int_{4m_s^2}^{\infty} \frac{ds}{\pi} G_{\gamma^*(q^2) \rightarrow s\bar{s}}(s) \Psi_\phi(s) \\ &\times \sqrt{\frac{s - 4m_s^2}{s}} \left(\frac{8}{3} m_s^2 + \frac{4}{3} s \right), \end{aligned} \quad (7.204)$$

with $Z_\phi = 1/3$.

In the loop diagram of Fig. 7.15 we use a normal vertex for the transition $\gamma^* \rightarrow q\bar{q}$ which results in a dominant $^3S_1 q\bar{q}$ state production in the intermediate state; the transition into $^3D_1 q\bar{q}$ -state is small, we neglect it. So, the normalisation condition for the vector meson wave functions has the form:

$$\frac{1}{16\pi} \int_{4m^2}^{\infty} \frac{ds}{\pi} \Psi_V^2(s) \sqrt{\frac{s - 4m^2}{s}} \left(\frac{8}{3} m^2 + \frac{4}{3} s \right) = 1. \quad (7.205)$$

Here, for ω , ρ and $\phi(1020)$ we use wave functions parametrised in the exponential form:

$$\begin{aligned}\Psi_V(s) &= C_V \exp(-b_V s) , \\ b_{\omega,\rho} &= 2.2 \text{ GeV}^{-2}, \quad C_{\omega,\rho} = 95.1 \text{ GeV}^{-2}, \\ b_\phi &= 2.5 \text{ GeV}^{-2}, \quad C_\phi(s\bar{s}) = 374.8 \text{ GeV}^{-2}.\end{aligned}\quad (7.206)$$

Within the used parametrisation the vector mesons are characterised by the following mean radii: $R_{\omega,\rho} = 3.2 (\text{GeV}/c)^{-1}$ and $R_\phi = 3.3 (\text{GeV}/c)^{-1}$. These values are in a qualitative agreement with those obtained in the spectral integral solution (see Chapter 8): $R_{\omega,\rho} \simeq 3.5 (\text{GeV}/c)^{-1}$ and $R_\phi \simeq 4.0 (\text{GeV}/c)^{-1}$.

7.5.2.2 The ratio $R(s) = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ at energies below the open charm production

At high energies but below the open charm production, $E_{e^+e^-} = \sqrt{s} < 3.7$ GeV, the ratio $R(s)$ is determined by the sum of quark charges squared in the transition $e^+e^- \rightarrow \gamma^* \rightarrow u\bar{u} + d\bar{d} + s\bar{s}$ multiplied by the factor $N_c = 3$:

$$R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = N_c(e_u^2 + e_d^2 + e_s^2) = 2. \quad (7.207)$$

We can introduce $R_v(s)$ as follows:

$$R_v(s) = 3(e_u^2 + e_d^2)G_\gamma^2(s) + 3e_s^2G_{\gamma(s\bar{s})}^2(s) = \frac{5}{3}G_\gamma^2(s) + \frac{1}{3}G_{\gamma(s\bar{s})}^2(s). \quad (7.208)$$

Since $G_\gamma(s)$ and $G_{\gamma(s\bar{s})}(s)$ are normalised as $G_\gamma(s) = G_{\gamma(s\bar{s})}(s) = 1$ at $s \rightarrow \infty$, we can relate $R(s)$ and $R_v(s)$ at large s .

$$R(s) \simeq R_v(s). \quad (7.209)$$

Following this equality, we determine the energy region where the hard components in $G_\gamma(s)$, $G_{\gamma(s\bar{s})}(s)$ begin to dominate.

7.5.3 Photon wave function

To determine the photon wave function, we use:

- (i) transition widths $\pi^0, \eta, \eta' \rightarrow \gamma\gamma^*(Q^2)$,
- (ii) partial decay widths $\omega, \rho^0, \phi \rightarrow e^+e^-, \mu^+\mu^-$,
- (iii) the ratio $R(s) = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$.

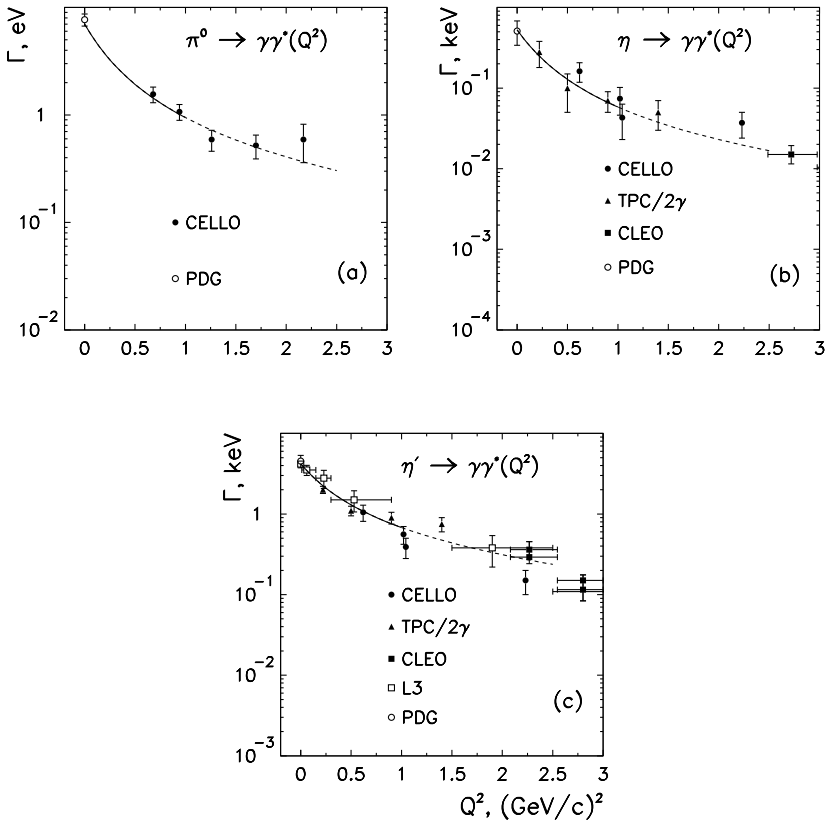


Fig. 7.16 Data for $\pi^0 \rightarrow \gamma\gamma^*$, $\eta \rightarrow \gamma\gamma^*$ and $\eta' \rightarrow \gamma\gamma^*$ vs the calculated curves (see also [48]).

Transition vertices for $u\bar{u}, d\bar{d} \rightarrow \gamma$ and $s\bar{s} \rightarrow \gamma$ have been chosen in the following form:

$$\begin{aligned}
 G_{\gamma \rightarrow q\bar{q}}(s) &= C_{\gamma} \left(e^{-b_{\gamma}^{(1)} s} + C_{\gamma}^{(2)} e^{-b_{\gamma}^{(2)} s} \right) + \frac{1}{1 + e^{-b_{\gamma}^{(0)}(s - s_{\gamma}^0)}}, \\
 G_{\gamma \rightarrow s\bar{s}}(s) &= C_{\gamma(s\bar{s})} e^{-b_{\gamma(s\bar{s})}^{(1)} s} + \frac{1}{1 + e^{-b_{\gamma(s\bar{s})}^{(0)}(s - s_{\gamma(s\bar{s})}^0)}}.
 \end{aligned} \tag{7.210}$$

Recall that the photon wave function is determined in (7.188).

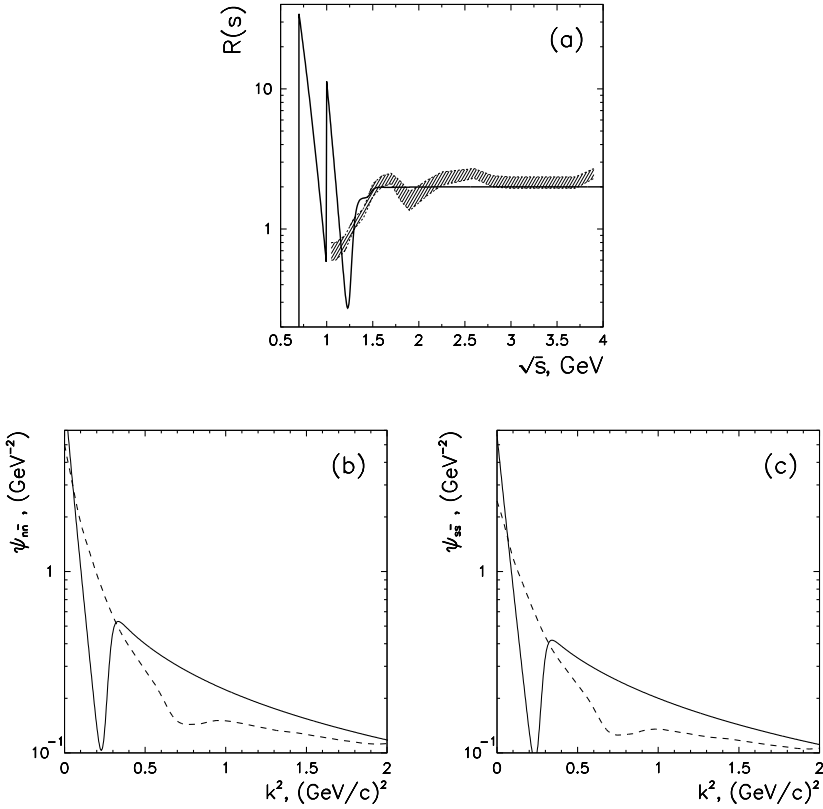


Fig. 7.17 a) $R_V(s)$ (solid line, Eq. (7.208)) vs $R(s) = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(\mu^+\mu^- \rightarrow \text{hadrons})$ (hatched area). b,c) The k^2 -dependence of photon wave functions (k^2 is relative quark momentum squared): we show $\Psi_{\gamma \rightarrow n\bar{n}}(4m^2 + 4k^2)$ and $\Psi_{\gamma \rightarrow s\bar{s}}(4m_s^2 + 4k^2)$. Solid curves stand for the wave functions determined by Eqs. (7.210) and (7.211), while the dashed lines for that found in the old fit [38].

Fitting to data [48], the following parameter values have been found:

$$\begin{aligned}
 u\bar{u}, d\bar{d}: \quad & C_\gamma = 32.577, \quad C_\gamma^{(2)} = -0.0187, \quad b_\gamma^{(1)} = 4 \text{ GeV}^{-2}, \quad b_\gamma^{(2)} = 0.8 \text{ GeV}^{-2}, \\
 & b_\gamma^{(0)} = 15 \text{ GeV}^{-2}, \quad s_\gamma^0 = 1.62 \text{ GeV}^2, \\
 s\bar{s}: \quad & C_{\gamma(s\bar{s})} = 310.55, \quad b_{\gamma(s\bar{s})}^{(1)} = 4 \text{ GeV}^{-2}, \quad b_{\gamma(s\bar{s})}^{(0)} = 15 \text{ GeV}^{-2}, \\
 & s_{\gamma(s\bar{s})}^0 = 2.15 \text{ GeV}^2.
 \end{aligned} \tag{7.211}$$

Let us present now the results of the fit in more details.

Figure 7.16 shows the data for $\pi^0 \rightarrow \gamma\gamma^*(Q^2)$ [6, 43], $\eta \rightarrow \gamma\gamma^*(Q^2)$ [6, 43, 44, 45] and $\eta' \rightarrow \gamma\gamma^*(Q^2)$ [6, 41, 43, 44, 45]. The fitting procedure is performed in the interval $0 \leq Q^2 \leq 1 \text{ (GeV/c)}^2$, the fitting curves are shown by solid lines. The continuation of the curves into the neighbouring region $1 \leq Q^2 \leq 2 \text{ (GeV/c)}^2$ (dashed lines) demonstrates that the description of the data is also reasonable there.

The calculation results for the $V \rightarrow e^+e^-$ decay partial widths *versus* the data [6] are given below (in keV):

$$\begin{aligned}
 \Gamma_{\rho^0 \rightarrow e^+e^-}^{\text{calc}} &= 7.50 \quad , & \Gamma_{\rho^0 \rightarrow e^+e^-}^{\text{exp}} &= 6.77 \pm 0.32 \quad , \\
 \Gamma_{\omega \rightarrow e^+e^-}^{\text{calc}} &= 0.796 \quad , & \Gamma_{\omega \rightarrow e^+e^-}^{\text{exp}} &= 0.60 \pm 0.02 \quad , \\
 \Gamma_{\phi \rightarrow e^+e^-}^{\text{calc}} &= 1.33 \quad , & \Gamma_{\phi \rightarrow e^+e^-}^{\text{exp}} &= 1.32 \pm 0.06 \quad , \\
 \Gamma_{\rho^0 \rightarrow \mu^+\mu^-}^{\text{calc}} &= 7.48 \quad , & \Gamma_{\rho^0 \rightarrow \mu^+\mu^-}^{\text{exp}} &= 6.91 \pm 0.42 \quad , \\
 \Gamma_{\phi \rightarrow \mu^+\mu^-}^{\text{calc}} &= 1.33 \quad , & \Gamma_{\phi \rightarrow \mu^+\mu^-}^{\text{exp}} &= 1.65 \pm 0.22 \quad . \quad (7.212)
 \end{aligned}$$

Figure 7.17a demonstrates the data for $R(s)$ [49] at $E_{e^+e^-} > 1 \text{ GeV}$ (dashed area) *versus* $R_v(s)$ with parameters (7.211) (solid line).

In Fig. 7.17b,c one can see the k^2 -dependence ($s = 4m^2 + 4k^2$) of the photon wave functions for the non-strange and strange components found in the latest fit [48] (solid line) and that found in [38] (dashed lines). One may see that in the region $0 \leq k^2 \leq 2.0 \text{ (GeV/c)}^2$, the fits in some points differ considerably, though in the average the old and new wave functions almost coincide. In the next section we compare the results obtained for the two-photon decays of scalar and tensor mesons, $S \rightarrow \gamma\gamma$ and $T \rightarrow \gamma\gamma$, calculated with old and new wave functions. This comparison shows that for physically defensible meson wave functions (when mean radii of the $q\bar{q}$ systems are of the order of $3 - 4 \text{ GeV}^{-1}$) the results of two calculations of the two-photon decay amplitudes lead to quite comparable values.

7.5.4 Transitions $S \rightarrow \gamma\gamma$ and $T \rightarrow \gamma\gamma$

As was mentioned above, the old [38] and new [48] photon wave functions are, in the average, close to each other, though they differ in details in the region $0 \leq k^2 \leq 2.0 \text{ (GeV/c)}^2$. Therefore, it would be useful to understand to what extent this difference influences the calculation results for the two-photon decays of scalar and tensor mesons (the corresponding formulae for transition amplitudes $S \rightarrow \gamma\gamma$ and $T \rightarrow \gamma\gamma$ are presented in Appendix 7.B).

The calculation of the two-photon decays of scalar mesons $f_0(980) \rightarrow \gamma\gamma$ and $a_0(980) \rightarrow \gamma\gamma$ have been performed in [20, 37] with the old wave

function, assuming that $f_0(980)$ and $a_0(980)$ are $q\bar{q}$ systems. The results for $a_0(980) \rightarrow \gamma\gamma$ are shown in Fig. 7.18 (dashed line). The solid curve shows the values found with the new photon wave function, Eqs. (7.210) and (7.211); for $a_0(980)$, the new wave function reveals a stronger dependence on the radius squared as compared to the old wave function. In the region $R_{a_0(980)}^2 \sim R_\pi^2 = 10 \text{ (GeV}/c)^{-2}$ the value $\Gamma_{a_0(980) \rightarrow \gamma\gamma}$ calculated with the new wave function becomes 1.5–2 times smaller than with the old one. We should stress, however, that neither of the definitions of the photon wave function contradicts the data: the error bars in the partial width $\Gamma_{a_0(980) \rightarrow \gamma\gamma}$ are rather large. A more precise definition of the photon wave function needs more precise measurements.

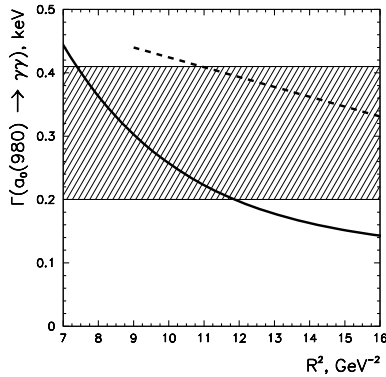


Fig. 7.18 Partial width $\Gamma_{a_0(980) \rightarrow \gamma\gamma}$ calculated under the assumption that $a_0(980)$ is a $q\bar{q}$ system, being a function of the radius squared of $a_0(980)$. The solid curve stands for the calculation with the new photon wave function, the dotted curve stands for the old one. The shaded area corresponds to the values allowed by the data [6].

For the flavour wave function of $f_0(980)$ we use here, as previously, the definition $n\bar{n} \cos \varphi + s\bar{s} \sin \varphi$. In Fig. 7.19, the calculated areas are shown for the region $\varphi < 0$. We see that the data agree with the calculated values at $-50^\circ \lesssim \varphi \lesssim -40^\circ$ in both versions.

The $f_0(980)$, being a $q\bar{q}$ system, is characterised by two parameters: the mean radius squared and the mixing angle φ . In Fig. 7.20 the areas allowed for these parameters are shown; they were obtained for the processes $f_0(980) \rightarrow \gamma\gamma$ and $\phi(1020) \rightarrow \gamma f_0(980)$ with both the old (Fig. 7.20a) and the new photon wave function (Fig. 7.20b). The change of the allowed areas $(R_{f_0(980)}^2, \varphi)$ for the reaction $f_0(980) \rightarrow \gamma\gamma$, though being noticeable, does not lead to drastic alterations of the parameter values.

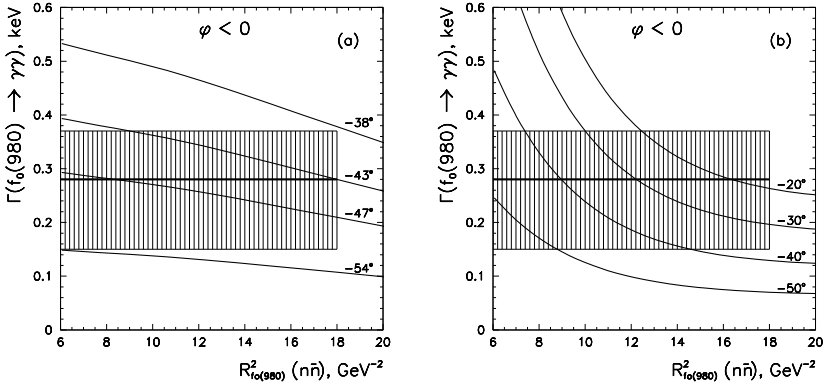


Fig. 7.19 Partial width $\Gamma_{f_0(980) \rightarrow \gamma\gamma}$ calculated under the assumption that $f_0(980)$ is a $q\bar{q}$ system, $q\bar{q} = n\bar{n} \cos \varphi + s\bar{s} \sin \varphi$, depending on the radius squared of the $q\bar{q}$ system: (a) with the old photon wave function, (b) with the new one. Calculations were carried out for different values of the mixing angle φ in the region $\varphi < 0$. The shaded area shows the allowed experimental values [6].

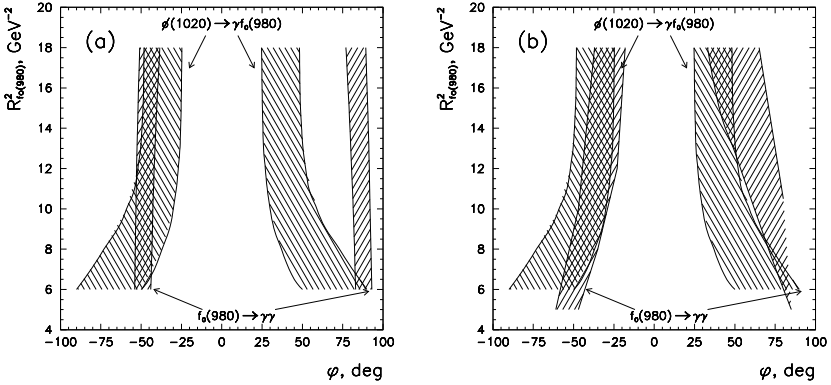


Fig. 7.20 Combined presentation of the $(R^2_{f_0(980)}, \varphi)$ areas allowed by the experiment for the decays $f_0(980) \rightarrow \gamma\gamma$ and $\phi(1020) \rightarrow \gamma f_0(980)$ with the old (a) and new (b) photon wave functions.

Another set of reactions calculated with the photon wave function is the two-photon decay of tensor mesons as follows: $a_2(1320) \rightarrow \gamma\gamma$, $f_2(1270) \rightarrow \gamma\gamma$ and $f_2(1525) \rightarrow \gamma\gamma$. The calculations of $a_2(1320) \rightarrow \gamma\gamma$ with old and new wave functions are shown in Fig. 7.21 (dotted and solid lines, respectively). Experimental data [6, 9] are presented also in

Fig. 7.21 (shaded areas). The data are described by form factors calculated at $R_{a_2(1320)}^2 \sim 8 \text{ (GeV}/c)^{-2}$: in this region the difference between the calculated values of the partial widths owing to the change of wave functions is of the order of 10–20%.

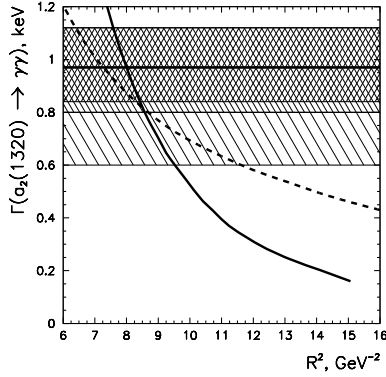


Fig. 7.21 Calculated curves *vs* experimental data (shaded areas) for $\Gamma_{a_2(1320) \rightarrow \gamma\gamma}$. The solid curve stands for the new photon wave function and the dotted line for the old one.

The amplitude of the transition $f_2 \rightarrow \gamma\gamma$ is determined by four form factors related to the existence of two flavour components and two spin structures (which correspond to different orbital momenta, $L = 1, 3$, see Appendix 7.B and [20, 37] for details). The calculations of these four form factors with old and new wave functions are shown in Fig. 7.22. We see that at $R_T^2 \sim 8\text{--}10 \text{ (GeV}/c)^{-2}$ the difference is not large, it is of the order of 10–20%. In Fig. 7.23, we show the allowed areas (R_T^2, φ_T) obtained in the description of experimental widths $\Gamma_{f_2(1270) \rightarrow \gamma\gamma}$ and $\Gamma_{f_2(1525) \rightarrow \gamma\gamma}$ [6] with old (Fig. 7.23a) and new (Fig. 7.23b) wave functions. The new photon wave function results in a more strict constraint for the areas (R_T^2, φ_T), though there is no qualitative change in the description of data.

The data give us two solutions for the (R_T^2, φ_T)-parameters:

$$(R_T^2, \varphi_T)^{\text{I}} \simeq (8 \text{ GeV}^{-2}, 0), \quad (R_T^2, \varphi_T)^{\text{II}} \simeq (8 \text{ GeV}^{-2}, 25^\circ). \quad (7.213)$$

The solution with $\varphi \simeq 0$, when $f_2(1270)$ is a nearly pure $n\bar{n}$ state and $f_2(1525)$ is an $s\bar{s}$ system, is more preferable from the point of view of the hadronic decays and the analysis [50].

Miniconclusion

Meson–photon transition form factors have been widely discussed in various approaches such as the perturbative QCD formalism [51, 52], QCD

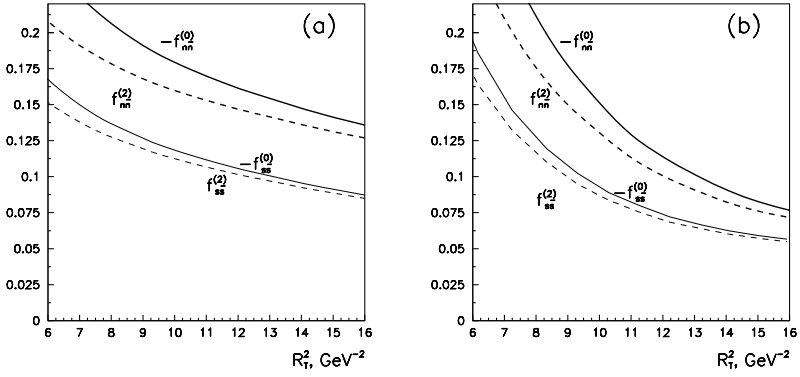


Fig. 7.22 Transition form factors in the decay of tensor quark-antiquark states $1^3P_2 n\bar{n} \rightarrow \gamma\gamma$ and $1^3P_2 s\bar{s} \rightarrow \gamma\gamma$ as functions of the radius squared of the $q\bar{q}$ system calculated with old (a) and new (b) photon wave functions.

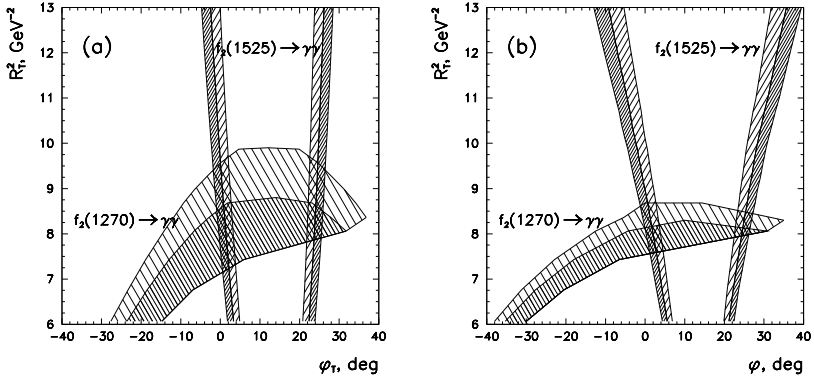


Fig. 7.23 Allowed areas $(R_{f_0(980)}^2, \varphi)$ for partial widths $\Gamma_{f_2(1270) \rightarrow \gamma\gamma}$ and $\Gamma_{f_2(1525) \rightarrow \gamma\gamma}$ calculated with old (a) and new (b) photon wave functions. The mixing angle φ_T defines the flavour content of mesons as follows: $f_2(1270) = n\bar{n} \cos \varphi_T + s\bar{s} \sin \varphi_T$ and $f_2(1525) = -n\bar{n} \sin \varphi_T + s\bar{s} \cos \varphi_T$.

sum rules [53, 54, 55], versions of the light-cone quark model [38, 56, 57, 58, 59, 60]. A distinctive feature of the quark model approach [38] consists in taking into account the soft interaction of quarks in the $\gamma \rightarrow q\bar{q}$ subprocess, that is, the account of the production of vector mesons in the intermediate state: $\gamma \rightarrow V \rightarrow q\bar{q}$.

We have reanalysed the quark components of the photon wave function

(the $\gamma^*(Q^2) \rightarrow u\bar{u}, d\bar{d}, s\bar{s}$ transitions) on the basis of data on the reactions $\pi^0, \eta, \eta' \rightarrow \gamma\gamma^*(Q^2)$, $e^+e^- \rightarrow \rho^0, \omega, \phi$ and $e^+e^- \rightarrow \text{hadrons}$. On a qualitative level, the obtained wave functions coincide with that defined before [20, 37, 38] by using the transitions $\pi^0, \eta, \eta' \rightarrow \gamma\gamma^*(Q^2)$ only. The data on the reactions $e^+e^- \rightarrow \rho^0, \omega, \phi$ and $e^+e^- \rightarrow \text{hadrons}$ allowed us to get the wave function structure more precisely, in particular, in the region of the relative quark momenta $k \sim 0.4 - 1.0 \text{ GeV}/c$. However, this fact does not lead to a cardinal change in the description of two-photon decays of the basic scalar and tensor mesons. Still, a more detailed definition of the photon wave function is important for the calculations of the decays of a loosely bound $q\bar{q}$ state such as a radial excitation state or reactions with virtual photons, $q\bar{q} \rightarrow \gamma^*(Q_1^2)\gamma^*(Q_2^2)$.

7.6 Nucleon Form Factors

We have already considered the relativistic description of the interaction of a composite system with an external field based on the spectral integral representation. We are now going to apply this technique to the calculation of nucleon form factors.

7.6.1 Quark–nucleon vertex

We start with constructing a four-fermion vertex, which describes the transition of three quarks into a hadron state with nucleon quantum numbers (that is, a non-strange spinor–isospinor state). Nucleons and quarks are described by the Dirac spinors with an additional isotopic index: $\tilde{N} \equiv (\tilde{p}, \tilde{n})$ for nucleons and $q \equiv (u, d)$ for non-strange quarks. Hereafter we omit the colour degrees of freedom, since the colour structure for all colourless qqq states is the same ($\epsilon_{abc}q^aq^bq^c$) and gives trivial contributions to all relevant expressions. The general form of the quark–nucleon vertex is [61]:

$$\begin{aligned} & \bar{N}q(1) \cdot \bar{q}_c(2)q(3) \cdot (f_1^s - f_1^\lambda - 3(f_2^\lambda + f_3^\lambda)) \\ & + \bar{N}\gamma^\mu q(1) \cdot \bar{q}_c(2)\gamma_\mu q(3) \cdot \left(\frac{1}{4}(f_1^s - f_2^s) + f_1^\lambda \right) \\ & + \frac{1}{2}\bar{N}\sigma^{\mu\nu}q(1) \cdot \bar{q}_c(2)\sigma_{\mu\nu}q(3) \cdot \left(\sqrt{3}f_2^\rho + f_1^a + \frac{2}{\sqrt{3}}f_4^\rho \right) \\ & + \bar{N}\gamma^5\gamma^\mu q(1) \cdot \bar{q}_c(2)\gamma_\mu\gamma^5 q(3) \cdot \left(\sqrt{3}f_3^\rho - \frac{3}{2}(f_1^a + f_2^a) + 2\sqrt{3}f_4^\rho \right) \end{aligned}$$

$$\begin{aligned}
& + \bar{N} \gamma^5 q(1) \cdot \bar{q}_c(2) \gamma^5 q(3) \cdot (f_2^s + f_1^\lambda - 3(f_2^\lambda - f_3^\lambda)) \\
& + \bar{N} \tau^a q(1) \cdot \bar{q}_c(2) \tau^a q(3) \cdot \left(\sqrt{3} f_2^\rho + \frac{1}{\sqrt{3}} (f_3^\rho - f_1^\rho) f_2^\lambda \right) \\
& + \bar{N} \tau^a \gamma^\mu q(1) \cdot \bar{q}_c(2) \tau^a \gamma_\mu q(3) \cdot \left(\frac{1}{\sqrt{3}} (2f_3^\rho + f_1^\rho) + \frac{1}{2} (f_1^a + f_2^a) \right) \\
& + \frac{1}{2} \bar{N} \tau^a \sigma^{\mu\nu} q(1) \cdot \bar{q}_c(2) \tau^a \sigma_{\mu\nu} q(3) \cdot \left(\frac{1}{6} (f_1^s + f_2^s) + f_2^\lambda - \frac{2}{3} f_4^\lambda \right) \\
& + \bar{N} \tau^a \gamma^5 \gamma^\mu q(1) \cdot \bar{q}_c(2) \tau^a \gamma_\mu \gamma^5 q(3) \cdot \left(\frac{1}{4} (f_1^s - f_2^s) + f_3^\lambda - 2f_4^\lambda \right) \\
& + \bar{N} \tau^a \gamma^5 q(1) \cdot \bar{q}_c(2) \tau^a \gamma^5 q(3) \cdot \left(\sqrt{3} f_2^\rho - \frac{1}{\sqrt{3}} (f_3^\rho - f_1^\rho) - (2f_1^a + f_2^a) \right) \\
& + \frac{4}{3m_q - M} \left(A^{(0)} \cdot \sqrt{3} f_4^\rho + A^{(1)} \cdot f_4^\lambda \right). \tag{7.214}
\end{aligned}$$

Recall that $\bar{q}_c = q^\top C \gamma_5 \tau_2$, where $C = i\gamma_0 \gamma_2$ is the charge conjugation matrix; τ_i are ordinary Pauli matrices operating in the isotopic space; M and m_q are masses of the nucleon and dressed quark, respectively, and

$$\begin{aligned}
A^{(0)} &= \bar{N} \gamma^5 \gamma^\mu q(1) \cdot \bar{q}_c(2) \gamma^5 q(3) \cdot (k_3 - k_2)_\mu \\
&+ \bar{N} \gamma^5 q(1) \cdot \bar{q}_c(2) \gamma^5 \gamma^\mu q(3) \cdot (P + k_1)_\mu, \\
A^{(1)} &= \bar{N} \tau^a \gamma^5 \gamma^\mu q(1) \cdot \bar{q}_c(2) \tau^a \gamma^5 q(3) \cdot (k_3 - k_2)_\mu \\
&+ \bar{N} \tau^a \gamma^5 q(1) \cdot \bar{q}_c(2) \tau^a \gamma^5 \gamma^\mu q(3) \cdot (P + k_1)_\mu; \tag{7.215}
\end{aligned}$$

$f_i^{(b)}$ ($b = s; \rho, \lambda; a$) in (7.214) are eight scalar functions with appropriate symmetry properties (s — symmetric, ρ, λ — mixed, and a — antisymmetric) with respect to permutations of the momenta k_2 and k_3 . Hereafter we use the standard notations ρ and λ for the mixed-type symmetry functions in the three-body system:

$$|\rho\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle); \quad |\lambda\rangle = \frac{1}{\sqrt{6}}(|2\rangle + |3\rangle - 2|1\rangle), \tag{7.216}$$

where the vector $|i\rangle$ characterises an isolated state of the particle i in the three-body system. The whole vertex (7.214) has to be symmetric with respect to all possible permutations of momentum, spin, and isospin quark variables (remember that we omitted all colour indices, which provide the required antisymmetry of the vertex for the complete set of quark variables). The functions $f_i^{(b)}$ depend on the relative momenta of the quarks $k_{ij}^2 = (k_i - k_j)^2$; in terms of these relative momenta we can single out the factors

responsible for the types of symmetries:

$$f_i^\rho = \frac{k_{13}^2 - k_{12}^2}{\sqrt{2}} \varphi_i, \quad f_i^\lambda = \frac{1}{\sqrt{6}} (k_{13}^2 + k_{12}^2 - 2k_{23}^2) \varphi_i, \quad (i = 1, 2, 3, 4)$$

$$f_i^a = (k_{12}^2 - k_{13}^2)(k_{13}^2 - k_{23}^2)(k_{23}^2 - k_{12}^2) \tilde{\varphi}_i, \quad (i = 1, 2), \quad (7.217)$$

where φ_i and $\tilde{\varphi}_i$ (and, of course, f_i^s) are completely symmetric functions under any permutation of the momenta k_1, k_2, k_3 .

Let us emphasise here that the vertex (7.214) describes not nucleons only, but also all (uud) states with the same quantum numbers. Different states correspond to different relative contributions of $f_i^{(b)}$ to the total vertex.

Nucleons are the lowest (qqq) state, and the relative quark momenta in the nucleon are rather small. We can expand $f_i^{(b)}$ with respect to relative quark momenta $k_{ij}^2 \equiv (k_i - k_j)^2$ and neglect all non-leading terms. In this case only the symmetric functions $f_1^s(s_{12}, s_{13}, s_{23})$ and $f_2^s(s_{12}, s_{13}, s_{23})$ (where $s_{ij} = k_{ij}^2$) survive. The vertex (7.214) in this approximation assumes the form

$$\begin{aligned} & \bar{N}q(1) \cdot \bar{q}_c(2)q(3) \cdot f_1^s + \bar{N}\gamma^\mu q(1) \cdot \bar{q}_c(2)\gamma_\mu q(3) \cdot \frac{1}{4}(f_1^s - f_2^s) \\ & + \bar{N}\gamma^5 q(1) \cdot \bar{q}_c(2)\gamma^5 q(3) \cdot f_2^s + \frac{1}{2}\bar{N}\tau^a \sigma^{\mu\nu} q(1) \cdot \bar{q}_c(2)\tau^a \sigma_{\mu\nu} q(3) \cdot \frac{1}{6}(f_1^s + f_2^s) \\ & + \bar{N}\tau^a \gamma^5 \gamma^\mu q(1) \cdot \bar{q}_c(2)\tau^a \gamma_\mu \gamma^5 q(3) \cdot \frac{1}{4}(f_1^s - f_2^s). \end{aligned} \quad (7.218)$$

The three first terms in (7.218) describe the nucleon state with the isoscalar (isospin $I = 0$) diquark q_2q_3 ; the remaining two terms correspond to the isovector ($I = 1$) diquark. The spin state of the diquark is determined by the γ -matrix structure of $\bar{q}_c(2)$ and $q(3)$ in (7.218).

The isoscalar diquark can have a total spin-parity $S^P = 0^+, 0^-, 1^-$ which means $\bar{q}_c(2)q(3) \rightarrow d(0^+)$, $\bar{q}_c(2)\gamma_\mu q(3) \rightarrow d(1^-)$, $\bar{q}_c(2)\gamma_5 q(3) \rightarrow d(0^-)$, while the isovector diquark can have $S^P = 1^+, 1^-$: $\bar{q}_c(2)\sigma_{\mu\nu} q(3) \rightarrow d(1^-)$, $d(1^+)$ and $\bar{q}_c(2)\gamma_\mu \gamma_5 q(3) \rightarrow d(1^-)$. The total angular momentum of the diquark q_2q_3 may be of arbitrary value, since we have to sum the spin structures $S = 0^+, 0^-, 1^-, 1^+$ with orbital momenta corresponding to the blocks $f_n^{(s)}(s_{12}, s_{23}, s_{13})$, which then lead to the total angular momentum of the diquark block $|\vec{S} + \vec{\ell}| = J_{23}$.

All $f_i^{(b)}$ in (7.214) can be related to the coordinate parts of the non-relativistic wave functions for three-fermion states with definite total spin S and orbital momentum L usually denoted as $|^{2S+1}L_b\rangle$, ($b = s, m, a$) (see *e.g.* [62]). Only eight colourless states with total angular momentum $1/2$

and total isospin 1/2 can be constructed:

$$\begin{aligned} &|{}^2S_s\rangle, |{}^2S_m\rangle, |{}^4D_m\rangle, |{}^2P_a\rangle, \\ &|{}^2S_a\rangle, |{}^2P_s\rangle, |{}^2P_m\rangle, |{}^4P_m\rangle. \end{aligned} \quad (7.219)$$

In terms of these states the leading non-relativistic terms in (7.218) assume the form

$$\begin{aligned} |f_1^s\rangle &= -i\frac{5}{\sqrt{2}}|{}^2S_s\rangle f_1 + \frac{i}{4m_q^2} \left(\frac{5}{\sqrt{6}}|{}^2S_s\rangle f_1 + \frac{4}{\sqrt{3}}|{}^2S_m\rangle f_1 \right. \\ &\quad \left. + 2\sqrt{\frac{5}{6}}|{}^4D_m\rangle f_1 + \sqrt{3}|{}^2P_a\rangle f_1 \right) + O\left(\frac{\mathbf{k}^4}{m_q^4}\right) \\ |f_2^s\rangle &= \frac{i}{\sqrt{2}}|{}^2S_s\rangle f_2 + \frac{i}{4m_q^2} \left(-\frac{1}{\sqrt{6}}|{}^2S_s\rangle f_2 + \frac{4}{\sqrt{3}}|{}^2S_m\rangle f_2 \right. \\ &\quad \left. - 10\sqrt{\frac{5}{6}}|{}^4D_m\rangle f_2 + 3\sqrt{3}|{}^2P_a\rangle f_2 \right) + O\left(\frac{\mathbf{k}^4}{m_q^4}\right). \end{aligned} \quad (7.220)$$

We can see from (7.220) that even the leading non-relativistic terms in the quark–nucleon vertex contain contributions corresponding to various types of symmetry of the spin–coordinate wave function, or, in terms of SU(6) multiplets, to multiplets other than the ground-state one $[56, 0^+]$. In other words, we should expect a certain configurational mixing in the nucleon wave function.

Such a configurational mixing is quite usual in potential models of three-fermion bound systems, which include spin–spin and spin–orbital pair interactions (see *e.g.* [62].) With a sufficient number of free parameters in such models, it is possible to reproduce the mass spectrum of the system and some other static features like magnetic moments *etc.* However, certain quantities which are determined by the structure details of the composite system (like structure functions and form factors) might be described inadequately. We faced such a situation when considering the radiative decays of vector mesons. Therefore, it is reasonable to choose a different way of investigation: we can try to determine the wave function of the composite system from the data on electromagnetic (or electroweak) interactions and then reconstruct the constituent interaction in such a way that both the mass spectrum and the internal structure of the composite system are adequately described. In a certain respect this task is similar to the well-known inverse scattering problem in physics of atoms and nuclei, when we try to reconstruct the effective potential from the data on the phase shift. The example considered below should be considered as a first step in this way

– we describe the nucleon form factors introducing phenomenological wave functions.

7.6.2 Nucleon form factor — relativistic description

In the lowest electromagnetic order the nucleon–photon vertex is described by the triangle diagram of Fig. 7.24 where, in the most general case, the *composite system–constituents* vertices can be written in the form (7.214) (or (7.218) in the leading non-relativistic approximation).

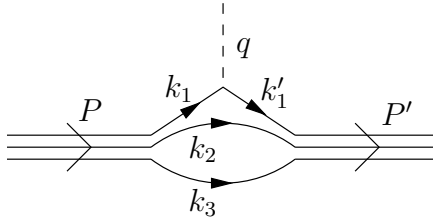


Fig. 7.24 The dispersion triangle diagram for the nucleon–photon interaction.

The nucleon matrix element of the electromagnetic current has the general form

$$\begin{aligned} \langle P' | J_\mu^{e.m.} | P \rangle &= e \bar{N}(P', M) \left[\frac{(P + P')_\mu}{2M} F_1^N(q^2) + \frac{i \sigma_{\mu\nu} q^\nu}{2M} F_2^N(q^2) \right] N(P, M) \\ &\equiv N(P') \Gamma_\mu(P', P | q) N(P). \end{aligned} \quad (7.221)$$

Here $N = (p, n)$ describes either a proton p , or a neutron n . The form factors F_1^N and F_2^N are related to the Sachs electric and magnetic form factors usually measured in the experiments by the relations

$$G_e(q^2) = \left(1 - \frac{q^2}{4M^2} \right) F_1^N(q^2) + \frac{q^2}{4M^2} F_2^N(q^2), \quad G_m(q^2) = F_2^N(q^2). \quad (7.222)$$

The triangle diagram in Fig. 7.24 can be calculated using the developed dispersion relation technique. In our case, when the constituents and the composite system are fermions, we have to single out, first, all the relevant spinor structures and take care about the proper choice of subtraction terms in the dispersion representation of the obtained scalar functions.

The double spectral integral for the nucleon form factors takes the form

[61]:

$$\begin{aligned}
G_e^{(I)}(q^2) &= \int \frac{ds}{\pi(s - M^2)} \frac{ds'}{\pi(s' - M^2)} \\
&\times \left[\left(1 - \frac{q^2}{2(s' + s)} \right) disc_s disc_{s'} F_1^{(I)}(s', s, q^2) \right. \\
&\quad \left. + \frac{q^2}{2(s' + s)} disc_s disc_{s'} F_2^{(I)}(s', s, q^2) \right], \\
disc_s disc_{s'} F_{1,2}^{(I)}(s', s, q^2) &= \sum_{i,j} f_i^{(I)}(s) f_j^{(I)}(s') disc_s disc_{s'} F_a^{ij}(s', s, q^2), \\
G_m^{(I)}(q^2) &= \int \frac{ds}{\pi(s - M_n^2)} \frac{ds'}{\pi(s' - M_n^2)} disc_s disc_{s'} F_2^{(I)}(s', s, q^2). \quad (7.223)
\end{aligned}$$

In the previous formulae we wrote wave functions and vertices for the proton and the neutron, here we write them for the isospin states of the diquark: the index $I = 0$ means that the diquark $q_2 q_3$ is an isoscalar, $I = 1$ stands for an isovector diquark. In other words, up to now we considered two states, the proton and the neutron; now the classification goes according to the two isospin states. Hence

$$G_{e,m}^p = 2G_{e,m}^{I=0}, \quad G_{e,m}^n = 3G_{e,m}^{I=1} - G_{e,m}^{I=0} \quad (7.224)$$

are proton and neutron Sachs form factors.

The detailed calculations of the double spectral densities in (7.223) and the final expressions for the form factors (which are rather cumbersome) can be found in [61]. Figure 7.25 (data are taken from [63]) illustrates the numerical results for the form factors obtained in [61] with the appropriate choice of two unknown functions in (7.220):

$$\begin{aligned}
f_i &= \frac{C_i}{s - 9m^2 + \Delta_i^2 \alpha_i}; \quad m = 0.42 \text{ GeV} \\
C_1 &= 1; \quad \alpha_1 = 2.5; \quad \Delta_1 = 0.7 \text{ GeV}^2 \\
C_2 &= 3.32; \quad \alpha_2 = 2.5; \quad \Delta_2 = 3 \text{ GeV}^2. \quad (7.225)
\end{aligned}$$

Let us underline once more that this calculation leads to a non-vanishing electric form factor of the neutron, which should be identically zero for the neutron state from the lowest $[56, 0^+]$ SU(6) multiplet owing to the complete symmetry of the coordinate part of the wave function. The calculations look more transparent in the non-relativistic description of the nucleon form factor considered in detail below.

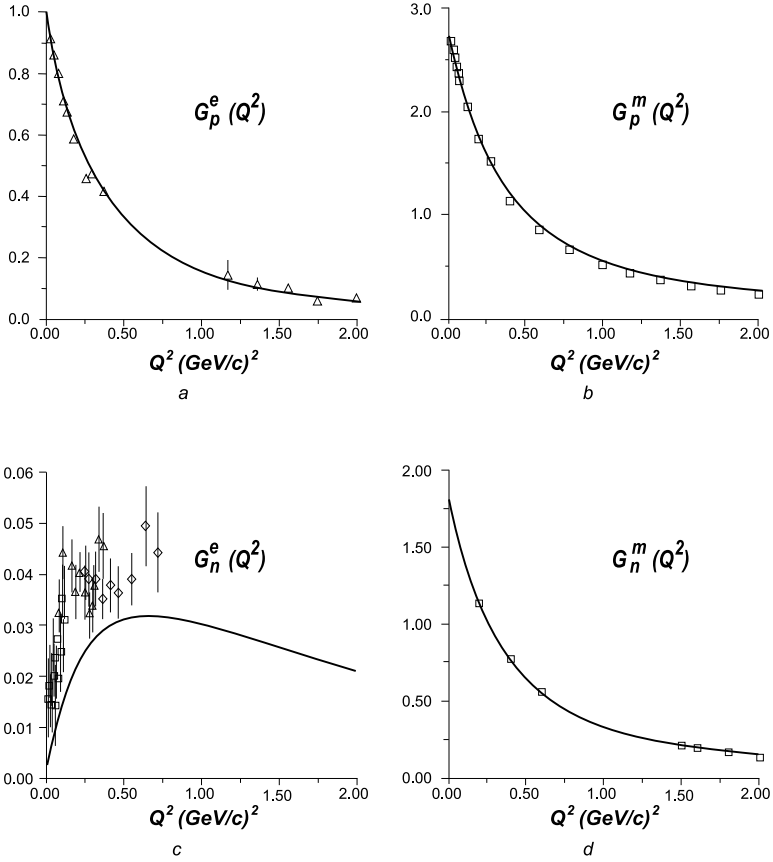


Fig. 7.25 The Sachs form factors of the proton (a, b) and the neutron (c, d).

7.6.3 Nucleon form factors — non-relativistic calculation

We understand now that the nucleon wave function is not a pure SU(6) multiplet state, but a mixture. This can be formulated using diquark states. Another possibility is to consider the mixing of various SU(6) multiplets. As we have seen, the relativistic expression for the quark–nucleon vertex (7.214) results in the configurational mixing in the nucleon wave function even in the lowest order with respect to the relative momentum of constituents (7.218). In the non-relativistic approach, we can arbitrarily insert any admixture of states belonging to higher SU(6) multiplets, for example, by introducing interaction breaking SU(6) symmetry in the constituent in-

teraction potential. The authors of [62] took into account the spin–spin interaction of quarks and obtained a nucleon wave function where the ground state $[56, 0^+]_{N=0}$ is mixed with the excitations $[56, 0^+]_{N=2}$ and $[70, 0^+]_{N=2}$.

The following example is aimed merely at illustrating the effect of the configurational mixing on nucleon (in particular, neutron) form factors. Therefore, we shall not specify the parameters of quark–quark interaction, but rather start with the nucleon wave function constructed as a mixture of two SU(6) multiplets, $[56, 0^+]$ and $[70, 0^+]$:

$$|N\rangle = \cos\phi|S_s\rangle + \sin\phi|S_m\rangle. \quad (7.226)$$

Here $|S_s\rangle$ describes the component with the completely symmetric coordinate part of the wave function, while $|S_m\rangle$ corresponds to the component with the mixed symmetry.

Remark that we do not adhere to any specific potential model here; therefore the subscript N , which enumerates the excitation level, is omitted from the wave function (7.226). Thus, states $|S_s\rangle$ and $|S_m\rangle$ should be understood as mixtures of various excited states with identical symmetry of wave functions rather than states from a certain SU(6) multiplet.

Using our usual notations $u^\uparrow, u^\downarrow$ etc., we can write an explicit expression for the wave function (7.226). For example, the state of the proton with spin projection $+1/2$ takes the form

$$\begin{aligned} |p^\uparrow\rangle = & u^\uparrow \left(\alpha \frac{u^\uparrow d^\downarrow + d^\downarrow u^\uparrow}{\sqrt{2}} + \beta \frac{u^\downarrow d^\uparrow + d^\uparrow u^\downarrow}{\sqrt{2}} + \gamma \frac{d^\downarrow u^\uparrow - u^\uparrow d^\downarrow}{\sqrt{2}} \right) \\ & + u^\downarrow \left(\gamma \frac{u^\uparrow d^\uparrow - d^\uparrow u^\uparrow}{\sqrt{2}} - (\alpha + \beta) \frac{u^\uparrow d^\downarrow + d^\downarrow u^\uparrow}{\sqrt{2}} \right) \\ & + d^\uparrow \left(\gamma \frac{u^\uparrow u^\downarrow - u^\downarrow u^\uparrow}{\sqrt{2}} - (\alpha + \beta) \frac{u^\uparrow u^\downarrow + u^\downarrow u^\uparrow}{\sqrt{2}} \right) \\ & + d^\downarrow \left(\sqrt{2}(\alpha + \beta) u^\uparrow u^\uparrow \right) \end{aligned} \quad (7.227)$$

where we assume that the first quark interacts. The coefficients α, β , and γ are built of the coordinate wave functions with appropriate symmetry properties with respect to permutations of the particles 2 and 3:

$$\alpha = \frac{1}{3} \cos\phi\Psi_s + \frac{1}{3\sqrt{2}} \sin\phi\Psi_\lambda; \quad \beta = -\frac{1}{3} \cos\phi\Psi_s - \frac{\sqrt{2}}{3} \sin\phi\Psi_\lambda; \quad \gamma = \frac{1}{\sqrt{6}} \Psi_\rho.$$

Introducing the relative momenta $\mathbf{p}_\rho = (\mathbf{k}_2 - \mathbf{k}_3)/\sqrt{2}$ and $\mathbf{p}_\lambda = (\mathbf{k}_2 + \mathbf{k}_3 - 2\mathbf{k}_1)\sqrt{6}$, we can single out the factors responsible for the symmetry properties from the functions Ψ_a (similarly to (7.217)):

$$\Psi_s = \Psi(\mathbf{p}_\rho^2 + \mathbf{p}_\lambda^2), \quad \Psi_\rho = (\mathbf{p}_\rho \mathbf{p}_\lambda) \Phi(\mathbf{p}_\rho^2 + \mathbf{p}_\lambda^2), \quad \Psi_\lambda = (\mathbf{p}_\lambda^2 - \mathbf{p}_\rho^2) \Phi(\mathbf{p}_\rho^2 + \mathbf{p}_\lambda^2).$$

In this case, by introducing $G_{ab}(Q^2) = \langle \Psi_a(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) | \Psi_b(\mathbf{k}_1 + \mathbf{q}, \mathbf{k}_2, \mathbf{k}_3) \rangle$ for $a, b = s, \rho, \lambda$ (here $Q^2 \equiv \mathbf{q}^2$), we write the electric form factors of the proton and the neutron as:

$$\begin{aligned} G_p^e(Q^2) &= \cos^2 \phi G_{ss}(Q^2) + \frac{1}{2} \sin^2 \phi (G_{\rho\rho}(Q^2) + G_{\lambda\lambda}(Q^2)) \\ &\quad + \frac{1}{\sqrt{2}} \sin(2\phi) G_{s\lambda}(Q^2), \\ G_n^e(Q^2) &= -\frac{1}{\sqrt{2}} \sin(2\phi) G_{s\lambda}(Q^2). \end{aligned} \quad (7.228)$$

The form factors $G_{ab}(Q^2)$ are represented by the triangle diagram of Fig. 7.24 with vertices determined by the corresponding parts of the wave function (7.226). We are going to calculate the form factor using the same spectral integration technique as in the relativistic case; therefore, it is convenient to express the functions Ψ_a in terms of invariant quantities, that is, to perform a “trivial relativisation” of the non-relativistic expression (7.226):

$$\begin{aligned} \Psi_s &= R_s(k_1, k_2, k_3) \Phi_s(s), & \Psi_\rho &= R_\rho(k_1, k_2, k_3) \Phi_m(s), \\ \Psi_\lambda &= R_\lambda(k_1, k_2, k_3) \Phi_m(s), \end{aligned} \quad (7.229)$$

where $s = (k_1 + k_2 + k_3)^2$, $k_{ij} = k_i - k_j$ and

$$R_s \equiv 1, \quad R_\rho = (k_{23}^2 - k_{13}^2)/\sqrt{2}, \quad R_\lambda = (k_{13}^2 + k_{23}^2 - 2k_{12}^2)/\sqrt{6}. \quad (7.230)$$

With this parametrisation, we arrive at the following double spectral integral for $G_{ab}(Q^2)$:

$$\begin{aligned} G_{ab}(Q^2) &= g_q(Q^2) \int ds ds' \Phi_a(s) \Phi_b(s') \Delta_{ab}(s, s', Q^2), \\ \Delta_{ab}(s, s', Q^2) &= \int dk_1 dk_2 dk_3 dk'_1 \delta(k_1^2 - m^2) \delta(k_2^2 - m^2) \\ &\quad \times \delta(k_3^2 - m^2) \delta(k_1'^2 - m^2) \delta(P - k_1 - k_2 - k_3) \delta(k'_1 - k_1 - q) R_a(k_1, k_2, k_3) \\ &\quad \times R_b(k'_1, k_2, k_3) Q^2 \frac{P^2 + P'^2 + Q^2 + 2(m^2 - (k_1 + k_2)^2)}{(P^2 - P'^2)^2 + 2Q^2(P^2 + P'^2) + Q^2}, \end{aligned} \quad (7.231)$$

where $P^2 = s$, $P'^2 = s' = (k'_1 + k_2 + k_3)^2$, $q = P - P'$, $q^2 = -Q^2$. We introduced also the quark form factor $g_q(Q^2)$ in the quark-photon vertex assuming the small but finite size of the constituent quark.

Similarly to the relativistic calculations (7.223), (7.224), the appropriate choice of two unknown functions $\Phi_s(s)$ and $\Phi_m(s)$ enabled the authors of [64] to describe proton and neutron form factors in agreement with the data (the non-relativistic curves for G_p^e and G_n^e are virtually the same as

those in Fig. 7.25, calculated in an explicitly relativistic formalism). The mixing parameter obtained in [64] is $\sin \phi = -0.45$; this corresponds to an approximately 20% admixture of the $[70, 0^+]$ state in the non-relativistic nucleon wave function.

7.7 Appendix 7.A: Pion Charge Form Factor and Pion $q\bar{q}$ Wave Function

Here, based on the data for pion charge form factor at $0 \leq Q^2 \leq 1$ (GeV/c)², we give the two-exponential parametrisation of the pion $q\bar{q}$ wave function.

First, recall the formulae we use. The structure of the amplitude of pion-photon interaction is as follows:

$$A_\mu^{(\pi)} = e(p_\mu + p'_\mu)F_\pi(Q^2), \quad (7.232)$$

where e is the absolute value of the electron charge, p and p' are the pion incoming-outgoing momenta. We are working in the space-like region of the momentum transfer, so $Q^2 = -q^2$, where $q = p - p'$. The amplitude $A_\mu^{(\pi)}$ is the transverse one: $q_\mu A_\mu^{(\pi)} = 0$.

The pion form factor in the additive quark model is defined as a process shown in Fig. 7.11a: the photon interacts with one of the constituent quarks. In the spectral integration technique, the method of calculation of the diagram of Fig. 7.11a is as follows: we consider the spectral integrals over masses of incoming and outgoing $q\bar{q}$ states, corresponding cuttings of the triangle diagram are shown in Fig. 7.11b. In this way we calculate the double discontinuity of the triangle diagram, $disc_s disc_{s'} F_\pi(s, s', Q^2)$, where s and s' are the energies squared of the $q\bar{q}$ systems before and after the photon emission, $P^2 = s$ and $P'^2 = s'$ (in the dispersion relation technique the momenta of intermediate particles do not coincide with the external momenta, $p \neq P$ and $p' \neq P'$). The double discontinuity is defined by three factors:

- (i) the product of the pion vertex functions and the quark charge: $e_q G_\pi(s) G_\pi(s')$ where, due to (7.232), e_q is given in the units of the charge e ,
- (ii) the phase space of the triangle diagram (Fig. 7.11b) at $s \geq 4m^2$ and $s' \geq 4m^2$: $d\Phi_{tr} = d\Phi_2(P; k_1, k_2) d\Phi_2(P'; k'_1, k'_2) (2\pi)^3 2k_{20} \delta^{(3)}(\mathbf{k}_2 - \mathbf{k}'_2)$,
- (iii) the spin factor $S_\pi(s, s', Q^2)$ determined by the trace of the triangle diagram process of Fig. 7.11b:

$$-\text{Sp} \left[i\gamma_5 (m - \hat{k}_2) i\gamma_5 (m + \hat{k}'_1) \gamma_\mu^{\perp q} (m + \hat{k}_1) \right] = (P + P')_\mu^{\perp q} S_\pi(s, s', Q^2). \quad (7.233)$$

The spin factor $S_\pi(s, s', Q^2)$ reads:

$$\begin{aligned} S_\pi(s, s', Q^2) &= 2 [(s + s' + Q^2) \alpha(s, s', Q^2) - Q^2] , \\ \alpha(s, s', Q^2) &= \frac{s + s' + Q^2}{2(s + s') + (s' - s)^2/Q^2 + Q^2} . \end{aligned} \quad (7.234)$$

As a result, the double discontinuity of the diagram with a photon emitted by quark is determined as:

$$disc_s disc_{s'} F_\pi(s, s', Q^2) = G_\pi(s) G_\pi(s') S_\pi(s, s', Q^2) d\Phi_{tr} . \quad (7.235)$$

Here we take into account that the total charge factor for the π^+ is unity, $e_u + e_{\bar{d}} = 1$. The form factor $F_\pi(Q^2)$ is defined as a double dispersion integral as follows:

$$\begin{aligned} F_\pi(Q^2) &= \int_{4m^2}^{\infty} \frac{ds}{\pi} \frac{ds'}{\pi} \frac{disc_s disc_{s'} F_\pi(s, s', Q^2)}{(s' - m_\pi^2)(s - m_\pi^2)} \\ &= \int_{4m^2}^{\infty} \frac{ds}{\pi} \frac{ds'}{\pi} \Psi_\pi(s) \Psi_\pi(s') S_\pi(s, s', Q^2) \frac{\Theta(s' s Q^2 - m^2 \lambda(s, s', -Q^2))}{16\sqrt{\lambda(s, s', -Q^2)}} . \end{aligned} \quad (7.236)$$

Remind that the presented spectral integral for the form factor appears after the integration in (7.237) over the momenta of constituents by removing the δ -functions in the phase space $d\Phi_{tr}$; we have $\lambda(s, s', -Q^2) = (s' - s)^2 + 2Q^2(s' + s) + Q^4$, while $\Theta(X) = 1$ at $X \geq 0$ and $\Theta(X) = 0$ at $X < 0$. The pion wave function is defined as follows:

$$\Psi_\pi(s) = \frac{G_\pi(s)}{s - m_\pi^2} . \quad (7.237)$$

In accordance with different goals where the $q\bar{q}$ system is involved, there are different ways to work with formula (7.237). Another way to present the form factor is to remove the integration over the energy squared of the quark–antiquark systems, s and s' , by using δ -functions entering $d\Phi_{tr}$. Then we have the formula for the pion form factor in light-cone variables:

$$\begin{aligned} F_\pi(Q^2) &= \frac{1}{16\pi^3} \int_0^1 \frac{dx}{x(1-x)^2} \int d^2 k_\perp \Psi_\pi(s) \Psi_\pi(s') S_\pi(s, s', Q^2) , \\ s &= \frac{m^2 + k_\perp^2}{x(1-x)} , \quad s' = \frac{m^2 + (\mathbf{k}_\perp - x\mathbf{Q})^2}{x(1-x)} , \end{aligned} \quad (7.238)$$

where $\mathbf{k}_\perp$ and x are the light-cone quark characteristics (the transverse momentum of the quark and a part of momentum along the z -axis).

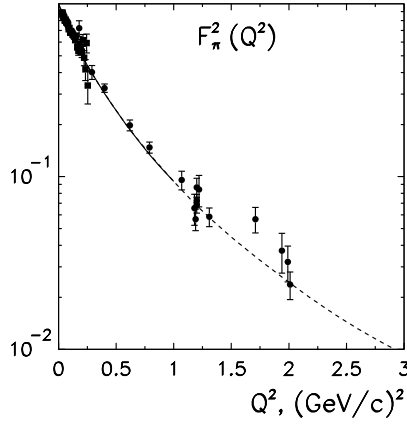


Fig. 7.26 Description of the experimental data on the pion charge form factor with the pion wave function given by (7.240).

Fitting the formula for the pion form factor to the data at $0 \leq Q^2 \leq 1$ $(\text{GeV}/c)^2$ with a two-exponential parametrisation of the wave function Ψ_π :

$$\Psi_\pi(s) = c_\pi [\exp(-b_1^\pi s) + \delta_\pi \exp(-b_2^\pi s)] , \quad (7.239)$$

we obtain the following values for the pion wave function parameters:

$$\begin{aligned} c_\pi &= 209.36 \text{ GeV}^{-2}, & \delta_\pi &= 0.01381, \\ b_1^\pi &= 3.57 \text{ GeV}^{-2}, & b_2^\pi &= 0.4 \text{ GeV}^{-2} . \end{aligned} \quad (7.240)$$

Figure 7.26 demonstrates the description of the data by the formula (7.237) (or (7.238)) with the pion wave function given by (7.239), (7.240).

The region $1 \leq Q^2 \leq 2$ $(\text{GeV}/c)^2$ was not used for the determination of parameters of the pion wave function: one could suppose that at $Q^2 \geq 1$ $(\text{GeV}/c)^2$ the predictions of the additive quark model fail. However, we see that the calculated curve fits the data reasonably in the neighbouring region $1 \leq Q^2 \leq 2$ $(\text{GeV}/c)^2$ too (dashed curve in Fig. 7.26).

The constraint $F_\pi(0) = 1$ serves us as a normalisation condition for the pion wave function. In the low- Q^2 region we have: $F_\pi(Q^2) \simeq 1 - \frac{1}{6} R_\pi^2 Q^2$ with $R_\pi^2 \simeq 10$ $(\text{GeV}/c)^{-2}$. The pion radius is just the characteristics which will be used later on for comparative estimates of the wave function parameters for other low-lying $q\bar{q}$ states.

7.8 Appendix 7.B: Two-Photon Decay of Scalar and Tensor Mesons

The transition form factors $q\bar{q}$ -meson $\rightarrow \gamma^*(q_1^2)\gamma^*(q_2^2)$ in the region of moderately small $q_i^2 \equiv -Q_i^2$ are determined by the quark loop diagrams of Figs. 7.12a, b which are convolutions of the $q\bar{q}$ -meson and photon wave functions, $\Psi_{q\bar{q}\text{-meson}} \otimes \Psi_{\gamma^*(q_i^2) \rightarrow q\bar{q}}$. The calculation of the processes of Fig. 7.12a, b, being performed in terms of the double spectral representation, gives valuable information about wave function of the studied $q\bar{q}$ -meson.

7.8.1 Decay of scalar mesons

We present here the formulae for the decay of scalar mesons $a_0 \rightarrow \gamma\gamma$ and $f_0 \rightarrow \gamma\gamma$. In their main points, the formulae for $f_0 \rightarrow \gamma\gamma$ coincide with those for $a_0 \rightarrow \gamma\gamma$.

The amplitude for the two-photon decay of the scalar meson has the following structure:

$$A_{\mu\nu}^{S \rightarrow \gamma\gamma} = e^2 g_{\mu\nu}^{\perp\perp} F_{S \rightarrow \gamma\gamma}(0, 0). \quad (7.241)$$

Here $e^2/4\pi = \alpha = 1/137$ and $F_{S \rightarrow \gamma\gamma}(0, 0)$ is the form factor for the transition $S \rightarrow \gamma(Q_1^2)\gamma(Q_2^2)$ at $Q_1^2 \rightarrow 0$ and $Q_2^2 \rightarrow 0$.

The partial width, $\Gamma_{S \rightarrow \gamma\gamma}$, is determined as

$$m_S \Gamma_{S \rightarrow \gamma\gamma} = \frac{1}{2} \int d\Phi_2(p_S; q_1, q_2) \sum_{\mu\nu} |A_{\mu\nu}|^2 = \pi \alpha^2 |F_{S \rightarrow \gamma\gamma}(0, 0)|^2. \quad (7.242)$$

The summation is carried out over the outgoing photon polarisations; the photon identity factor, $\frac{1}{2}$, is written explicitly.

In terms of the spectral integrals over the (s, s') variables, the transition form factor for the decay $S \rightarrow \gamma^*(Q_1^2)\gamma^*(Q_2^2)$ in the additive quark model (see Fig. 7.12a, b) reads:

$$\begin{aligned} F_{S \rightarrow \gamma^*\gamma^*}(Q_1^2, Q_2^2) &= \zeta_{S \rightarrow \gamma\gamma} \frac{\sqrt{N_c}}{16} \int_{4m^2}^{\infty} \frac{ds}{\pi} \frac{ds'}{\pi} \Psi_S(s) \\ &\times \left[\frac{\Theta(s'sQ_1^2 - m^2\lambda(s, s', -Q_1^2))}{\sqrt{\lambda(s, s', -Q_1^2)}} S_{S \rightarrow \gamma^*\gamma^*}(s, s', -Q_1^2) \frac{G_{\gamma^*}(s')}{s' + Q_2^2} \right. \\ &\left. + \frac{\Theta(s'sQ_2^2 - m^2\lambda(s, s', -Q_2^2))}{\sqrt{\lambda(s, s', -Q_2^2)}} S_{S \rightarrow \gamma^*\gamma^*}(s, s', -Q_2^2) \frac{G_{\gamma^*}(s')}{s' + Q_1^2} \right], \quad (7.243) \end{aligned}$$

where $\lambda(s, s', -Q_i^2)$ is determined in (7.134), the charge factors for isovector and isoscalar mesons are equal to:

$$I = 1 : \quad \zeta_{a_0^0 \rightarrow \gamma\gamma} = \frac{e_u^2 - e_d^2}{\sqrt{2}} = \frac{1}{3\sqrt{2}}, \quad (7.244)$$

$$I = 0 : \quad \zeta_{f_0(n\bar{n}) \rightarrow \gamma\gamma} = \frac{e_u^2 + e_d^2}{\sqrt{2}} = \frac{5}{9\sqrt{2}}, \quad \zeta_{f_0(s\bar{s}) \rightarrow \gamma\gamma} = e_s^2 = \frac{1}{9},$$

and the spin factor looks as follows:

$$S_{S \rightarrow \gamma^* \gamma^*}(s, s', q^2) = -2m \left[4m^2 - s + s' + q^2 - \frac{4ss'q^2}{2(s+s')q^2 - (s-s')^2 - q^4} \right]. \quad (7.245)$$

Remind that for the transversely polarised photons the spin structure factor is fixed by the quark loop trace:

$$\text{Sp}[\gamma_\nu^{\perp\perp}(\hat{k}'_1 + m)\gamma_\mu^{\perp\perp}(\hat{k}_1 + m)(\hat{k}_2 - m)] = S_{S \rightarrow \gamma^* \gamma^*}(s, s', q^2) g_{\mu\nu}^{\perp\perp}, \quad (7.246)$$

where $\gamma_\nu^{\perp\perp}$ and $\gamma_\mu^{\perp\perp}$ stand for photon vertices, and $\gamma_\mu^{\perp\perp} = g_{\mu\beta}^{\perp\perp} \gamma_\beta$.

Standard calculations of form factor in the limit $Q_1^2, Q_2^2 \rightarrow 0$ result in:

$$F_{S \rightarrow \gamma\gamma}(0, 0) = Z_{S \rightarrow \gamma\gamma} \sqrt{N_c} m^2 \int_{4m^2}^{\infty} \frac{ds}{4\pi^2} \Psi_S(s) \Psi_{\gamma \rightarrow q\bar{q}}(s) \times \left(\sqrt{s(s-4m^2)} - 2m^2 \ln \frac{\sqrt{s} + \sqrt{s-4m^2}}{\sqrt{s} - \sqrt{s-4m^2}} \right), \quad (7.247)$$

where $Z_{S \rightarrow \gamma\gamma} = 2\zeta_{S \rightarrow \gamma\gamma}$; normalisation of $\Psi_S(s)$ is given by (7.155).

7.8.2 Tensor-meson decay amplitudes for the process $q\bar{q} (2^{++}) \rightarrow \gamma\gamma$

We present here formulae for the amplitudes of the radiative decays of the $q\bar{q}$ tensor mesons with dominant $n^3P_2 q\bar{q}$ and $n^3F_2 q\bar{q}$ states. The corresponding vertices, $\hat{G}_{\mu_1\mu_2}^{(S,L,J)}$, are determined in (7.166). Calculations of amplitudes for transitions $T(L) \rightarrow \gamma\gamma$ are performed in a quite analogous way as for pseudoscalar and scalar mesons.

(i) Spin-momentum structure of the decay amplitude.

The decay amplitude for the process $q\bar{q} (2^{++}) \rightarrow \gamma\gamma$ has the following structure:

$$A_{\mu\nu, \alpha\beta}^{(T \rightarrow \gamma\gamma)} = e^2 \left[S_{\mu\nu, \alpha\beta}^{(0)}(p, q) F_{T \rightarrow \gamma\gamma}^{(0)}(0, 0) + S_{\mu\nu, \alpha\beta}^{(2)}(p, q) F_{T \rightarrow \gamma\gamma}^{(2)}(0, 0) \right], \quad (7.248)$$

where, as usually, $e^2/4\pi = \alpha = 1/137$. Here $S_{\mu\nu,\alpha\beta}^{(0)}$ and $S_{\mu\nu,\alpha\beta}^{(2)}$ are the moment operators for helicities $H = 0, 2$; the indices α, β refer to photons and μ, ν to the tensor meson. The transition form factors for photons with the transverse polarisation $T \rightarrow \gamma_\perp(q_1^2)\gamma_\perp(q_2^2)$: $F_{T \rightarrow \gamma\gamma}^{(0)}(q_1^2, q_2^2)$ and $F_{T \rightarrow \gamma\gamma}^{(2)}(q_1^2, q_2^2)$, depend on the photon momenta squared q_1^2 and q_2^2 ; recall that the two-photon decay corresponds to the limiting values $q_1^2 = 0$ and $q_2^2 = 0$.

The moment operators for real photons with the notations $p = q_1 + q_2$ and $q = (q_1 - q_2)/2$ have the form:

$$\begin{aligned} S_{\mu\nu,\alpha\beta}^{(0)}(p, q) &= g_{\alpha\beta}^{\perp\perp} \left(\frac{q_\mu q_\nu}{q^2} - \frac{1}{3} g_{\mu\nu}^{\perp\perp} \right) \\ S_{\mu\nu,\alpha\beta}^{(2)}(p, q) &= g_{\mu\alpha}^{\perp\perp} g_{\nu\beta}^{\perp\perp} + g_{\mu\beta}^{\perp\perp} g_{\nu\alpha}^{\perp\perp} - g_{\mu\nu}^{\perp\perp} g_{\alpha\beta}^{\perp\perp}, \end{aligned} \quad (7.249)$$

where the metric tensors $g_{\mu\nu}^{\perp\perp}$ and $g_{\alpha\beta}^{\perp\perp}$ are determined in a standard way: $g_{\mu\nu}^{\perp\perp} = g_{\mu\nu} - p_\mu p_\nu / p^2$ and $g_{\alpha\beta}^{\perp\perp} = g_{\alpha\beta} - q_\alpha q_\beta / q^2 - p_\alpha p_\beta / p^2$. The moment operators are orthogonal to each other in the space of photon polarisations:

$$S_{\mu\nu,\alpha\beta}^{(0)} S_{\mu'\nu',\alpha\beta}^{(2)} = 0. \quad (7.250)$$

(ii) Partial width for the decay $T \rightarrow \gamma\gamma$.

The partial width for the decay process $T \rightarrow \gamma\gamma$ is defined by two transition amplitudes with the helicities $H = 0, 2$:

$$\Gamma(T \rightarrow \gamma\gamma) = \frac{4}{5} \frac{\pi \alpha^2}{m_T} \left[\frac{1}{6} \left| \sum_{l=1,3} F_{T(L) \rightarrow \gamma\gamma}^{(0)} \right|^2 + \left| \sum_{l=1,3} F_{T(L) \rightarrow \gamma\gamma}^{(2)} \right|^2 \right]. \quad (7.251)$$

Here we have taken into account that the considered tensor meson can be a mixture of the quark–antiquark states with $L = 1$ and $L = 3$, so we write: $F_{T \rightarrow \gamma\gamma}^{(H)} = F_{T(1) \rightarrow \gamma\gamma}^{(H)} + F_{T(3) \rightarrow \gamma\gamma}^{(H)}$.

(iii) Form factors for $T \rightarrow \gamma\gamma$.

The form factor with fixed L and H reads:

$$F_{T(L) \rightarrow \gamma\gamma}^{(H)} = Z_{T \rightarrow q\bar{q}} \sqrt{N_c} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{T(L)}(s) \Psi_{\gamma \rightarrow q\bar{q}}(s) S_{T(L) \rightarrow \gamma\gamma}^{(H)}(s). \quad (7.252)$$

The charge factor $Z_{T \rightarrow q\bar{q}} = 2\zeta_{T \rightarrow q\bar{q}}$ depends on the isospin of the decaying meson only, see (7.244). The spin factors for the triangle diagrams (the additive quark model) are calculated for vertices (7.166) in a standard way.

For $H = 0$ they are as follows:

$$\begin{aligned}
 S_{T(1) \rightarrow \gamma\gamma}^{(0)}(s) &= -\frac{4}{\sqrt{3}} \sqrt{s(s-4m^2)} (12m^2 + s) \\
 &\quad + \frac{8m^2}{\sqrt{3}} (4m^2 + 3s) \ln \frac{s + \sqrt{s(s-4m^2)}}{s - \sqrt{s(s-4m^2)}}, \\
 S_{T(3) \rightarrow \gamma\gamma}^{(0)}(s) &= -\frac{2\sqrt{2s(s-4m^2)}}{5} (72m^4 + 8m^2s + s^2) \\
 &\quad + \frac{12\sqrt{2}}{5} m^2 (8m^4 + 4m^2s + s^2) \ln \frac{s + \sqrt{s(s-4m^2)}}{s - \sqrt{s(s-4m^2)}}, \quad (7.253)
 \end{aligned}$$

and for $H = 2$:

$$\begin{aligned}
 S_{T(1) \rightarrow \gamma\gamma}^{(2)}(s) &= \frac{8\sqrt{s(s-4m^2)}}{3\sqrt{3}} (5m^2 + s) \\
 &\quad - \frac{8m^2}{\sqrt{3}} (2m^2 + s) \ln \frac{s + \sqrt{s(s-4m^2)}}{s - \sqrt{s(s-4m^2)}}, \\
 S_{T(3) \rightarrow \gamma\gamma}^{(2)}(s) &= \frac{2\sqrt{2s(s-4m^2)}}{15} (30m^4 - 4m^2s + s^2) \\
 &\quad - \frac{2\sqrt{2}}{5} m^2 (12m^4 - 2m^2s + s^2) \ln \frac{s + \sqrt{s(s-4m^2)}}{s - \sqrt{s(s-4m^2)}}. \quad (7.254)
 \end{aligned}$$

The normalisation of $\psi_{T(L)}(s)$ is determined by (7.173).

7.9 Appendix 7.C: Comments about Efficiency of QCD Sum Rules

Various versions of QCD sum rules [65] have been extensively applied to the calculation of hadron parameters, such as masses, leptonic constants, form factors, etc. The extraction of a ground-state parameter within the method of sum rules consists of the two following steps (i) the construction of the OPE for a relevant correlator of quark currents in QCD and (ii) the application of certain cutting procedures to extract the parameters of the individual hadron state from the OPE series which involves the contribution of infinitely many states.

The main emphasis of the initial papers on QCD sum rules was the demonstration of the sensitivity of the Borel-transformed OPE series to the parameters of the ground state. Then, the manipulations with the OPE allow one to obtain numerical estimates for ground-state hadron parameters with an expected accuracy of 20-30%. However, in later applications of

the method the emphasis was shifted to the attempts to obtain hadron parameters with a better and controlled accuracy. Specific criteria have been worked out and it was believed that these criteria in fact allow one to extract hadron parameters and to obtain error estimates for the extracted values. Unfortunately, the efficiency of these procedures was neither proven nor tested in models where the exact solution is known.

Recently, a systematic study of the accuracy of different versions of QCD sum rules for hadron observables was performed in [66, 67, 68, 69, 70, 71, 72]. In these papers (a) Shifman–Vainshtein–Zakharov (SVZ) sum rules for leptonic constants and (b) light-cone sum rules for heavy-to-light weak transition form factors were analysed.

In [66, 67, 68, 69] the systematic errors of the ground-state parameters obtained by SVZ sum rules from two-point correlators were studied. The harmonic-oscillator potential model was used as an example: in this case the exact solution for the polarisation operator is known, which allows one to obtain both the OPE to any order and the parameters (masses and leptonic constants) of the bound states. The parameters of the ground state were extracted by applying the standard procedures adopted in the method of QCD sum rules, and the obtained results were compared with their known exact values. It was shown that the knowledge of the correlator in a limited range of the Borel parameter with any accuracy does not allow one to gain control over the systematic errors of the extracted ground-state parameters.

(b) A systematic study of the light-cone expansion of heavy-to-light transition form factors within the method of light-cone sum rules (LCSR) was performed in [70, 71, 72]. In these papers, a cut heavy-to-light correlator, relevant for the extraction of the transition form factor, was analysed in a model with scalar constituents interacting via massless boson exchange. The correlator was calculated in two different ways: by making use of the Bethe–Salpeter wave function of the light bound state and by performing the light-cone expansion. It was shown that, in distinction to the often claims in the literature, the higher-twist off-light-cone contributions are not suppressed compared to the light-cone one by any large parameter. Numerically, the difference between the full cut correlator and the light-cone contribution to this correlator was found to be about 20-30% in a wide range of masses of the particles involved in the decay process. These results show that the application of LCSRs to hadron form factors suffers from two sources of systematic errors: (i) the uncontrolled errors in the correlator itself related to higher-twist effects, (ii) the errors related to the

extraction of the ground-state parameters from the correlator known in the limited range of the Borel parameter.

This analysis explicitly demonstrates the limited potential for the use of QCD sum rules in problems, where rigorous control of the accuracy of the extracted hadron parameters is necessary: QCD sum rules share the same difficulties as other approaches to non-perturbative QCD such as effective constituent quark models.

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Chapter 8

Spectral Integral Equation

Considering soft processes, we deal with all the problems connected with strong interactions, and, first of all, the phenomenon of quark confinement. It follows from the proposed theory [1, 2] formulated as a quantum theory containing both perturbative and non-perturbative phenomena that spectroscopy, the account of levels and wave functions is in fact a search for confinement-related interactions; our aim is to find the corresponding singularities.

We know that the hypothesis of the constituent quark structure (owing to which a baryon is a three-quark system and a meson is a two-quark one) works well for the low-lying hadrons. This hypothesis can successfully explain data for high energy collisions (see Chapter 1) and radiative hadron decays (Chapter 7).

The successful systematisation of mesons on (n, M^2) -planes where n is the radial quantum number of the $q\bar{q}$ composite systems tells us that in the mass region ≤ 2500 MeV the hypothesis of the constituent quark structure of hadrons can be applied for highly excited states as well (Chapter 2). In the (n, M^2) systematics we observe two remarkable features:

- (i) Meson trajectories with fixed IJ^{PC} are linear in the studied region (≤ 2500 MeV);
- (ii) Practically all observed mesons find a place on these trajectories not leaving room for candidates to hybrid-like or four-quark states (the number of such exotic states, if they exist, should be large).

These features allow us to suggest that between a quark (colour number $\mathbf{3}$) and an antiquark (colour number $\bar{\mathbf{3}}$) certain long-range universal forces exist which form meson levels at large masses putting them on linear (n, M^2) -trajectories. This suggestion is supported by the fact that baryon states with fixed IJ^{PC} are also lying on linear trajectories with the same

slope. We are able to explain this behaviour of the baryon levels by accepting the quark–diquark structure of the excited states and their formation by the same type of forces as it is for excited mesons (the colour number of a diquark coincides with that of an antiquark, $\bar{\mathbf{3}}$). It looks very natural to suppose that the discussed long-range universal forces are responsible for the confinement of colour objects too.

We have now enough data for the quantitative study of the universal forces. As we see it, this means that we have good perspectives for extracting the confinement singularity.

8.1 Basic Standings in the Consideration of Light Meson Levels in the Framework of the Spectral Integral Equation

The spectral integral method applied to the analysis of the quark–antiquark systems is a direct generalisation of the dispersion N/D method [3] for the case of separable vertices (see Chapter 3). In the framework of this method the two-nucleon systems and their interactions with the electromagnetic field (in particular, the form factors of the deuteron [4] and the deuteron photodisintegration amplitude) were analysed [5] (see Chapter 4). In this method there were no problems with the description of the high-spin particles.

The method has been generalised [6] aiming to describe the quark–antiquark systems. As a result, the equation was written for the quark wave function, its form being similar to the Bethe–Salpeter equation. There is, however, an important difference between the standard Bethe–Salpeter equation [7] and that written in terms of the spectral integral. In the dispersion relation technique the constituents in the intermediate state are mass-on-shell, $k_i^2 = m^2$, while in the Feynman technique, which is used in the Bethe–Salpeter equation, $k_i^2 \neq m^2$. So, in the spectral integral equation, when the high spin state structures are calculated, we have a simple numerical factor $k_i^2 = m^2$, while in the Feynman technique one has $k_i^2 = m^2 + (k_i^2 - m^2)$. The first term in the right-hand side of this equality provides us with a contribution similar to that obtained in the spectral integration technique, while the second term cancels one of the denominators in the kernel of the Bethe–Salpeter equation. This results in penguin (or tadpole) type diagrams — we call them zoo-diagrams (or animal-like ones). A particular feature of the spectral integral technique is

the exclusion of these diagrams from the equation for a composite system.

The absence of zoo-diagrams in the used equations makes it difficult to compare directly the spectral integral calculations with those of the standard Bethe–Salpeter technique. In particular, the interactions reconstructed by these two methods may differ. Therefore, one may compare the final results only (masses of levels, radiative decay widths).

To reconstruct the interaction, one needs to know the positions of levels and wave functions of composite systems [6]. Information about wave functions can be obtained from radiative decays (in other words, from the form factors of the composite particles).

The analyses of the light $q\bar{q}$ systems and heavy $Q\bar{Q}$ quarkonia in terms of the spectral integral equation differ from one another in a certain respect, because the corresponding experimental data are different: in the $Q\bar{Q}$ systems only the masses of low-lying states are known, except for the 1^{--} quarkonia (Υ and ψ) where a long series of vector states was discovered in the e^+e^- annihilation. At the same time, for the low-lying heavy quark states there exists a rich set of data on radiative decays: $(Q\bar{Q})_{\text{in}} \rightarrow \gamma + (Q\bar{Q})_{\text{out}}$ and $(Q\bar{Q})_{\text{in}} \rightarrow \gamma\gamma$. For the light quark sector there is an abundance of information on the masses of highly excited states with different J^{PC} , but we have rather poor data for radiative decays.

Despite the scarcity of data on radiative decays, we apply the method to the study of light quarkonia, relying on our knowledge of linear trajectories in the (n, M^2) -plane that may, we hope, compensate the lack of information about the wave functions. In the fitting procedure we pay the main attention to states with large masses, which are essentially formed, as we suppose, by the confinement interaction.

Here we consider the light-quark (u, d, s) mesons with masses $M \leq 3$ GeV following results obtained in [8] for the mesons lying on linear trajectories in the (n, M^2) -planes. Calculations are performed for $q\bar{q}$ states with one component in the flavour space such as:

$\pi(0^{-+}), \rho(1^{--}), \omega(1^{--}), \phi(1^{--}), a_0(0^{++}), a_1(1^{++}), a_2(2^{++}), b_1(1^{+-}), f_2(2^{++}), \pi_2(2^{-+}), \rho_3(3^{--}), \omega_3(3^{--}), \phi_3(3^{--}), \pi_4(4^{-+})$ at $n \leq 6$.

The fit performed in [8] gives us wave functions and mass values of mesons lying on the (n, M^2) trajectories. The obtained trajectories are linear, in agreement with the data.

The calculated widths for the two-photon decays $\pi \rightarrow \gamma\gamma, a_0(980) \rightarrow \gamma\gamma, a_2(1320) \rightarrow \gamma\gamma, f_2(1285) \rightarrow \gamma\gamma, f_2(1525) \rightarrow \gamma\gamma$ and radiative transitions $\rho \rightarrow \gamma\pi, \omega \rightarrow \gamma\pi$ agree qualitatively with the experiment.

On this basis the singular parts of the quark–antiquark long-range in-

teractions which correspond to the confinement are singled out. The description of the data requires the presence of strong leading singularities for both scalar and vector t -channel exchanges:

$$[I \otimes I - \gamma_\mu \otimes \gamma_\mu]_{t\text{-channel}} \quad (8.1)$$

At small momentum transfer the singular interaction behaves as $\sim 1/q^4$ or, in the coordinate representation, as $\sim r$. Along with the confinement singularities, in the fitting procedure the one-gluon t -channel exchange was included. The one-gluon coupling is provided to be approximately of the same order for all quarkonium sectors $q\bar{q}$, $c\bar{c}$ and $b\bar{b}$, namely, $\alpha_s \simeq 0.4$. The universal stability of α_s for all quarkonium sectors (see Appendices 8.A and 8.B as well as [9, 10]) raises doubts about the validity of the hypothesis of a frozen α_s in the soft region.

In Appendices 8.A and 8.B we present results for the sectors of heavy quarkonia, $b\bar{b}$ and $c\bar{c}$ obtained in terms of the spectral integral equations.

The $b\bar{b}$ sector, studied in [9], is discussed in Appendix A. The $b\bar{b}$ interaction is reconstructed on the basis of data for the bottomonium levels with $J^{PC} = 0^{-+}, 1^{--}, 0^{++}, 1^{++}, 2^{++}$ as well as the data for the radiative transitions $\Upsilon(3S) \rightarrow \gamma\chi_{bJ}(2P)$ and $\Upsilon(2S) \rightarrow \gamma\chi_{bJ}(1P)$ with $J = 0, 1, 2$. We calculate the bottomonium levels with the radial quantum numbers $n \leq 6$ and their wave functions as well as corresponding radiative transitions. The ratios $\text{Br}[\chi_{bJ}(2P) \rightarrow \gamma\Upsilon(2S)]/\text{Br}[\chi_{bJ}(2P) \rightarrow \gamma\Upsilon(1S)]$ for $J = 0, 1, 2$ are found in agreement with the data. The $b\bar{b}$ component of the photon wave function is determined using the data for the e^+e^- annihilation, $e^+e^- \rightarrow \Upsilon(9460), \Upsilon(10023), \Upsilon(10036), \Upsilon(10580), \Upsilon(10865), \Upsilon(11019)$, and predictions are made for partial widths of the two-photon decays $\eta_{b0} \rightarrow \gamma\gamma, \chi_{b0} \rightarrow \gamma\gamma, \chi_{b2} \rightarrow \gamma\gamma$ (for the radial excitation states below the $B\bar{B}$ threshold, $n \leq 3$).

Appendix 8.B is devoted to the results obtained for charmonium ($c\bar{c}$) states [10]. The interaction in the $c\bar{c}$ -sector is reconstructed on the basis of data for the charmonium levels with $J^{PC} = 0^{-+}, 1^{--}, 0^{++}, 1^{++}, 2^{++}, 1^{+-}$ as well as radiative transitions $\psi(2S) \rightarrow \gamma\chi_{c0}(1P), \gamma\chi_{c1}(1P), \gamma\chi_{c2}(1P), \gamma\eta_c(1S)$ and $\chi_{c0}(1P), \chi_{c1}(1P), \chi_{c2}(1P) \rightarrow \gamma J/\psi$. In [10] the $c\bar{c}$ levels and their wave functions are calculated for $n \leq 6$. Also, the $c\bar{c}$ component of the photon wave function is determined by using the e^+e^- annihilation data: $e^+e^- \rightarrow J/\psi(3097), \psi(3686), \psi(3770), \psi(4040), \psi(4160), \psi(4415)$. This makes it possible to perform the calculations of the partial widths of the two-photon decays for the $n = 1$ states: $\eta_{c0}(1S), \chi_{c0}(1P), \chi_{c2}(1P) \rightarrow \gamma\gamma$, and the $n = 2$ states: $\eta_{c0}(2S) \rightarrow \gamma\gamma, \chi_{c0}(2P), \chi_{c2}(2P) \rightarrow \gamma\gamma$.

Owing to the large mass of the heavy quarks and the rather restricted amount of studied states, these sectors do not supply us with conclusive information about confinement forces. Moreover, the large mass of quarks suggests that for these systems the spectral integral equation can be transformed with a reasonably good accuracy into a non-relativistic quark model equation – a similar transformation, one could think, may be performed with the standard Bethe-Salpeter equation as well. So, the calculations in heavy quarkonium sectors are interesting for comparing results obtained by different groups in different approaches.

Another point of interest in the sectors of heavy quarks is the fitting program for composite systems. Just performing a fit of the $b\bar{b}$ and $c\bar{c}$ states, one can check the stability of the fit to the inclusion (or exclusion) of some data.

Appendix 8.C is devoted to some technical problems of the fitting procedure related to the calculation of the loop diagrams for high spin composite particles.

In Appendix 8.D, using the simple example of a spinless constituent, we demonstrate that to extract the interaction, we have to know not only the levels of the bound states but also their wave functions. Just this point compels us to present wave functions for the calculated $q\bar{q}$ state (Appendix 8.E).

8.2 Spectral Integral Equation

Let us remind here some points related to the spectral integral equation presented in Chapters 3 and 4, as well as notations used for quark–antiquark systems.

We denote the wave function of the $q\bar{q}$ meson as $\hat{\Psi}_{(n)\mu_1\ldots\mu_J}^{(S,J)}(k_\perp)$, with $k_\perp$ being the relative quark momentum and the indices $\mu_1, \ldots, \mu_J$ are related to the total momentum. For the one-flavour $q\bar{q}$ system the spectral integral equation reads:

$$\begin{aligned} (s - M^2) \hat{\Psi}_{(n)\mu_1\ldots\mu_J}^{(S,J)}(k_\perp) &= \int_{4m^2}^{\infty} \frac{ds'}{\pi} \int d\Phi_2(P'; k'_1, k'_2) \hat{V}(s, s', (k_\perp k'_\perp)) \\ &\times (\hat{k}'_1 + m) \hat{\Psi}_{(n)\mu_1\ldots\mu_J}^{(S,J)}(k'_\perp) (-\hat{k}'_2 + m) . \quad (8.2) \end{aligned}$$

Here the quarks are on the mass shell, $k_1^2 = k_1'^2 = k_2^2 = k_2'^2 = m^2$. The

phase space factor in the intermediate state is determined in the standard way:

$$d\Phi_2(P'; k'_1, k'_2) = \frac{1}{2} \frac{d^3 k'_1}{(2\pi)^3 2k'_{10}} \frac{d^3 k'_2}{(2\pi)^3 2k'_{20}} (2\pi)^4 \delta^{(4)}(P' - k'_1 - k'_2) . \quad (8.3)$$

The following notations are used:

$$\begin{aligned} k_\perp &= \frac{1}{2} (k_1 - k_2), \quad P = k_1 + k_2, \quad k'_\perp = \frac{1}{2} (k'_1 - k'_2), \quad P' = k'_1 + k'_2, \\ P^2 &= s, \quad P'^2 = s', \quad g_{\mu\nu}^\perp = g_{\mu\nu} - \frac{P_\mu P_\nu}{s}, \quad g_{\mu\nu}^{\prime\perp} = g_{\mu\nu} - \frac{P'_\mu P'_\nu}{s'}, \end{aligned} \quad (8.4)$$

so one can write $k_\mu^\perp = k_\nu g_{\nu\mu}^\perp$ and $k'_\mu^\perp = k'_\nu g_{\nu\mu}^{\prime\perp}$. In the c.m. system the integration may be rewritten as

$$\int_{4m^2}^{\infty} \frac{ds'}{\pi} \int d\Phi_2(P'; k'_1, k'_2) \longrightarrow \int \frac{d^3 k'}{(2\pi)^3 k'_0}, \quad (8.5)$$

where k' is the momentum of one of the quarks.

For the fermion–antifermion system with definite J, S and L we introduce the momentum operators $G_{\mu_1 \dots \mu_J}^{(S, L, J)}(k_\perp)$ defined as follows:

$$\begin{aligned} G_{\mu_1 \mu_2 \dots \mu_J}^{(0, J, J)}(k_\perp) &= i\gamma_5 X_{\mu_1 \dots \mu_J}(k^\perp) \sqrt{\frac{2J+1}{\alpha_J}}, \\ G_{\mu_1 \dots \mu_J}^{(1, J, J)}(k_\perp) &= \frac{i\varepsilon_{\alpha\eta\xi\gamma} \gamma_\eta k_\xi^\perp P_\gamma Z_{\mu_1 \dots \mu_J}^\alpha(k^\perp)}{\sqrt{s}} \sqrt{\frac{(2J+1)J}{(J+1)\alpha_J}}, \\ G_{\mu_1 \dots \mu_J}^{(1, J+1, J)}(k_\perp) &= \gamma_\alpha X_{\alpha\mu_1 \dots \mu_J}(k^\perp) \sqrt{\frac{J+1}{\alpha_J}}, \\ G_{\mu_1 \dots \mu_J}^{(1, J-1, J)}(k_\perp) &= \gamma_\alpha Z_{\mu_1 \dots \mu_J}^\alpha(k^\perp) \sqrt{\frac{J}{\alpha_J}}. \end{aligned} \quad (8.6)$$

The operators obey the normalisation condition:

$$\begin{aligned} \int \frac{d\Omega}{4\pi} \text{Sp}[G_{\mu_1 \dots \mu_L}^{(0, J, J)}(m + \hat{k}_1) G_{\nu_1 \dots \nu_L}^{(0, J, J)}(m - \hat{k}_2)] &= -2s \mathbf{k}^{2J} (-1)^J O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}(\perp P), \\ \int \frac{d\Omega}{4\pi} \text{Sp}[G_{\mu_1 \dots \mu_J}^{(1, J, J)}(m + \hat{k}_1) G_{\nu_1 \dots \nu_J}^{(1, J, J)}(m - \hat{k}_2)] &= -2s \mathbf{k}^{2J} (-1)^J O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}(\perp P), \end{aligned}$$

$$\begin{aligned}
& \int \frac{d\Omega}{4\pi} \text{Sp}[G_{\mu_1 \dots \mu_n}^{(1, J+1, J)}(m + \hat{k}_1) G_{\nu_1 \dots \nu_J}^{(1, J+1, J)}(m - \hat{k}_2)] \\
&= \left(\frac{8(J+1)\mathbf{k}^2}{2J+1} - 2s \right) \mathbf{k}^{2(J+1)} (-1)^J O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}(\perp P) , \\
& \int \frac{d\Omega}{4\pi} \text{Sp}[G_{\mu_1 \dots \mu_J}^{(1, J-1, J)}(m + \hat{k}_1) G_{\nu_1 \dots \nu_J}^{(1, J-1, J)}(m - \hat{k}_2)] \\
&= \left(\frac{8J\mathbf{k}^2}{2J+1} - 2s \right) \mathbf{k}^{2(J-1)} (-1)^J O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}(\perp P) , \\
& \int \frac{d\Omega}{4\pi} \text{Sp}[G_{\mu_1 \dots \mu_J}^{(1, J-1, J)}(m + \hat{k}_1) G_{\nu_1 \dots \nu_J}^{(1, J+1, J)}(m - \hat{k}_2)] \\
&= -8 \frac{\sqrt{J(J+1)}}{2J+1} \mathbf{k}^{2(J+1)} (-1)^J O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}(\perp P) . \quad (8.7)
\end{aligned}$$

Let us remind that $O_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n}(\perp P)$ is the projection operator to a state with the momentum J and $s = 4m^2 + 4\mathbf{k}^2$.

In terms of the momentum operators (8.6), the wave functions read:

$$\begin{aligned}
S = 0, 1 \text{ and } J = L : \quad & \widehat{\Psi}_{(n) \mu_1 \dots \mu_J}^{(S, J)}(k_\perp) = G_{\mu_1 \dots \mu_J}^{(S, J, J)}(k_\perp) \psi_n^{(S, L=J, J)}(k_\perp^2), \\
S = 0, 1 \text{ and } J \neq L : \quad & \widehat{\Psi}_{(n) \mu_1 \dots \mu_J}^{(S, J)}(k_\perp) = G_{\mu_1 \dots \mu_J}^{(S, J+1, J)}(k_\perp) \psi_n^{(S, L=J+1, J)}(k_\perp^2) \\
& + G_{\mu_1 \dots \mu_J}^{(S, J-1, J)}(k_\perp) \psi_n^{(S, L=J-1, J)}(k_\perp^2), \quad (8.8)
\end{aligned}$$

where functions $\psi_n^{(S, L, J)}(k_\perp^2)$ depend on $k_\perp^2 = -\mathbf{k}^2$ only.

The wave functions with $L = J$ are normalised as follows:

$$1 = \int \frac{d^3k}{(2\pi^3)k_0} 2s |\mathbf{k}|^{2J} |\psi_n^{(S, L=J, J)}(k_\perp^2)|^2 , \quad (8.9)$$

while for $L = J \pm 1$ the normalisation reads:

$$\begin{aligned}
1 &= W_{J+1, J+1} + W_{J+1, J-1} + W_{J-1, J-1}, \\
W_{J+1, J+1} &= \int \frac{d^3k}{(2\pi^3)k_0} |\psi_n^{(S, J+1, J)}(k_\perp^2)|^2 \left(2s - \frac{8(J+1)\mathbf{k}^2}{2J+1} \right) \mathbf{k}^{2(J+1)}, \\
W_{J+1, J-1} &= \int \frac{d^3k}{(2\pi^3)k_0} 16 \frac{\sqrt{J(J+1)}}{2J+1} \mathbf{k}^{2(J+1)} \psi_n^{(S, J+1, J)}(k_\perp^2) \psi_n^{*(S, J-1, J)}(k_\perp^2), \\
W_{J-1, J-1} &= \int \frac{d^3k}{(2\pi^3)k_0} |\psi_n^{(S, J-1, J)}(k_\perp^2)|^2 \left(2s - \frac{8J\mathbf{k}^2}{2J+1} \right) \mathbf{k}^{2(J-1)} . \quad (8.10)
\end{aligned}$$

Generally, the interaction block is a full set of the t -channel operators $\widehat{O}_I$:

$$\widehat{O}_I = \mathbf{I}, \quad \gamma_\mu, \quad i\sigma_{\mu\nu}, \quad i\gamma_\mu\gamma_5, \quad \gamma_5 ,$$

$$\widehat{V}(s, s', (k_\perp k'_\perp)) = \sum_I V_I(s, s', (k_\perp k'_\perp)) \widehat{O}_I \otimes \widehat{O}_I . \quad (8.11)$$

The t -channel operators (8.11) can be, with the help of the Fierz transformation, reorganised into a set of the s -channel operators — for details of this procedure see Appendix 8.C.

The equation (8.2) is written in momentum representation, and it was solved in [8] also in momentum representation. The equation (8.2) allows one to use the instantaneous interaction, or to take into account the retardation effects. In the instantaneous approximation one has:

$$\widehat{V}(s, s', (k_{\perp} k'_{\perp})) \longrightarrow \widehat{V}(t_{\perp}), \quad t_{\perp} = (k_{1\perp} - k'_{1\perp})_{\mu} (-k_{2\perp} + k'_{2\perp})_{\mu}. \quad (8.12)$$

The retardation effects are taken into account when the momentum transfer squared t in the interaction block depends on the time components of the quark momentum (for more details see the discussion in [6, 11, 12, 13, 14]). Then

$$\widehat{V}(s, s', (k_{\perp} k'_{\perp})) \longrightarrow \widehat{V}(t), \quad t = (k_1 - k'_1)_{\mu} (-k_2 + k'_2)_{\mu}. \quad (8.13)$$

In [8] both types of interactions, the instantaneous and retardation ones, were used in the fitting procedures. The description of the experimental situation is approximately of the same accuracy level in both approaches. Indeed, the existing data do not allow us to prefer either approach. We present here the results obtained by using the instantaneous interaction: the main reason is that in this case we construct mesons as pure $q\bar{q}$ states. The interaction with retardation, depending on zero momentum components, $(k_{i0} - k'_{i0})$, gives us in the ladder diagrams not only the two-quark intermediate states but also multipartical ones, see discussion in Chapter 3 (section 3).

Fitting to quark–antiquark states, we expand the interaction blocks using the following t -dependent terms:

$$\begin{aligned} I_{-1} &= \frac{4\pi}{\mu^2 - t}, \quad I_0 = \frac{8\pi\mu}{(\mu^2 - t)^2}, \\ I_1 &= 8\pi \left(\frac{4\mu^2}{(\mu^2 - t)^3} - \frac{1}{(\mu^2 - t)^2} \right), \quad I_2 = 96\pi\mu \left(\frac{2\mu^2}{(\mu^2 - t)^4} - \frac{1}{(\mu^2 - t)^3} \right), \\ I_3 &= 96\pi \left(\frac{16\mu^4}{(\mu^2 - t)^5} - \frac{12\mu^2}{(\mu^2 - t)^4} + \frac{1}{(\mu^2 - t)^3} \right), \end{aligned} \quad (8.14)$$

or, in the general case,

$$I_N = \frac{4\pi(N+1)!}{(\mu^2 - t)^{N+2}} \sum_{n=0}^{N+1} (\mu + \sqrt{t})^{N+1-n} (\mu - \sqrt{t})^n. \quad (8.15)$$

Traditionally, the interaction of quarks in the instantaneous approximation is represented in terms of the potential $V(r)$. It is also convenient to

work with such a representation in the case of the spectral integral equation. But one should keep in mind that the interaction used in the spectral integrals does not coincide literally with that of the Bethe–Salpeter equation. In the spectral integral technique, the interaction is given by the N -function represented as an infinite sum of separable vertices, see Chapter 3. The N -function at small s is defined by the t -channel one-pole exchange diagrams, so it can be compared with the potential terms of the standard Bethe–Salpeter equation at large distances. However, at large s , where the multiple t -channel exchanges dominate (in the region of small distances), the N -functions cannot be reduced to the standard potentials. To underline this difference we call the instantaneous interaction in the r -representation, used in the spectral integral technique, as a "quasi-potential".

The form of the quasi-potential can be obtained with the help of the Fourier transform of (8.14) in the centre-of-mass system. Thus, we have

$$t_{\perp} = -(\mathbf{k} - \mathbf{k}')^2 = -\mathbf{q}^2, \\ I_N^{(\text{coord})}(r, \mu) = \int \frac{d^3 q}{(2\pi)^3} e^{-i\mathbf{q} \cdot \mathbf{r}} I_N(t_{\perp}), \quad (8.16)$$

that gives

$$I_N^{(\text{coord})}(r, \mu) = r^N e^{-\mu r}. \quad (8.17)$$

In the fitting procedure [8] the following types of $V(r)$ were used:

$$V(r) = a + b r + c e^{-\mu_c r} + d \frac{e^{-\mu_d r}}{r}, \quad (8.18)$$

where the constant and linear (confinement) terms read:

$$a \rightarrow a I_0^{(\text{coord})}(r, \mu_{\text{constant}} \rightarrow 0), \\ b r \rightarrow b I_1^{(\text{coord})}(r, \mu_{\text{linear}} \rightarrow 0). \quad (8.19)$$

The limits $\mu_{\text{constant}}, \mu_{\text{linear}} \rightarrow 0$ mean that in the fitting procedure the parameters μ_{constant} and μ_{linear} are chosen to be small enough, of the order of 1–10 MeV. It was checked that the solution for the states with $n \leq 6$ is stable, when μ_{constant} and μ_{linear} change within this interval.

8.3 Light Quark Mesons

In this section we study the light quark systems with a single flavour component. These are, first, systems with unity isospin, ($I = 1, J^{PC}$). Second, among the systems with zero isospin, ($I = 0, J^{PC}$), there are also one-component states, $s\bar{s}$ or $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$, which are considered as well.

We mean the ϕ and ω mesons, $\phi(1^{--})$, $\phi_3(3^{--})$ and $\omega(1^{--})$, $\omega_3(3^{--})$. Besides, in the $f_2(2^{++})$ -mesons at $M \lesssim 2400$ MeV the components $n\bar{n}$ and $s\bar{s}$ are separated with a good accuracy [15]; below, all the f_2 -mesons are assumed to be pure flavour states.

Considering the trajectory $\pi(140)$, $\pi(1300)$, $\pi(1800)$, $\pi(2070)$, $\pi(2360)$, we fix our attention on the excited states $\pi(1300)$, $\pi(1800)$, $\pi(2070)$, $\pi(2360)$. As concerns the lightest pion $\pi(140)$, this particle is a singular state in many respects, and we intend to get only a qualitative agreement with the data (a good quantitative description of the 0^{+-} states, which requires the study of the role of the instanton-induced forces, is beyond the scope of the present approach).

We investigate the $q\bar{q}$ -mesons with the masses $\lesssim 3000$ MeV and characterise these states by the following wave functions:

$$\begin{array}{c|c|c}
 L=0 & 0^{-+} & i\gamma_5 \psi_n^{(0,0,0)}(k^2) \\
 & 1^{--} & \gamma_\mu^\perp \psi_n^{(1,0,1)}(k^2) \\
 \hline
 L=1 & 0^{++} & m \psi_n^{(1,1,0)}(k^2) \\
 & 1^{++} & \sqrt{3/2}s \cdot i \varepsilon_{\gamma P k \mu} \psi_n^{(1,1,1)}(k^2) \\
 & 2^{++} & \sqrt{3/4} \cdot \left[k_{\mu_1} \gamma_{\mu_2}^\perp + k_{\mu_2} \gamma_{\mu_1}^\perp - \frac{2}{3} \hat{k} g_{\mu_1 \mu_2}^\perp \right] \psi_n^{(1,1,2)}(k^2) \\
 & 1^{+-} & \sqrt{3} i \gamma_5 k_\mu \psi_n^{(0,1,1)}(k^2) \\
 \hline
 L=2 & 1^{--} & 3/\sqrt{2} \cdot \left[k_\mu \hat{k} - \frac{1}{3} k^2 \gamma_\mu^\perp \right] \psi_n^{(1,2,1)}(k^2) \\
 & 2^{--} & \sqrt{20/9}s \cdot i \varepsilon_{\gamma P k \alpha} Z_{\mu_1 \mu_2, \alpha}^{(2)}(k_\perp) \psi_n^{(1,2,2)}(k^2) \\
 & 3^{--} & \sqrt{6/5} \cdot \gamma_\alpha Z_{\mu_1 \mu_2 \mu_3, \alpha}^{(2)}(k_\perp) \psi_n^{(1,2,3)}(k^2) \\
 & 2^{-+} & \sqrt{10/3} \cdot i \gamma_5 X_{\mu_1 \mu_2}^{(2)}(k_\perp) \psi_n^{(0,2,2)}(k^2) \\
 \hline
 L=3 & 2^{++} & \sqrt{2} \cdot \gamma_\alpha X_{\mu_1 \mu_2 \alpha}^{(3)}(k_\perp) \psi_n^{(1,3,2)}(k^2) \\
 & 3^{++} & \sqrt{21/10}s \cdot i \varepsilon_{\gamma P k \alpha} Z_{\mu_1 \mu_2 \mu_3, \alpha}^{(2)}(k_\perp) \psi_n^{(1,3,3)}(k^2) \\
 & 4^{++} & \sqrt{36/35} \cdot \gamma_\alpha Z_{\mu_1 \mu_2 \mu_3 \mu_4, \alpha}^{(3)}(k_\perp) \psi_n^{(1,3,4)}(k^2) \\
 & 3^{+-} & \sqrt{14/5} \cdot i \gamma_5 X_{\mu_1 \mu_2 \mu_3}^{(3)}(k_\perp) \psi_n^{(0,3,3)}(k^2) \\
 \hline
 L=4 & 3^{--} & \sqrt{8/7} \cdot \gamma_\alpha X_{\mu_1 \mu_2 \mu_3 \alpha}^{(4)}(k_\perp) \psi_n^{(1,4,3)}(k^2) \\
 & 4^{--} & \sqrt{288/175}s \cdot i \varepsilon_{\gamma P k \alpha} Z_{\mu_1 \mu_2 \mu_3 \mu_4, \alpha}^{(4)}(k_\perp) \psi_n^{(1,4,4)}(k^2) \\
 & 5^{--} & \sqrt{16/35} \cdot \gamma_\alpha Z_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5, \alpha}^{(2)}(k_\perp) \psi_n^{(1,4,5)}(k^2) \\
 & 4^{-+} & \sqrt{81/35} \cdot i \gamma_5 X_{\mu_1 \mu_2 \mu_3 \mu_4}^{(4)}(k_\perp) \psi_n^{(0,4,4)}(k^2).
 \end{array} \tag{8.20}$$

Generally speaking, the 1^{--} , 2^{++} , 3^{--} states are mixtures of waves with different angular momenta. However, the investigation of the bottomonium and charmonium states (see Appendices 8.A and 8.B) shows that the angular momentum is a good quantum number for these types of

states. Here we use the one-component ansatz for the light-quark systems too: we describe these states by one-component wave functions.

In (8.20) we label the group of mesons by the index L . Recall that the index L does not select a pure angular momentum state, for example, the wave function $\gamma_\mu^\perp \psi_n^{(1,0,1)}(k^2)$ given in (8.20) for a $(1^{--}, L=0)$ -system, being dominantly an S -wave state, contains an admixture of the D -wave.

As our calculations show, the ansatz (8.20) works well for the considered mesons.

8.3.1 Short-range interactions and confinement

In [8] two types of the t -channel exchange interactions are used: scalar, $(I \otimes I)$, and vector, $(\gamma_\nu \otimes \gamma_\mu)$. We classify the interactions as being effectively short-range,

$$V_{sh}(r) = a + c e^{-\mu_c r} + d \frac{e^{-\mu_d r}}{r}, \quad (8.21)$$

and long-range ones:

$$V_{\text{conf}}(r) = b r \quad (8.22)$$

which are responsible for confinement.

The states with different L are fitted to the (n, M^2) trajectories separately, assuming that the leading (confinement) singularity is common for all states (it i.e. b in (8.22) is universal for all L) while the short-range interactions may depend on L . For the short-range interaction we adopt here, in fact, the ideology of the dispersion relation N/D -method where the N -function may be different for each wave. Thus, we project $V_{sh}(r)$ on states with different L ,

$$\langle L | V_{sh}(r) | L \rangle, \quad (8.23)$$

and fit separately to each group of mesons.

The fitting procedure carried out in [8] resulted in the following parameters for $L = 1, 2, 3, 4$ (all values in GeV).

For the scalar interaction, $(I \otimes I)$, we have:

Wave	a	b	c	μ_c	d	μ_d
$L = 0$	-2.860	0.150	5.037	0.410	0.221	0.410
$L = 1$	-0.398	0.150	5.362	0.410	-2.270	0.210
$L = 2$	8.407	0.150	6.866	0.110	-1.250	0.210
$L = 3$	-0.281	0.150	5.243	0.110	-32.507	0.410
$L = 4$	-1.912	0.150	3.8574	0.010	-3.3175	0.110 ,

(8.24)

and for the vector one, $(\gamma_\mu \otimes \gamma_\mu)$:

Wave	a	b	c	μ_c	d	μ_d	
$L = 0$	0.180	0.150	0.060	0.610	0.656	0.10	
$L = 1$	0.971	0.150	-0.188	0.610	0.664	0.10	
$L = 2$	1.804	0.150	-2.135	0.610	0.405	0.10	(8.25)
$L = 3$	1.239	0.150	-12.823	0.710	0.558	0.10	
$L = 4$	1.548	0.150	-2.5458	0.210	0.536	0.10	.

The fit requires the confinement singularity $V_{conf}(r) \sim br$ for both scalar ($I \otimes I$) and vector ($\gamma_\mu \otimes \gamma_\mu$) t -channel exchanges, and the coefficients b turn out to be approximately of the same value but different in sign: $b_S \simeq -b_V$. In the final fit the slopes were fixed to be equal to each other, thus resulting in

$$b_S = -b_V = 0.15 \text{ GeV}^2. \quad (8.26)$$

We see that the spin structure of the t -channel exchange (or, confinement) singularity has the following form:

$$[I \otimes I - \gamma_\mu \otimes \gamma_\mu]_{t\text{-channel}}. \quad (8.27)$$

Along with the confinement singularities, in the interaction studied in [8] the one-gluon t -channel exchange was included. The one-gluon coupling ($\alpha_s = \frac{3}{4}d_V$) turns out to be of the same order for all L , namely, $\alpha_s \simeq 0.4$. This value looks quite reasonable and agrees with other estimates for the soft region, see, for example, [16]. Moreover, calculations performed for the $b\bar{b}$ and $c\bar{c}$ sectors (see Appendices 8.A and 8.B as well as [10, 9]) also give $\alpha_s \simeq 0.4$. This substantiates the hypothesis of a frozen α_s in the soft region.

For the masses of the constituent quarks the following values were used: $m_u = m_d = 400 \text{ MeV}$ and $m_s = 500 \text{ MeV}$. The mass of the light constituent quark is larger than that applied usually in the quark models. But one should keep in mind that the mass of the constituent quark is the mean value of the self-energy part of the quark propagator for the considered region. This value can be different in the different energy (mass) regions; correspondingly, the “mass” of the constituent quark can be different for low-lying and highly excited states. Therefore, the value 400 MeV can be understood as an average quark mass value over the region 500–2500 MeV. Note that the increase of the constituent quark mass for the highly excited meson states was discussed earlier, in [17].

Let us remind that rather large parameter values, $a = 8.407 \text{ GeV}$ and $d = 6.886$, were obtained in the scalar sector at $L = 2$. Such values do not

violate any general principles; still, this point requires certain additional investigations. In the first place, we have to see whether there exists some other solution in the $L = 2$ sector.

8.3.2 Masses and mean radii squared of mesons with $L \leq 4$

Here the results of calculations for the masses and mean radii squared of the mesons with $L = 1, 2, 3, 4$ are presented. The mean radius squared of a quark–antiquark system is a rather interesting characteristics, especially for highly excited states, which are formed by the confinement forces. (It is useful to keep in mind that for the pion $R^2 \simeq 10 \text{ GeV}^{-2} = 0.39 \text{ fm}^2$). By listing the experimentally observed states, we follow Table 2.1 of Chapter 2.

8.3.2.1 Mesons of the ($L = 0$) group

The calculation of the ($L = 0$) states leads to the following masses (column "Mass", values in MeV) and mean radii squared (R^2 in GeV^{-2}) for the $(10^{-+}, L = 0)$ and $(11^{--}, L = 0)$ mesons, with different radial quantum numbers n :

n	Meson	Mass	R^2	Meson	Mass	R^2
1	$\pi(140)$	546	12.91	$\rho(775)$	778	12.77
2	$\pi(1300)$	1309	33.94	$\rho(1460)$	1473	12.34
3	$\pi(1800)$	1771	62.26	$\rho(1870)$	1763	18.08
4	$\pi(2070)$	2009	— — —	$\rho(2110)$	2158	45.71
5	$\pi(2360)$	2429	— — —	$\rho(\mathbf{2430})$	2363	60.30
6	— — —	3075	— — —	— — —	2675	— — —

(8.28)

The column "Meson" shows the masses which were used for the fit: within the error bars they coincide with those given in Chapter 2, the states predicted by the (n, M^2) systematics are drawn by bold characters.

Equation (8.28) demonstrates results obtained without including the instanton-induced forces giving the pion mass $\sim 500 \text{ MeV}$. The inclusion of the instanton-induced interaction (see below) leads to $M_{\text{pion}} \simeq 140 \text{ MeV}$.

For isoscalar $(01^{--}, L = 0)$ mesons, we assume ω and ϕ to be pure

flavour states ($\omega = (u\bar{u} + d\bar{d})/\sqrt{2}$ and $\phi = s\bar{s}$) and obtain:

n	Meson	Mass	R^2	Meson	Mass	R^2
1	$\omega(782)$	778	12.77	$\phi(1020)$	938	16.20
2	$\omega(1430)$	1473	12.34	$\phi(1650)$	1541	21.29
3	$\omega(\mathbf{1830})$	1763	18.08	$\phi(\mathbf{1970})$	1907	22.18
4	$\omega(2205)$	2158	45.71	$\phi(\mathbf{2300})$	2327	31.88
5	—	2363	60.30	—	2601	78.03
6	—	2675	—	—	2757	— .

(8.29)

In the sector $L = 0$, one can see the rapid growth of R^2 in the region of large masses ($R^2[\omega(2363)] \simeq 60 \text{ GeV}^{-2}$, $R^2[\phi(2601)] \simeq 78 \text{ GeV}^{-2}$). It is difficult to say now whether this growth reflects a certain physical phenomenon, or just uncertainties inherent to calculations near the upper edge of the mass spectrum.

8.3.2.2 Mesons of the ($L = 1$) group

In the isovector sector, the following ($10^{++}, L = 1$) and ($11^{++}, L = 1$) mesons were obtained:

n	Meson	Mass	R^2	Meson	Mass	R^2
1	$a_0(980)$	1035	7.19	$a_1(1230)$	1151	6.88
2	$a_0(1474)$	1496	13.57	$a_1(1640)$	1562	13.67
3	$a_0(\mathbf{1780})$	1884	21.63	$a_1(1930)$	1923	21.95
4	$a_0(2025)$	2208	30.72	$a_1(2270)$	2231	42.81
5	—	2488	42.78	—	2305	48.03
6	—	2777	—	—	2682	— ;

(8.30)

for ($12^{++}, L = 1$) and ($11^{+-}, L = 1$) we have:

n	Meson	Mass	R^2	Meson	Mass	R^2
1	$a_2(1320)$	1356	7.08	$b_1(1229)$	1168	7.01
2	$a_2(1732)$	1641	13.89	$b_1(1620)$	1567	13.67
3	$a_2(1950)$	1963	22.05	$b_1(1960)$	1928	21.73
4	$a_2(2175)$	2260	31.76	$b_1(2240)$	2240	31.03
5	—	2517	43.70	—	2548	34.70
6	—	2810	—	—	2927	— .

(8.31)

The fitting to the ($02^{++}, L = 1$) mesons is performed separately for $n\bar{n}$ and $s\bar{s}$ systems: the analysis of the decay couplings $f_2 \rightarrow K\bar{K}, \pi\pi, \eta\eta, \eta\eta'$ [18] tells that f_2 mesons at $M \leq 2400 \text{ MeV}$ are nearly pure $n\bar{n}$ or $s\bar{s}$ states.

Analogous arguments follow from the data on $\pi^-p \rightarrow \phi\phi p$ [19]. The fit resulted in:

n	Meson ($n\bar{n}$)	Mass	R^2	Meson ($s\bar{s}$)	Mass	R^2
1	$f_2(1275)$	1356	7.08	$f_2(1525)$	1608	6.52
2	$f_2(1580)$	1641	13.89	$f_2(1755)$	1855	11.88
3	$f_2(1920)$	1963	22.05	$f_2(2120)$	2162	18.76
4	$f_2(2240)$	2260	31.76	$f_2(2410)$	2454	26.27
5	—	2516	43.70	—	2731	36.42
6	—	2809	—	—	2990	—

Let us emphasise that the fit gives us comparatively small values for $R^2[f_2(1285)]$ and $R^2[f_2(1525)]$, (of the order of $\sim 7 \text{ GeV}^{-2}$): just such small values are required by the $\gamma\gamma$ decays of the tensor mesons, see Chapter 7 (and calculation in [20]).

8.3.2.3 Mesons of the ($L = 2$) group

The fit provided us with the following masses (in MeV) and mean radii squared (in GeV^{-2} units) for the ($L = 2$) sector.

For the ($^1D_2, I_{q\bar{q}} = 1$), ($^3D_1, I_{q\bar{q}} = 1$) states:

n	Meson	Mass	R^2	Meson	Mass	R^2
1	$\pi_2(1676)$	1700	5.81	$\rho(1700)$	1701	8.14
2	$\pi_2(2005)$	1937	11.53	$\rho(1970)$	1992	15.26
3	$\pi_2(2245)$	2348	16.44	$\rho(2265)$	2212	31.44
4	$\pi_2(\mathbf{2510})$	2637	22.98	—	2515	—
5	—	2914	—	—	2743	—

for the ($^3D_3, I_{q\bar{q}} = 1$), ($^3D_1, I_{q\bar{q}} = 0$) ones:

n	Meson	Mass	Meson	Mass
1	$\rho_3(1690)$	1671	$\omega(1670)$	1701
2	$\rho_3(1980)$	1987	$\omega(1960)$	1992
3	$\rho_3(2300)$	2376	$\omega(2330)$	2212
4	—	2705	—	2515
5	—	2991	—	2743

and for the ($^3D_3, I_{q\bar{q}} = 0$), ($^3D_3, s\bar{s}$) states:

n	Meson	Mass	Meson	Mass
1	$\omega_3(1667)$	1671	$\phi_3(1854)$	1850
2	$\omega_3(1980)$	1987	$\phi_3(\mathbf{2150})$	2150
3	$\omega_3(2285)$	2376	$\phi_3(\mathbf{2450})$	2450
4	—	2705	$\phi_3(\mathbf{2640})$	2654
5	—	2991	—	2797

8.3.2.4 Mesons of the ($L = 3$) group

Mesons of the ($L = 3$) group form $q\bar{q}$ states with a dominant F -wave. In the ($I = 1$) sector the following levels were obtained:

n	Meson	Mass	R^2	Meson	Mass
1	$a_2(2030)$	2019	10.88	$a_3(2030)$	2062
2	$a_2(2255)$	2263	22.42	$a_3(2275)$	2314
3	—	2460	29.63	—	2585
4	—	2847	37.00	—	2938
5	—	3360	—	—	3390

(8.36)

and

n	Meson	Mass	Meson	Mass
1	$b_3(2032)$	2013	$a_4(2005)$	2018
2	$b_3(2245)$	2291	$a_4(2255)$	2333
3	$b_3(\mathbf{2520})$	2538	—	2493
4	$b_3(\mathbf{2740})$	2706	—	2659
5	—	3065	—	3059

(8.37)

For the ($I = 0$) sector the fit gives:

n	$(n\bar{n})$ -meson	Mass	$(s\bar{s})$ -meson	Mass	$(n\bar{n})$ -meson	Mass
1	$f_2(2020)$	2018	$f_2(2340)$	2315	$f_4(2025)$	2014
2	$f_2(2300)$	2262	—	2498	$f_4(\mathbf{2150})$	2241
3	—	2460	—	2770	—	2336
4	—	2846	—	3136	—	2570
5	—	3360	—	3591	—	2941

(8.38)

In (8.38), the experimental mass values for mesons with dominant $(n\bar{n})$ and $(s\bar{s})$ components are taken from [18] and [19].

8.3.2.5 Mesons of the ($L = 4$) group

In the ($L = 4$) group the following mesons were obtained in the fit [8]:

n	Meson	Mass	Meson	Mass
1	$\rho_3(2240)$	2252	$\pi_4(2250)$	2257
2	—	2482	—	2516
3	—	2746	—	2842
4	—	3131	—	3268
5	—	3607	—	3760

(8.39)

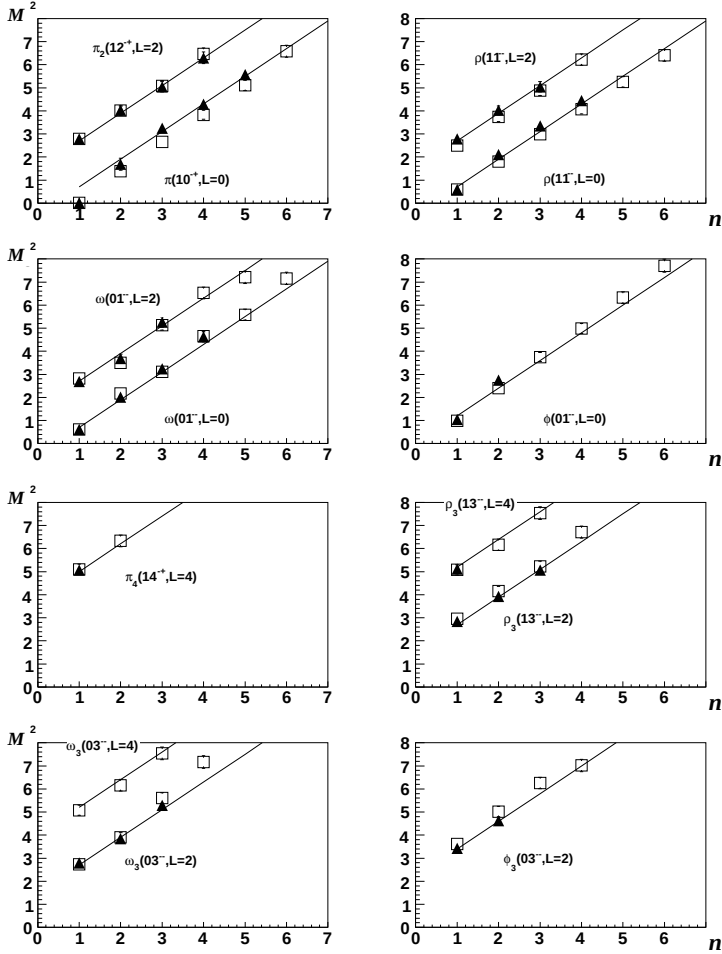


Fig. 8.1 The $(L = 0)$, $(L = 2)$ and $(L = 2)$, $(L = 4)$ trajectories on the (n, M^2) planes. Full triangles stand for the experimentally observed states and states from Table 1 of Chapter 2 while the open squares show the calculated masses in the fit (M^2 in GeV^2 units). Thin lines represent linear trajectories with $\mu = 1.2 \text{ GeV}^2$.

8.3.3 Trajectories on the (n, M^2) planes

In Figs. 8.1 and 8.2 one may see the $q\bar{q}$ trajectories on the (n, M^2) planes. In Fig. 8.1 we show the trajectories for the $(L = 0)$, $(L = 2)$ and $(L = 4)$ groups, in Fig. 8.2 we demonstrate the $L = 1$ and $L = 3$ trajectories.

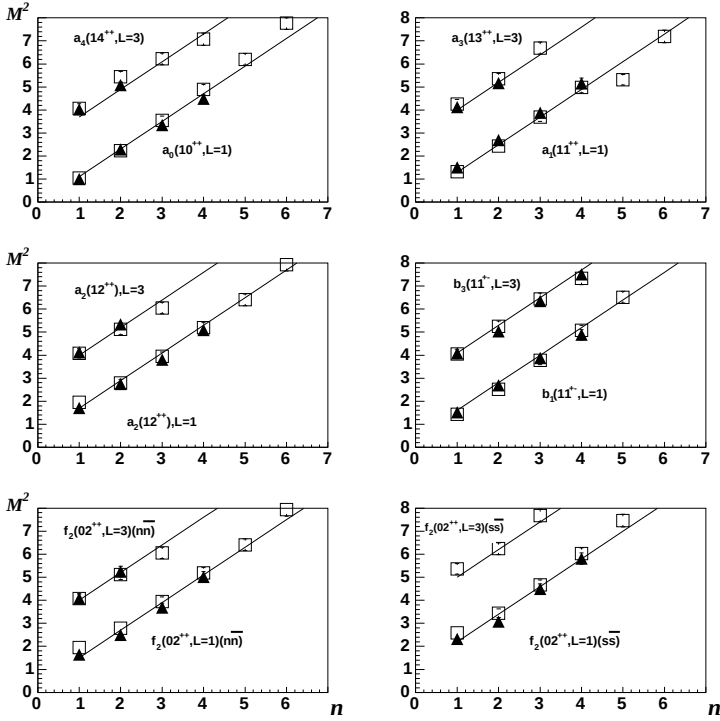


Fig. 8.2 The $(L = 1)$ and $(L = 3)$ trajectories on the (n, M^2) -planes. The notations are as in Fig. 8.1.

In line with the observation [21] (see Chapter 2 for details), all trajectories are linear with a good accuracy:

$$M^2 = M_0^2 + \mu^2(n - 1), \quad (8.40)$$

and have a universal slope:

$$\mu^2 \simeq 1.2 \text{ GeV}^2. \quad (8.41)$$

8.4 Radiative decays

Information about the wave functions of the $q\bar{q}$ states can be obtained from their radiative decays, mainly two-photon meson decays. The two-photon decay amplitude is the convolution of the meson wave function and the quark component of the photon wave function, see Chapter 7. As was stressed in Chapter 7, at present the light-quark component of the

photon can be rather reliably determined (see also [20, 22]): the basis is a description of the experimental data for $V \rightarrow e^+e^-$. Below we demonstrate the description of the available data for $V \rightarrow e^+e^-$ with wave functions found in [8] as solutions of the spectral integral equation:

Process	Data	Fit	
$\rho(770) \rightarrow e^+e^-$	7.02 ± 0.11	7.260	
$\rho(1450) \rightarrow e^+e^-$	—	3.280	
$\rho(1830) \rightarrow e^+e^-$	—	2.790	
$\rho(2110) \rightarrow e^+e^-$	—	2.431	
$\omega(780) \rightarrow e^+e^-$	0.60 ± 0.02	0.776	(8.42)
$\omega(1420) \rightarrow e^+e^-$	—	0.388	
$\omega(1800) \rightarrow e^+e^-$	—	0.326	
$\omega(2150) \rightarrow e^+e^-$	—	0.255	
$\phi(1020) \rightarrow e^+e^-$	1.27 ± 0.04	1.353	
$\phi(1657) \rightarrow e^+e^-$	—	0.985	

Recall that these decays are determined by the convolution of the vector meson wave functions and the $\gamma \rightarrow q\bar{q}$ vertex.

With the obtained photon wave function, the widths of the two-photon decays of mesons were calculated in [8]:

$q\bar{q}$ -State	Process	Data, keV	Fit, keV	
1^1S_0	$\pi(140) \rightarrow \gamma\gamma$	0.007	0.005	
1^1S_0	$\pi(1300) \rightarrow \gamma\gamma$	—	3.742	
1^1S_0	$\pi(1800) \rightarrow \gamma\gamma$	—	8.466	
1^3P_0	$a_0(980) \rightarrow \gamma\gamma$	0.300 ± 0.100	0.340	
1^3P_0	$a_0(1474) \rightarrow \gamma\gamma$	—	0.224	
1^3P_0	$a_0(1830) \rightarrow \gamma\gamma$	—	0.186	
1^3P_2	$a_2(1320) \rightarrow \gamma\gamma$	1.00 ± 0.06	1.045	
1^3P_2	$a_2(1660) \rightarrow \gamma\gamma$	—	0.821	(8.43)
1^3P_2	$a_2(1950) \rightarrow \gamma\gamma$	—	0.699	
$1^3P_2 n\bar{n}$	$f_2(1275) \rightarrow \gamma\gamma$	2.71 ± 0.25	2.946	
$1^3P_2 n\bar{n}$	$f_2(1580) \rightarrow \gamma\gamma$	—	2.396	
$1^3P_2 n\bar{n}$	$f_2(1920) \rightarrow \gamma\gamma$	—	1.971	
$1^3P_2 s\bar{s}$	$f_2(1525) \rightarrow \gamma\gamma$	0.10 ± 0.01	0.135	
$1^3P_2 s\bar{s}$	$f_2(1755) \rightarrow \gamma\gamma$	—	0.118	
$1^3P_2 s\bar{s}$	$f_2(2120) \rightarrow \gamma\gamma$	—	0.097	

Concerning the measured widths, one can see a good agreement with the calculated magnitudes.

Let us emphasise the proximity of the calculated width $\Gamma(\pi^0 \rightarrow \gamma\gamma) \simeq 0.005$ keV and the experimental value (in the calculation [8], the real mass of the pion was taken for the phase space). This proximity tells us that the calculated wave function is close to the real one, despite a large difference between real and calculated pion masses. The information on the radiative decays of vector mesons means the same:

Process	Data, keV	Fit, keV	
$\rho^+(770) \rightarrow \gamma\pi^+(140)$	68 ± 7	67.1	(8.44)
$\omega(780) \rightarrow \gamma\pi^0(140)$	758 ± 25	604	

We see an agreement with the data (the difference of amplitudes is of the order of 10%); still, let us underline that the decays $V \rightarrow \gamma P$ are determined by the $M1$ transitions, which are sensitive to the presence of the small contributions initiated by the anomalous magnetic moment, *e.g.* see the discussion in [23, 24]). One may think that the corrections to the $\pi(140)$ mass and its wave function can be easily reached in the standard way, with instanton-induced interaction (*e.g.* see [25, 26] and references therein). From this point of view, typical are the results obtained in [26], where the bootstrap quark model was considered for the three lowest meson nonets $^1S_0, ^3S_0, ^3P_0$. Without instanton-induced interactions, the pion mass was obtained to be equal ~ 500 MeV, while the input of these forces in the calculation made the pion mass to be near 140 MeV.

(i) Instanton-induced interaction and pion.

Let us demonstrate the change in the description of $\pi(140)$ after including the instanton-induced interaction. Namely, let us include the s -channel vertex in the spectral integral equation for the pion ($L = 0$ in (8.23)):

$$[\gamma_5 \otimes \gamma_5]_{s\text{-channel}} g \exp[-\mu_{II}r], \quad g = -0.072, \quad \mu_{II} = 0.001. \quad (8.45)$$

Here g and μ_{II} are parameters (in GeV units); g was found from fitting to the data while μ_{II} was fixed. In this way the following values were obtained:

	Calculation	Data
M_{pion}	141 MeV,	140 MeV
$\Gamma(\pi_0(135) \rightarrow \gamma\gamma)$	0.005 keV,	0.007 keV
$\Gamma(\rho_0(770) \rightarrow \gamma\pi^+(140))$	67.5 keV,	68 ± 7 keV
$\Gamma(\omega_0(780) \rightarrow \gamma\pi^0(135))$	607 keV,	758 ± 25 keV.

(8.46)

The pion wave function corresponding to the inclusion of the vertex (8.45) is shown in Fig. 8.5 (see Appendix 8.E) by the dotted line (it almost coincides with the solid line).

This example is indeed a good illustration of the fact that the description of the pion does not face problems after the inclusion of the instanton-induced interaction.

8.4.1 *Wave functions of the quark–antiquark states*

The fit [8] provided us with a sufficiently good description of mesons treated as bound states of constituent quarks: the masses of mesons with one flavour component lay on the linear trajectories in the (n, M^2) plane. Also, we have quite a good coincidence of the measured and calculated widths for the radiative decays.

The main purpose of the investigation of the light quark sector is to determine the characteristics of the leading t -channel singularities (confinement singularities or, in the language of potentials, the confinement potentials $V_{\text{conf}}(r) \sim br$). Solution [8] requires the scalar and vector t -channel exchanges; in the color space this is an exchange of the quantum numbers $c = 1 + 8$ (basing on the fit of only the meson sector, we cannot determine the ratio of the singlet and octet forces).

The data require confinement singularities both in the scalar and vector t -channels. The confinement singularity couplings appeared to be equal to each other, $b_S = |b_V|$. We do not know precisely the possible deviations from this equality: for such a study, more data are needed, first of all, data on radiative decays. The version with $|b_V| \ll b_S$ is, however, definitely excluded.

We pay special attention to the obtained wave functions. The problem is that the knowledge on the masses only is not enough for reconstructing the interaction — one should also know the wave functions of mesons (see the discussion in [6]). Because of that, a simultaneous presentation of the calculated levels and their wave functions is absolutely necessary for both the understanding of the results and the verification of the predictions.

We present the wave functions of the calculated $q\bar{q}$ states in Appendix 8.E.

8.5 **Appendix 8.A: Bottomonium States Found from Spectral Integral Equation and Radiative Transitions**

Here we present the results of the calculation for bottomonium states [9]: masses of bottomonia and partial widths of their radiative transitions.

Performing the fit, we suppose that the confinement interaction in this sector is the same as in the light quark sector.

8.5.1 Masses of the $b\bar{b}$ states

The data in the $b\bar{b}$ sector are described by two types of t -channel exchanges, the scalar and vector ones: $I \otimes I$, $\gamma_\mu \otimes \gamma_\mu$. The addition of pseudoscalar exchanges like $\gamma_5 \otimes \gamma_5$ does not improve the results.

The fitting procedure prefers the mass value $m_b = 4.5$ GeV for the constituent b -quark. This value looks quite reasonable if we take into account that the mass difference of the constituent and QCD quarks is of the order of 200–350 MeV (the QCD estimates [27] give the constraint $4.0 \leq m_b(QCD) \leq 4.5$ GeV).

For bottomonia we have data for two L -sectors only — $L = 1, 2$. These data are well described by instantaneous interactions with parameters common for both L -sectors:

$$a(b\bar{b}) + b(b\bar{b})r + ce^{-\mu_c(b\bar{b})r} + \frac{d(b\bar{b})}{r}e^{-\mu_d(b\bar{b})r}. \quad (8.47)$$

The parameters for scalar and vector exchange interactions, $I \otimes I$ and $\gamma_\mu \otimes \gamma_\mu$, are as follows (all values are in GeV units):

Interaction	$a(b\bar{b})$	$b(b\bar{b})$	$c(b\bar{b})$	$\mu_c(b\bar{b})$	$d(b\bar{b})$	$\mu_d(b\bar{b})$
$(I \otimes I)$	0.911	0.150	−0.377	0.401	−0.201	0.401
$(\gamma_\mu \otimes \gamma_\mu)$	1.178	−0.150	−1.356	0.201	0.500	0.001

(8.48)

As for the light quark sector, in the fitting procedure the confinement terms were used in the form $a \rightarrow aI_0^{(\text{coord})}(r, \mu_{\text{constant}} \rightarrow 0)$ and $br \rightarrow bI_1^{(\text{coord})}(r, \mu_{\text{linear}} \rightarrow 0)$ (functions $I_N^{(\text{coord})}(r, \mu_N)$ are given in (8.17)). The limits $\mu_{\text{constant}} \rightarrow 0$ and $\mu_{\text{linear}} \rightarrow 0$ mean that in the fitting procedure the parameters μ_{constant} and μ_{linear} are chosen to be of the order of 1–10 MeV.

In the solution [9] the vector-exchange forces $V_{\text{short}}^{V(b\bar{b})}(r) = 1.355 \exp(-0.5r) - 0.500/r$ (in GeV units) contain the one-gluon exchange term $-0.500/r$ which corresponds to a rather large coupling $\alpha_s \simeq 0.38$ fitting the data.

The masses of $b\bar{b}$ states for $n = 1, 2, 3, 4, 5, 6$ (experimental values and those obtained in the fit) are given below, in (8.49), (8.50) and (8.51). The bold numbers stand for the masses which are included in the fitting procedure. In parentheses we show the dominant wave for the $b\bar{b}$ state (S or D for 1^{--} and P or F for 2^{++}).

We have the following masses (in GeV) for 1^{--} states and for 2^{++} states:

1^{--}	Data	Fit	R^2	2^{++}	Data	Fit	R^2
$\Upsilon(1S)$	9.460	9.382 (<i>S</i>)	0.342	$\chi_{b2}(1P)$	9.912	9.911 (<i>P</i>)	0.956
$\Upsilon(2S)$	10.023	10.027 (<i>S</i>)	1.632	$\chi_{b2}(2P)$	10.268	10.262 (<i>P</i>)	2.782
$\Upsilon(1D)$	10.150	10.158 (<i>D</i>)	0.342	$\chi_{b2}(1F)$	—	10.347 (<i>F</i>)	0.956
$\Upsilon(3S)$	10.355	10.365 (<i>S</i>)	3.794	$\chi_{b2}(3P)$	—	10.535 (<i>P</i>)	5.361
$\Upsilon(2D)$	10.450	10.436 (<i>D</i>)	1.632	$\chi_{b2}(2F)$	—	10.592 (<i>F</i>)	2.782
$\Upsilon(4S)$	10.580	10.634 (<i>S</i>)	6.504	$\chi_{b2}(4P)$	—	10.773 (<i>P</i>)	8.573
$\Upsilon(3D)$	10.700	10.677 (<i>D</i>)	3.794	$\chi_{b2}(3F)$	—	10.813 (<i>F</i>)	5.361
$\Upsilon(5S)$	10.865	10.872 (<i>S</i>)	9.793	$\chi_{b2}(5F)$	—	10.994 (<i>P</i>)	18.995
$\Upsilon(4D)$	10.950	10.898 (<i>D</i>)	6.504	$\chi_{b2}(4F)$	—	11.020 (<i>F</i>)	8.573
$\Upsilon(6S)$	11.020	11.084 (<i>S</i>)	11.990	$\chi_{b2}(6F)$	—	11.196 (<i>P</i>)	13.978
$\Upsilon(5D)$	—	11.109 (<i>D</i>)	9.793	$\chi_{b2}(5F)$	—	11.221 (<i>F</i>)	18.995
$\Upsilon(6D)$	—	11.303 (<i>D</i>)	11.990	$\chi_{b2}(6F)$	—	11.411 (<i>F</i>)	13.978 ,

(8.49)

for 0^{-+} and 0^{++} states:

0^{-+}	Data	Fit	R^2	0^{++}	Data	Fit	R^2
$\eta_b(1S)$	9.300	9.322	0.922	$\chi_{b0}(1P)$	9.859	9.862	0.847
$\eta_b(2S)$	—	10.011	2.782	$\chi_{b0}(2P)$	10.232	10.236	2.632
$\eta_b(3S)$	—	10.355	5.781	$\chi_{b0}(3P)$	—	10.517	5.161
$\eta_b(4S)$	—	10.626	18.839	$\chi_{b0}(4P)$	—	10.759	8.053
$\eta_b(5S)$	—	10.864	13.699	$\chi_{b0}(5P)$	—	10.983	12.437
$\eta_b(6S)$	—	11.079	11.668	$\chi_{b0}(6P)$	—	11.185	19.969 ,

(8.50)

for 1^{++} and 1^{+-} states:

1^{++}	Data	Fit	R^2	1^{+-}	Data	Fit	R^2
$\chi_{b1}(1P)$	9.892	9.895	0.915	$h_b(1S)$	—	9.902	0.922
$\chi_{b1}(2P)$	10.255	10.252	2.777	$h_b(2S)$	—	10.255	2.782
$\chi_{b1}(3P)$	—	10.528	5.814	$h_b(3S)$	—	10.530	5.781
$\chi_{b1}(4P)$	—	10.767	18.944	$h_b(4S)$	—	10.768	18.839
$\chi_{b1}(5P)$	—	10.989	13.544	$h_b(5S)$	—	10.990	13.699
$\chi_{b1}(6P)$	—	11.191	11.702	$h_b(6S)$	—	11.192	11.668 .

(8.51)

8.5.2 Radiative decays $(b\bar{b})_{in} \rightarrow \gamma(b\bar{b})_{out}$

Figure 8.3 shows the radiative transitions which were included in the fitting procedure — the corresponding formulae are given in Chapter 7.

The fit resulted in the following values for the radiative decays of

Υ -mesons (partial widths are given in keV):

Process	Data	Fit	[28]	[29]
$\Upsilon(1S) \rightarrow \gamma\eta_{b0}(1S)$	—	0.0100	—	—
$\Upsilon(2S) \rightarrow \gamma\eta_{b0}(1S)$	—	0.0015	—	—
$\Upsilon(2S) \rightarrow \gamma\eta_{b0}(2S)$	—	0.0002	—	—
$\Upsilon(2S) \rightarrow \gamma\chi_{b0}(1P)$	1.7 ± 0.2	1.0669	1.62	1.41
$\Upsilon(2S) \rightarrow \gamma\chi_{b1}(1P)$	3.0 ± 0.5	2.3675	2.55	2.27
$\Upsilon(2S) \rightarrow \gamma\chi_{b2}(1P)$	3.1 ± 0.5	2.6674	2.51	2.24
$\Upsilon(3S) \rightarrow \gamma\eta_{b0}(1S)$	—	0.0007	—	—
$\Upsilon(3S) \rightarrow \gamma\eta_{b0}(2S)$	—	0.0000	—	—
$\Upsilon(3S) \rightarrow \gamma\eta_{b0}(3S)$	—	0.0001	—	—
$\Upsilon(3S) \rightarrow \gamma\chi_{b0}(2P)$	1.4 ± 0.2	1.3746	1.77	—
$\Upsilon(3S) \rightarrow \gamma\chi_{b1}(2P)$	3.0 ± 0.5	4.0831	2.88	—
$\Upsilon(3S) \rightarrow \gamma\chi_{b2}(2P)$	3.0 ± 0.5	4.7438	3.14	—

(8.52)

For illustration, in (8.52) we present the results of [28, 29].

The radiative decays of χ_{bJ} were not included in the fitting procedure. For the partial widths (in keV) the following predictions are given:

Process	Data	Fit
$\chi_{b0}(1P) \rightarrow \gamma\Upsilon(1S) < \Gamma_{\text{tot}}(\chi_{b0}(1P)) \cdot 6 \times 10^{-2}$		52.79
$\chi_{b1}(1P) \rightarrow \gamma\Upsilon(1S) \Gamma_{\text{tot}}(\chi_{b1}(1P)) \cdot (35 \pm 8) \times 10^{-2}$		63.77
$\chi_{b2}(1P) \rightarrow \gamma\Upsilon(1S) \Gamma_{\text{tot}}(\chi_{b2}(1P)) \cdot (22 \pm 4) \times 10^{-2}$		56.15
$\chi_{b0}(2P) \rightarrow \gamma\Upsilon(1S) \Gamma_{\text{tot}}(\chi_{b0}(2P)) \cdot (0.9 \pm 0.6) \times 10^{-2}$		9.25
$\chi_{b0}(2P) \rightarrow \gamma\Upsilon(2S) \Gamma_{\text{tot}}(\chi_{b0}(2P)) \cdot (4.6 \pm 2.1) \times 10^{-2}$		15.88
$\chi_{b1}(2P) \rightarrow \gamma\Upsilon(1S) \Gamma_{\text{tot}}(\chi_{b1}(2P)) \cdot (8.5 \pm 1.3) \times 10^{-2}$		16.85
$\chi_{b1}(2P) \rightarrow \gamma\Upsilon(2S) \Gamma_{\text{tot}}(\chi_{b1}(2P)) \cdot (21.0 \pm 4.0) \times 10^{-2}$		14.40
$\chi_{b2}(2P) \rightarrow \gamma\Upsilon(1S) \Gamma_{\text{tot}}(\chi_{b2}(2P)) \cdot (7.1 \pm 1.0) \times 10^{-2}$		20.58
$\chi_{b2}(2P) \rightarrow \gamma\Upsilon(2S) \Gamma_{\text{tot}}(\chi_{b2}(2P)) \cdot (16.2 \pm 2.4) \times 10^{-2}$		18.25
$\chi_{b0}(3P) \rightarrow \gamma\Upsilon(1S) - - -$		3.56
$\chi_{b1}(3P) \rightarrow \gamma\Upsilon(1S) - - -$		7.06
$\chi_{b2}(3P) \rightarrow \gamma\Upsilon(1S) - - -$		8.11
$\chi_{b0}(3P) \rightarrow \gamma\Upsilon(2S) - - -$		1.86
$\chi_{b1}(3P) \rightarrow \gamma\Upsilon(2S) - - -$		3.88
$\chi_{b2}(3P) \rightarrow \gamma\Upsilon(2S) - - -$		4.59
$\chi_{b0}(3P) \rightarrow \gamma\Upsilon(3S) - - -$		10.37
$\chi_{b1}(3P) \rightarrow \gamma\Upsilon(3S) - - -$		15.85
$\chi_{b2}(3P) \rightarrow \gamma\Upsilon(3S) - - -$		13.81

(8.53)

Let us stress that the total widths $\Gamma_{\text{tot}}(\chi_{bJ}(1P))$ and $\Gamma_{\text{tot}}(\chi_{bJ}(2P))$, with $J = 0, 1, 2$, have not been measured yet.

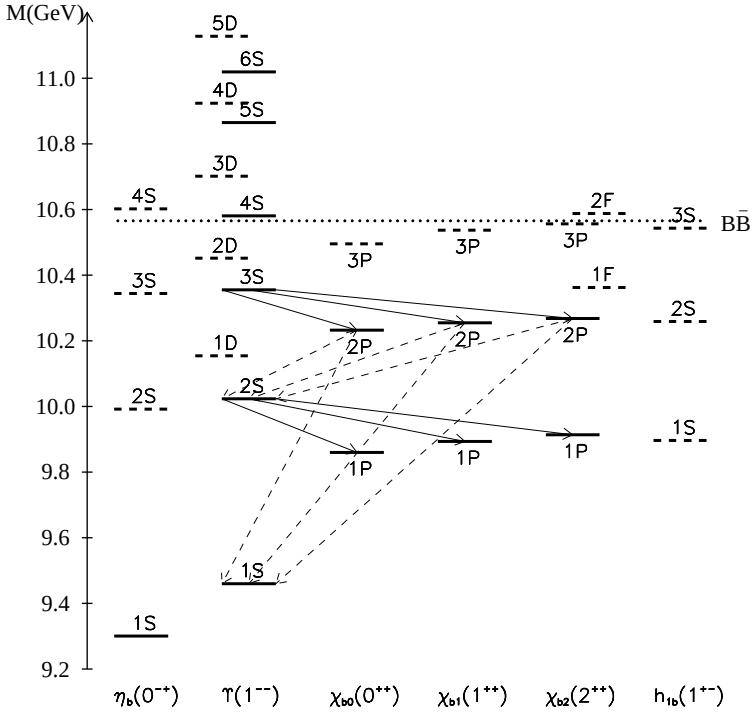


Fig. 8.3 Radiative decays of the bottomonium systems which were taken into account in the fit [9] are represented by solid lines. The dashed lines show radiative transitions with the known ratios for the branchings $\text{Br}[\chi_{bJ}(2P) \rightarrow \gamma \Upsilon(2S)]/\text{Br}[\chi_{bJ}(2P) \rightarrow \gamma \Upsilon(1S)]$; these ratios are not included in the fit.

The calculations performed on the basis of (8.53) give us the following estimates for the total widths (in keV):

$$\begin{aligned}
 \Gamma_{\text{tot}}(\chi_{b0}(1P)) &< 730, \\
 \Gamma_{\text{tot}}(\chi_{b1}(1P)) &\simeq 120 - 200, \\
 \Gamma_{\text{tot}}(\chi_{b2}(1P)) &\simeq 180 - 270, \\
 \Gamma_{\text{tot}}(\chi_{b0}(2P)) &\simeq 180 - 480, \\
 \Gamma_{\text{tot}}(\chi_{b1}(2P)) &\simeq 50 - 80, \\
 \Gamma_{\text{tot}}(\chi_{b2}(2P)) &\simeq 70 - 120.
 \end{aligned} \tag{8.54}$$

The values for the partial widths of the radiative decays of η_{b0} -mesons are given by the fit

Process	Data	Fit
$\eta_{b0}(2S) \rightarrow \gamma \Upsilon(1S)$	—	0.20
$\eta_{b0}(3S) \rightarrow \gamma \Upsilon(1S)$	—	0.18
$\eta_{b0}(3S) \rightarrow \gamma \Upsilon(2S)$	—	0.02

(8.55)

8.5.3 The $b\bar{b}$ component of the photon wave function and the $e^+e^- \rightarrow V(b\bar{b})$ and $b\bar{b}$ -meson $\rightarrow \gamma\gamma$ transitions

In the $b\bar{b}$ sector we have a large number of observed 1^{--} -states in the $e^+e^- \rightarrow V(b\bar{b})$ reaction (states with $n \leq 6$). This makes it possible to give a reliable determination of the photon vertex $\gamma^* \rightarrow b\bar{b}$ and to carry out subsequent calculations of the decay widths $b\bar{b}$ -meson $\rightarrow \gamma\gamma$.

8.5.3.1 Determination of the photon vertex $\gamma^* \rightarrow b\bar{b}$

The points on which the determination of the quark–antiquark vertex of the photon is based were given in Chapter 6. Here we remind some of them which are needed for our present considerations. The problem is that the data for extracting quark components are of different types in the heavy and light quark sectors. In the light quark sector the only reliably measured reactions $e^+e^- \rightarrow V$ are productions of ρ^0 , ω , and $\phi(1020)$, but there is a good set of data for $\gamma\gamma^*(Q^2) \rightarrow \pi^0$ [30], $\gamma\gamma^*(Q^2) \rightarrow \eta$ [30, 31, 32] and $\gamma\gamma^*(Q^2) \rightarrow \eta'$ [30, 31, 32, 33] at $Q^2 \equiv -q^2 \leq 2 \text{ GeV}^2$. Because of that flexible fitting strategies should be applied to these sectors.

To describe the transition $b\bar{b} \rightarrow \gamma^*$ we introduce the $b\bar{b}$ -component of the photon wave function as follows:

$$\frac{G_{\gamma \rightarrow b\bar{b}}(s)}{s - q^2} = \Psi_{\gamma^*(q^2) \rightarrow b\bar{b}}(s) . \quad (8.56)$$

Let us emphasise that such a wave function is determined at $s \gtrsim 4m_b^2$.

The vertex function $G_{\gamma \rightarrow b\bar{b}}(s)$ at $s \sim 4m_b^2$ is the superposition of vertices of the $V(n)$ -mesons:

$$G_{\gamma \rightarrow b\bar{b}}(s) \simeq \sum_n C_n(b\bar{b}) G_{V(n)}(s) , \quad s \sim 4m_b^2 , \quad (8.57)$$

where n is the radial quantum number of the V meson and C_n are the coefficients which should be determined in the fit.

At large s the vertex $b\bar{b} \rightarrow \gamma^*$ is a point-like one:

$$G_{\gamma \rightarrow b\bar{b}}(s) \simeq 1 \quad \text{at} \quad s > s_0 . \quad (8.58)$$

The parameter s_0 can be determined from the data on e^+e^- -annihilation into hadrons: it defines the energy range where the ratio $R(s) = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ reaches a regime of constant behaviour above the threshold of the production of $b\bar{b}$ -mesons. The data [43] give us $s_0 \sim (100\text{--}150) \text{ GeV}^2$ for the $b\bar{b}$ production.

The reactions $e^+e^- \rightarrow \gamma^* \rightarrow \Upsilon_n$ determine promptly the $b\bar{b}$ -component of the photon wave function. The transition $\gamma^* \rightarrow \Upsilon_n$ contains the loop diagram which is defined by the convolution of the vector meson wave function and the vertex $G_{\gamma \rightarrow b\bar{b}}$. One should take into account that the transition $\gamma \rightarrow b\bar{b}$ is determined by two spin structures, γ_α and $\frac{3}{2} \left[k_\alpha \hat{k} - \frac{1}{3} k^2 \gamma_\alpha^\perp \right]$, and, correspondingly, by two vertices:

$$\gamma_\alpha G_{\gamma \rightarrow b\bar{b}}^{(S)}(s), \quad \gamma_\xi X_{\xi\alpha}^{(2)} G_{\gamma \rightarrow b\bar{b}}^{(D)}(s). \quad (8.59)$$

This means that we take into account the normal quark-photon interaction γ_α , as well as the contribution of the anomalous magnetic moment.

For the vertex function of the transition $\gamma \rightarrow b\bar{b}$ the following fitting formula was used:

$$G_{\gamma \rightarrow b\bar{b}}^{(S)}(s) = \sum_{n=1}^6 C_{nS}(b\bar{b}) G_{V(nS)}(s) + \frac{1}{1 + \exp(-\beta_\gamma(b\bar{b})[s - s_0(b\bar{b})])},$$

$$G_{\gamma \rightarrow b\bar{b}}^{(D)}(s) = \sum_{n=1}^6 C_{nD}(b\bar{b}) G_{V(nD)}(s), \quad (8.60)$$

where $G_{V(nS)}(s) = \psi_n^{(101)}(s)(s - M_{V(nS)}^2)$ and $G_{V(nD)}(s) = \psi_n^{(121)}(s)(s - M_{V(nD)}^2)$.

The fitting to the reactions $\gamma^* \rightarrow b\bar{b}$ results in the following parameters C_n, β_γ, s_0 for $G_{\gamma \rightarrow b\bar{b}}^{S,D}(s)$, see (8.60) (all values in GeV units):

$$\begin{aligned} C_{1S}(b\bar{b}) &= -0.800, & C_{2S}(b\bar{b}) &= -0.303, \\ C_{3S}(b\bar{b}) &= 0.074, & C_{4S}(b\bar{b}) &= 0.197, \\ C_{5S}(b\bar{b}) &= -0.781, & C_{6S}(b\bar{b}) &= 2.000, \\ C_{1D}(b\bar{b}) &= -0.328, & C_{2D}(b\bar{b}) &= 0.233, \\ \beta_\gamma(b\bar{b}) &= 2.85, & s_0(b\bar{b}) &= 18.79. \end{aligned} \quad (8.61)$$

Experimental values of partial widths included in the fitting procedure as an input together with those obtained in the fitting procedure are shown

below:

Process	Data	Fit	[34]	
$\Upsilon(1S) \rightarrow e^+e^-$	1.314 ± 0.029	1.313	1.01	
$\Upsilon(2S) \rightarrow e^+e^-$	0.576 ± 0.024	0.575	0.35	
$\Upsilon(3S) \rightarrow e^+e^-$	0.476 ± 0.076	0.476	0.25	(8.62)
$\Upsilon(4S) \rightarrow e^+e^-$	0.248 ± 0.031	0.248	0.22	
$\Upsilon(5S) \rightarrow e^+e^-$	0.31 ± 0.07	0.310	0.18	
$\Upsilon(6S) \rightarrow e^+e^-$	0.130 ± 0.03	0.130	0.14	

Here the last column demonstrates the results of [34].

8.5.3.2 Photon-photon decays of $b\bar{b}$ -states

The predictions for the two-photon partial widths $\eta_{b0} \rightarrow \gamma\gamma$, $\chi_{b0} \rightarrow \gamma\gamma$, $\chi_{b2} \rightarrow \gamma\gamma$ are as follows [9]:

Process	Fit	[35]	[36]	[37]	[38]	[39]	[40]	
$\eta_{b0}(1S) \rightarrow \gamma\gamma$	1.851	0.35	0.22	0.46	0.46	0.45	0.17	
$\eta_{b0}(2S) \rightarrow \gamma\gamma$	2.296	0.11	—	0.20	0.21	0.13	—	
$\eta_{b0}(3S) \rightarrow \gamma\gamma$	2.547	0.10	0.084	—	—	—	—	
$\chi_{b0}(1P) \rightarrow \gamma\gamma$	0.029	0.038	0.024	0.080	0.043	—	—	(8.63)
$\chi_{b0}(2P) \rightarrow \gamma\gamma$	0.028	0.029	0.026	—	—	—	—	
$\chi_{b0}(3P) \rightarrow \gamma\gamma$	0.027	—	—	—	—	—	—	
$\chi_{b2}(1P) \rightarrow \gamma\gamma$	0.020	0.0080	0.0056	0.0080	0.0074	—	—	
$\chi_{b2}(2P) \rightarrow \gamma\gamma$	0.020	0.0060	0.0068	—	—	—	—	
$\chi_{b2}(3P) \rightarrow \gamma\gamma$	0.019	—	—	—	—	—	—	

Comparisons with other calculations are carried out, data for $\gamma\gamma$ decays are absent.

Miniconclusion

The spectral integral method, being in fact a version of the dispersion relation approach, allows us to describe reasonably well the bottomonium sector: the $b\bar{b}$ -levels and their radiative transitions such as $(b\bar{b})_{\text{in}} \rightarrow \gamma + (b\bar{b})_{\text{out}}$, $e^+e^- \rightarrow V(b\bar{b})$.

As was stressed in [9], the performed fit faces ambiguities when reconstructing the $b\bar{b}$ interaction in the soft region; this is related to the scarcity of the radiative decay data. To restore the $b\bar{b}$ interaction, one needs more data, in particular, on the two-photon reactions: $\gamma\gamma \rightarrow b\bar{b}\text{-meson}$, including the bottomonium production by virtual photons in $\gamma\gamma^*$ and $\gamma^*\gamma^*$ collisions.

The one-gluon coupling α_s , obtained in the fit, is not small. This reflects the importance of strong interactions in the $b\bar{b}$ sector.

8.6 Appendix 8.B: Charmonium States

In [10], the $c\bar{c}$ levels and their wave functions were calculated, using two types of the t -channel exchanges – those by scalar and vector states: $(I \otimes I)_{t\text{-channel}}$ and $(\gamma_\mu \otimes \gamma_\mu)_{t\text{-channel}}$. The calculations of the $c\bar{c}$ -systems have been carried out, similarly to the consideration of bottomonia [9], supposing the following interactions of quasi-potential type:

$$a(c\bar{c}) + b(c\bar{c})r + c(c\bar{c})e^{-\mu_c(c\bar{c})r} + \frac{d(c\bar{c})}{r}e^{-\mu_d(c\bar{c})r}. \quad (8.64)$$

The interaction parameters obtained in the fit are as follows (all values in GeV units):

Interaction	$a(c\bar{c})$	$b(c\bar{c})$	$c(c\bar{c})$	$\mu_c(c\bar{c})$	$d(c\bar{c})$	$\mu_d(c\bar{c})$	
$(I \otimes I)$	-0.300	0.150	-0.044	0.351	-0.245	0.201	(8.65)
$(\gamma_\mu \otimes \gamma_\mu)$	1.000	-0.150	-1.600	0.201	0.544	0.001	

Following the results of [8], the scalar and vector confinement forces have been included into the fit with $b_S = -b_V = 0.150 \text{ GeV}^2$.

The α_s coupling, being determined by the one-gluon exchange forces, is of the same order as in the $q\bar{q}$ and $c\bar{c}$ sectors: $\alpha_s = 3/4 \cdot d_V \simeq 0.38$.

The mass of the constituent c -quark is taken to be $m_c = 1.25 \text{ GeV}$. This mass value is consistent with the value provided by the heavy-quark effective theory [41, 42]: $1.0 \leq m_c \leq 1.4 \text{ GeV}$; a slightly larger interval for m_c is given by lattice calculations, $0.93 \leq m_c \leq 1.59 \text{ GeV}$, see [42] and references therein. The compilation [43] gives us $1.15 \leq m_c \leq 1.35 \text{ GeV}$.

8.6.0.3 Masses of $c\bar{c}$ states

The fitting procedure results in the following masses (in GeV units) for 1^{--} and 2^{++} states ($L = 1, 3$):

1^{--}	Data	Fit	R^2	2^{++}	Data	Fit	R^2
J/ψ	3.097	3.115 (<i>S</i>)	2.060	$\chi_{c2}(1P)$	3.556	3.508 (<i>P</i>)	5.008
$\psi(2S)$	3.686	3.635 (<i>S</i>)	6.897	$\chi_{c2}(2P)$	3.941	3.898 (<i>P</i>)	11.085
$\psi(1D)$	3.770	3.747 (<i>D</i>)	2.060	$\chi_{c2}(1F)$	—	3.946 (<i>F</i>)	5.008
$\psi(3S)$	4.040	4.009 (<i>S</i>)	12.636	$\chi_{c2}(3P)$	—	4.222 (<i>P</i>)	14.928
$\psi(2D)$	4.160	4.087 (<i>D</i>)	6.897	$\chi_{c2}(2F)$	—	4.260 (<i>F</i>)	11.085
$\psi(4S)$	4.415	4.290 (<i>S</i>)	17.227	$\chi_{c2}(4P)$	—	4.546 (<i>P</i>)	41.793
$\psi(3D)$	—	4.390 (<i>D</i>)	12.636	$\chi_{c2}(3F)$	—	4.558 (<i>F</i>)	14.928
$\psi(5S)$	—	4.566 (<i>S</i>)	32.968	$\chi_{c2}(5P)$	—	4.803 (<i>P</i>)	12.018
$\psi(4D)$	—	4.711 (<i>D</i>)	17.227	$\chi_{c2}(4F)$	—	4.937 (<i>F</i>)	41.793
$\psi(6S)$	—	4.993 (<i>S</i>)	23.372	$\chi_{c2}(6P)$	—	5.079 (<i>P</i>)	10.590
$\psi(5D)$	—	5.136 (<i>D</i>)	32.968	$\chi_{c2}(5F)$	—	5.429 (<i>F</i>)	12.018
$\psi(6D)$	—	5.819 (<i>D</i>)	23.372	$\chi_{c2}(6F)$	—	6.065 (<i>F</i>)	10.590

(8.66)

Bold numbers stand for the masses included in the fit as an input. The states 1^{--} are the mixture of S and D waves (in parentheses the dominant waves are shown, with indices (nS) and (nD)). The last column gives us the mean radii squared: R^2 GeV $^{-2}$.

For the other considered states the fit resulted in the following masses and R^2 (all values in GeV units).

For 0^{-+} states ($L = 0$) and for 0^{++} states ($L = 1$):

0^{-+}	Data	Fit	R^2	0^{++}	Data	Fit	R^2
$\eta_c(1S)$	2.979	3.016	1.682	$\chi_{c0}(1P)$	3.415	3.473	3.401
$\eta_c(2S)$	3.594	3.574	6.207	$\chi_{c0}(2P)$	—	3.850	8.777
$\eta_c(3S)$	—	3.958	11.813	$\chi_{c0}(3P)$	—	4.173	15.115
$\eta_c(4S)$	—	4.265	16.604	$\chi_{c0}(4P)$	—	4.493	22.156
$\eta_c(5S)$	—	4.555	30.919	$\chi_{c0}(5P)$	—	4.795	18.133
$\eta_c(6S)$	—	4.881	22.831	$\chi_{c0}(6P)$	—	5.067	13.806

(8.67)

For 1^{++} states ($L = 1$) and for 1^{+-} states ($L = 1$):

1^{++}	Data	Fit	R^2	1^{+-}	Data	Fit	R^2
$\chi_{c1}(1P)$	3.510	3.503	4.234	$h_c(1P)$	3.526	3.522	4.447
$\chi_{c1}(2P)$	3.872	3.880	9.861	$h_c(2P)$	—	4.013	10.199
$\chi_{c1}(3P)$	—	3.989	17.628	$h_c(3P)$	—	4.385	14.886
$\chi_{c1}(4P)$	—	4.228	24.460	$h_c(4P)$	—	4.696	19.976
$\chi_{c1}(5P)$	—	4.575	18.407	$h_c(5P)$	—	5.078	24.106
$\chi_{c1}(6P)$	—	4.819	13.345	$h_c(6P)$	—	5.531	15.336

(8.68)

for 2^{-+} states ($L = 2$):

2^{-+}	Data	Fit	R^2
$\eta_{c2}(1D)$	—	3.742	7.721
$\eta_{c2}(2D)$	—	4.087	14.387
$\eta_{c2}(3D)$	—	4.397	22.729
$\eta_{c2}(4D)$	—	4.713	18.708
$\eta_{c2}(5D)$	—	5.084	14.024
$\eta_{c2}(6D)$	—	5.546	12.227

(8.69)

In Fig. 8.4, the levels found as solutions of spectral integral equation are shown for the mass region $M < 4.5$ GeV. The wave functions may be found in [10].

8.6.1 Radiative transitions $(c\bar{c})_{in} \rightarrow \gamma + (c\bar{c})_{out}$

In Fig. 8.4 we show radiative decays which have been accounted for in the fitting procedure [10], the corresponding formulae are presented in Chapter 7 (see also [44]). For the levels below $D\bar{D}$ threshold the experimental data [43, 45, 46, 47] and the values of widths obtained in the fit [10] are as follows (in keV):

Note that a 20% accuracy is allowed for the transitions $\psi(2S) \rightarrow \gamma\chi_{cJ}(1P)$ and a 30% one for $\chi_{cJ}(1P) \rightarrow \gamma\psi(1S)$. The fit predicts also the widths of the decays $\eta_{c0} \rightarrow \gamma J/\psi$ and $\psi(2S) \rightarrow \gamma\eta_{c0}(2S)$. The calculated values in (8.70) agree rather reasonably with the data.

The predictions of widths of the levels above the $D\bar{D}$ threshold (see Fig. 8.4) are (in keV):

Process	Data	Fit	
$\chi_{c0}(2P) \rightarrow \gamma J/\psi$	—	0.468	
$\chi_{c1}(2P) \rightarrow \gamma J/\psi$	—	28.797	
$\chi_{c2}(2P) \rightarrow \gamma J/\psi$	—	31.331	
$\chi_{c0}(2P) \rightarrow \gamma\psi(2S)$	—	92.450	(8.71)
$\chi_{c1}(2P) \rightarrow \gamma\psi(2S)$	—	290.379	
$\chi_{c2}(2P) \rightarrow \gamma\psi(2S)$	—	197.162	

8.6.2 The $c\bar{c}$ component of the photon wave function and two-photon radiative decays

In the fitting procedure the vertex of the transition $\gamma \rightarrow c\bar{c}$ is approximated by the following formula:

$$G_{\gamma \rightarrow c\bar{c}(S)}(s) = \sum_{n=1}^6 C_{nS}(c\bar{c}) G_{V(nS)}(s) + \frac{1}{1 + \exp[-\beta_\gamma(c\bar{c})(s - s_0(c\bar{c}))]} ,$$

$$G_{\gamma \rightarrow c\bar{c}(D)}(s) = \sum_{n=1}^2 C_{nD}(c\bar{c}) G_{V(nD)}(s) , \quad (8.72)$$

where $G_{V(nS)}(s)$ is the vertex for the transition $\psi(nS) \rightarrow c\bar{c}$ and $G_{V(nD)}(s)$ is the vertex for the transition $\psi(nD) \rightarrow c\bar{c}$, see [9] for the details. The following parameters $C_{nS}(c\bar{c})$, $C_{nD}(c\bar{c})$, $\beta_\gamma(c\bar{c})$, $s_0(c\bar{c})$ have been found for the solution (in GeV):

Fitting results :

$$\begin{aligned}
C_{1S}(c\bar{c}) &= -3.852, & C_{2S}(c\bar{c}) &= 0.476, \\
C_{3S}(c\bar{c}) &= 0.325, & C_{4S}(c\bar{c}) &= 0.667, \\
C_{5S}(c\bar{c}) &= -2.571, & C_{6S}(c\bar{c}) &= -0.707, \\
C_{1D}(c\bar{c}) &= 0.080, & C_{2D}(c\bar{c}) &= -0.082, \\
\beta_\gamma(c\bar{c}) &= 2.85, & s_0(c\bar{c}) &= 18.79.
\end{aligned} \quad (8.73)$$

The experimental values of partial widths [43, 48, 49, 50, 51] are shown below (in keV) together with the widths obtained in the fitting procedure:

Process	Data	Fit
$J/\psi(1S) \rightarrow e^+e^-$	5.40 ± 0.22	5.403
$\psi(2S) \rightarrow e^+e^-$	2.14 ± 0.21	2.142
$\psi(1D) \rightarrow e^+e^-$	0.24 ± 0.05	0.240
$\psi(3S) \rightarrow e^+e^-$	0.75 ± 0.15	0.749
$\psi(2D) \rightarrow e^+e^-$	0.47 ± 0.10	0.469
$\psi(4S) \rightarrow e^+e^-$	0.77 ± 0.23	0.770

(8.74)

With the vertices determined for $G_{\gamma \rightarrow c\bar{c}}(s)$ one can obtain the widths of the two-photon decays. The comparison of experimentally measured widths with those obtained in calculations [10] is given as follows:

Process	Data	Fit
$\eta_{c0}(1S) \rightarrow \gamma\gamma$	7.0 ± 0.9	7.002
$\chi_{c0}(1P) \rightarrow \gamma\gamma$	2.6 ± 0.5	2.578
$\chi_{c2}(1P) \rightarrow \gamma\gamma$	$1.02 \pm 0.40 \pm 0.17(\text{L3})$ $1.76 \pm 0.47 \pm 0.40(\text{OPAL})$ $1.08 \pm 0.30 \pm 0.26(\text{CLEO})$ $0.33 \pm 0.08 \pm 0.06(\text{E760})$	0.068

(8.75)

Let us emphasise that the data do not tell us anything definite about the width $\chi_{c2}(3556) \rightarrow \gamma\gamma$. In the reaction $p\bar{p} \rightarrow \gamma\gamma$ the value $\Gamma(\chi_2(3556) \rightarrow \gamma\gamma) = 0.32 \pm 0.080 \pm 0.055$ keV was obtained in [51], while in direct measurements such as e^+e^- annihilation the width is much larger: $1.02 \pm 0.40 \pm 0.17$ keV [48], $1.76 \pm 0.47 \pm 0.40$ keV [49], $1.08 \pm 0.30 \pm 0.26$ keV [50]. The compilation [43] provides us with a value close to that of [51]. The value found in the fit [10] agrees with data reported by [48, 49, 50] and contradicts those from [51].

The predictions of widths $c\bar{c} \rightarrow \gamma\gamma$ for the levels below 4 GeV are as follows (see Table 8.1 summarising the world data together with our results):

Process	Data	Fit
$\eta_{c0}(2S) \rightarrow \gamma\gamma$	—	12.289
$\chi_{c0}(2P) \rightarrow \gamma\gamma$	—	2.276
$\chi_{c2}(2P) \rightarrow \gamma\gamma$	—	0.061

(8.76)

Table 8.1 Comparison of data on the decay widths for $(c\bar{c})_{\text{in}} \rightarrow \gamma + (c\bar{c})_{\text{out}}$, $\psi \rightarrow e^+e^-$ and $\psi \rightarrow e^+e^-$ with our results and calculations of other groups (the width is given in keV).

Decay	Data	Fit	LS(F)[52]	LS(C)[52]	RM(S)[28]	RM(V)[28]	NR[53]
$J/\psi(1S) \rightarrow \eta_{c0}(1S)\gamma$	1.1 ± 0.3	1.4	1.7–1.3	1.7–1.4	3.35	2.66	1.21
$\psi(2S) \rightarrow \chi_{c0}(1P)\gamma$	26 ± 4	12.2	31–47	26–31	31	32	19.4
$\psi(2S) \rightarrow \chi_{c1}(1P)\gamma$	25 ± 4	31.1	58–49	63–50	36	48	34.8
$\psi(2S) \rightarrow \chi_{c2}(1P)\gamma$	20 ± 4	40.2	48–47	51–49	60	35	29.3
$\psi(2S) \rightarrow \eta_{c0}(1S)\gamma$	0.8 ± 0.2	0.37	11–10	10–13	6	1.3	4.47
$\chi_{c0}(1P) \rightarrow J/\psi(1S)\gamma$	165 ± 50	273.8	130–96	143–110	140	119	147
$\chi_{c1}(1P) \rightarrow J/\psi(1S)\gamma$	295 ± 90	391.8	390–399	426–434	250	230	287
$\chi_{c2}(1P) \rightarrow J/\psi(1S)\gamma$	390 ± 120	312.3	218–195	240–218	270	347	393
Decay	Data	Fit	LS(F)[52]	LS(C)[52]	RM(S)[28]	RM(V)[28]	NR[53]
$J/\psi(1S) \rightarrow e^+e^-$	5.40 ± 0.22	5.403	5.26	5.26	8.05	9.21	12.2
$\psi(2S) \rightarrow e^+e^-$	2.14 ± 0.21	2.142	2.8–2.5	2.9–2.7	4.30	5.87	4.63
$\psi(1D) \rightarrow e^+e^-$	0.24 ± 0.05	0.240	2.0–1.6	2.1–1.8	3.05	4.81	3.20
$\psi(3S) \rightarrow e^+e^-$	0.75 ± 0.15	0.749	1.4–1.0	1.6–1.3	2.16	3.95	2.41
$\psi(2D) \rightarrow e^+e^-$	0.47 ± 0.10	0.469	—	—	—	—	—
$\psi(4S) \rightarrow e^+e^-$	0.77 ± 0.23	0.770	—	—	—	—	—
Decay	Data	Fit	LS [52]	[35]	[36]	[56]	[38]
$\eta_c(1S) \rightarrow \gamma\gamma$	7.0 ± 0.9	7.002	6.2–6.3 (F,C)	5.5	3.5	10.9	7.8
$\eta_c(2S) \rightarrow \gamma\gamma$	—	12.278	—	1.8	1.38	—	3.5
$\chi_{c0}(1P) \rightarrow \gamma\gamma$	2.6 ± 0.5	2.578	1.5–1.8 (F,C)	2.9	1.39	6.4	2.5
$\chi_{c0}(2P) \rightarrow \gamma\gamma$	—	2.276	—	1.9	1.11	—	—
$\chi_{c2}(1P) \rightarrow \gamma\gamma$	$1.02 \pm 0.40 \pm 0.17$ [48] $1.76 \pm 0.47 \pm 0.40$ [49] $1.08 \pm 0.30 \pm 0.26$ [50] $0.33 \pm 0.08 \pm 0.06$ [51]	0.069	0.3–0.4 (F,C)	0.50	0.44	0.57	0.28
$\chi_{c2}(2P) \rightarrow \gamma\gamma$	—	0.061	—	0.52	0.48	—	—

In [52], the calculated widths depend on a chosen gauge for the gluon exchange interaction — we demonstrate the results obtained for both the Feynman (F) and Coulomb (C) gauges.

In [28], the $c\bar{c}$ system was studied in terms of scalar (S) and vector (V) confinement forces — both versions are presented above. The results obtained in the non-relativistic approach to the $c\bar{c}$ system [53] are also shown.

There is a serious discrepancy between the data and the calculated values of $\psi(nS) \rightarrow e^+e^-$ in both the relativistic [28, 52] and the non-relativistic [53] approaches (in [52] the width of the transition $J/\psi \rightarrow e^+e^-$ was fixed using a subtraction parameter). The reason is that in [28, 52, 53] the soft interaction of quarks was not accounted for. In fact, the necessity of taking into consideration the low-energy quark interaction was understood decades ago; still, this procedure has not become commonly accepted even for light quarks (see, for example, [54, 55]).

Miniconclusion

The spectral integral technique gives a possibility to perform a successful description of both the $c\bar{c}$ levels and their radial excitation transitions.

However, we should realise that a good description of the observed $c\bar{c}$ levels obtained in the fit [10] does not mean a reliable restoration of the interaction at large distances: for this task we need much more data for the highly excited charmonium states.

Concerning short-range interactions, let us emphasise once more the equality of α_s obtained in fits of $q\bar{q}$, $b\bar{b}$ and $c\bar{c}$ states: this fact indicates that in the strong interaction region α_s becomes frozen: $\alpha_s \simeq 0.4$.

8.7 Appendix 8.C: The Fierz Transformation and the Structure of the t -Channel Exchanges

The t -channel interaction operator $\hat{V}(s, s', (kk'))$ can be decomposed into s -channel terms by using the Fierz transformation:

$$\hat{V}(s, s', (kk')) = \sum_I \sum_c \hat{V}_I^{(0)}(s, s', (kk')) C_{Ic}(\hat{F}_c \otimes \hat{F}_c), \quad (8.77)$$

where C_{Ic} are coefficients of the Fierz matrix:

$$C_{Ic} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & -\frac{1}{2} & 0 & \frac{1}{2} & -1 \\ \frac{3}{2} & 0 & -\frac{1}{2} & 0 & \frac{3}{2} \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}. \quad (8.78)$$

Denoting

$$V_c(s, s', (kk')) = \sum_I \hat{V}_I^{(0)}(s, s', (kk')) C_{Ic}, \quad (8.79)$$

we have

$$\begin{aligned} \hat{V}(s, s', (kk')) &= \sum_c (\hat{F}_c \otimes \hat{F}_c) V_c(s, s', (kk')) \\ &= (\mathbf{I} \otimes \mathbf{I}) V_S(s, s', (kk')) + (\gamma_\mu \otimes \gamma_\mu) V_V(s, s', (kk')) \\ &\quad + (i\sigma_{\mu\nu} \otimes i\sigma_{\mu\nu}) V_T(s, s', (kk')) \\ &\quad + (i\gamma_\mu \gamma_\nu \otimes i\gamma_\mu \gamma_\nu) V_A(s, s', (kk')) + (\gamma_5 \otimes \gamma_5) V_P(s, s', (kk')). \end{aligned} \quad (8.80)$$

Let us multiply Eq. (8.2) by the operator $Q_{\mu_1 \dots \mu_J}^{(S, L, J)}(k)$ and convolute over the spin-momentum indices:

$$\begin{aligned} &(s - M^2) \text{Sp} \left[\hat{\Psi}_{(n)\mu_1 \dots \mu_J}^{(S, L, J)}(k) (\hat{k}_1 + m) Q_{\mu_1 \dots \mu_J}^{(S, L, J)}(k) (-\hat{k}_2 + m) \right] \\ &= \sum_c \text{Sp} \left[\hat{F}_c (\hat{k}_1 + m) Q_{\mu_1 \dots \mu_J}^{(S, L, J)}(k) (-\hat{k}_2 + m) \right] \int \frac{d^3 k'}{(2\pi)^3 k'_0} V_c(s, s', (kk')) \\ &\quad \times \text{Sp} \left[(\hat{k}'_1 + m') \hat{F}_c (-\hat{k}'_2 + m') \hat{\Psi}_{(n)\mu_1 \dots \mu_J}^{(S, L, J)}(k') \right]. \end{aligned} \quad (8.81)$$

(i) The structure of pseudoscalar, scalar and vector exchanges.

The loop diagram that includes the interaction is given by the expression

$$\begin{aligned} &\text{Sp}[G_{\mu_1 \dots \mu_J}^{(S, L, J)}(m + \hat{k}_1) O_I(m + \hat{k}'_1) G_{\nu_1 \dots \nu_J}^{(S, L, J)}(m - \hat{k}'_2) O_I(m - \hat{k}_2)] \\ &= V_I^{(S, L, J)}(-1)^J O_{\nu_1 \dots \nu_J}^{\mu_1 \dots \mu_J}, \end{aligned} \quad (8.82)$$

where k_1, k_2 and k'_1, k'_2 are the momenta of particles before and after the interaction, respectively, and the operators O_I are given by (8.11).

For scalar, pseudoscalar and vector exchanges we obtain for the singlet ($S = 0$) states

$$\begin{aligned} V_I^{(0, J, J)} &= \sqrt{ss'} (4z\kappa - 4m^2 - \sqrt{ss'}) \kappa^J P_J(z), \\ V_{\gamma_5}^{(0, J, J)} &= \sqrt{ss'} (4z\kappa + 4m^2 - \sqrt{ss'}) \kappa^J P_J(z), \\ V_{\gamma_\mu}^{(0, J, J)} &= \sqrt{ss'} (4\sqrt{ss'} - 8m^2) \kappa^J P_J(z). \end{aligned} \quad (8.83)$$

Here $P_J(z)$ are Legendre polynomials depending on the angle between the final and initial particles and

$$\kappa = |\mathbf{k}||\mathbf{k}'| . \quad (8.84)$$

Near the threshold, the factor $\kappa = |\mathbf{k}||\mathbf{k}'|$ occurs in the pseudoscalar interaction in a higher order than in the scalar and vector interactions, thus suppressing the pseudoscalar contribution, and thus playing a minor role for mesons consisting of heavy quarks. In the lowest order of $|\mathbf{k}||\mathbf{k}'|$ the scalar and vector interactions are of equal absolute value but have opposite signs.

To obtain the expressions for triplet states, let us first calculate the trace with vertex functions taken as γ_μ . The general expression can be obtained by the convolution of the trace operators:

$$\begin{aligned} & \text{Sp}[\gamma_\mu (m + \hat{k}_1) O_I (m + \hat{k}'_1) \gamma_\nu (m - \hat{k}'_2) O_I (m - \hat{k}_2)] \\ &= (a_1^I + z\kappa a_2^I) g_{\mu\nu}^\perp + a_3^I k_\mu^\perp k_\nu^\perp + a_4^I k_\mu'^\perp k_\nu'^\perp \\ &+ (a_5^I + z\kappa a_6^I) k_\mu^\perp k_\nu'^\perp + a_7^I (k_\mu^\perp k_\nu'^\perp - k_\mu'^\perp k_\nu^\perp) . \end{aligned} \quad (8.85)$$

The coefficients a_i for the scalar, pseudoscalar and vector exchanges are

O_I	1	γ_5	γ_μ
a_1^I	$\sqrt{ss'}(4m^2 + \sqrt{ss'})$	$\sqrt{ss'}(4m^2 - \sqrt{ss'})$	$-2ss'$
a_2^I	$-4\sqrt{ss'}$	$+4\sqrt{ss'}$	$-8\sqrt{ss'}$
a_3^I	$+4s'$	$-4s'$	$-8s'$
a_4^I	$+4s$	$-4s$	$-8s$
a_5^I	$4(4m^2 - \sqrt{ss'})$	$4(4m^2 + \sqrt{ss'})$	$8(8m^2 - \sqrt{ss'})$
a_6^I	-16	$+16$	$+32$
a_7^I	$+4\sqrt{ss'}$	$-4\sqrt{ss'}$	$+8\sqrt{ss'}$

(8.86)

For $S = 1$ and $L = J$ states we obtain:

$$\begin{aligned} V_1^{(1,J,J)} &= \sqrt{ss'} \kappa^J \left[(4z\kappa - 4m^2 - \sqrt{ss'}) P_J(z) - \frac{4\kappa}{J+1} (zP_J(z) - P_{J-1}(z)) \right] , \\ V_{\gamma_5}^{(1,J,J)} &= \sqrt{ss'} \kappa^J \left[(\sqrt{ss'} - 4z\kappa - 4m^2) P_J(z) + \frac{4\kappa}{J+1} (zP_J(z) - P_{J-1}(z)) \right] , \\ V_{\gamma_\mu}^{(1,J,J)} &= \sqrt{ss'} \kappa^J \left[(2\sqrt{ss'} + 8z\kappa) P_J(z) - \frac{8\kappa}{J+1} (zP_J(z) - P_{J-1}(z)) \right] . \end{aligned} \quad (8.87)$$

Likewise, the states with $L = J \pm 1$ are expressed as follows:

$$V_I^{(1,L,L',J)} = \frac{-1}{2J+1} \kappa^{\frac{L+L'}{2}} \sum_{k=1}^7 a_k^I v_k^{(L,L')} . \quad (8.88)$$

We use the additional index (L') to describe transitions between states with $L^+ = J+1$ and $L^- = J-1$.

	$L^- \rightarrow L^-$	$L^+ \rightarrow L^+$	$L^- \rightarrow L^+$	$L^+ \rightarrow L^-$
$v_1^{(L,L')}$	$(2J+1)P_{J-1}(z)$	$(2J+1)P_{J+1}(z)$	0	0
$v_2^{(L,L')}$	$(2J+1)z\kappa P_{J-1}(z)$	$(2J+1)z\kappa P_{J+1}(z)$	0	0
$v_3^{(L,L')}$	$-JP_{J-1}(z) \mathbf{k} ^2$	$-(J+1)P_{J+1}(z) \mathbf{k} ^2$	$\Lambda \kappa P_{J+1}$	$\Lambda P_{J-1} \frac{ \mathbf{k} ^4}{\kappa}$
$v_4^{(L,L')}$	$-JP_{J-1}(z) \mathbf{k}' ^2$	$-(J+1)P_{J+1}(z) \mathbf{k}' ^2$	$\Lambda P_{J-1}(z) \frac{ \mathbf{k}' ^4}{\kappa}$	$\Lambda \kappa P_{J+1}(z)$
$v_5^{(L,L')}$	$-J\kappa P_J(z)$	$-(J+1)\kappa P_J(z)$	$\Lambda P_J(z) \mathbf{k}' ^2$	$\Lambda P_J(z) \mathbf{k} ^2$
$v_6^{(L,L')}$	$-Jz\kappa^2 P_J(z)$	$-(J+1)z\kappa^2 P_J(z)$	$\Lambda z\kappa P_J(z) \mathbf{k}' ^2$	$\Lambda z\kappa P_J(z) \mathbf{k} ^2$
$v_7^{(L,L')}$	$\frac{(2J+1)(1-J)}{2J-1} \kappa(P_J(z) - P_{J-2}(z))$	$(2J+1)\kappa(zP_{J+1}(z) - P_J(z))$	0	0 .

(8.89)

Here $\Lambda = \sqrt{J(J+1)}$ and κ are defined by (8.84).

8.8 Appendix 8.D: Spectral Integral Equation for Composite Systems Built by Spinless Constituents

Using this comparatively simple example, we present here a conceptual scheme of the fitting procedure. First, we consider the case of $L = 0$ for non-identical scalar constituents with equal masses. The bound system is treated as a composite system of these constituents. Further, the $L \neq 0$ case is considered in detail.

8.8.1 Spectral integral equation for a vertex function with $L = 0$

The equation for the vertex of transition of the composite system into two constituents, $G(s)$, reads:

$$G(s) = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \int d\Phi_2(P'; k'_1, k'_2) V(k_1, k_2; k'_1, k'_2) \frac{G(s')}{s' - M^2 - i0} , \quad (8.90)$$

where $V(k_1, k_2; k'_1, k'_2)$ is the interaction block and M is the mass of the composite scalar particle. Spinless constituents are not supposed to be identical, so we do not write an additional identity factor $1/2$ in the phase space.

Recall that Eq. (8.90) deals with the energy off-shell states $s' = (k'_1 + k'_2)^2 \neq M^2$, $s = (k_1 + k_2)^2 \neq M^2$ and $s \neq s'$; the constituents are on the

mass shell, $k_1'^2 = m^2$ and $k_2'^2 = m^2$. We can use an alternative expression for the phase space:

$$d\Phi_2(P'; k_1', k_2') = \rho(s') \frac{dz}{2} \equiv d\Phi(k') , \quad z = \frac{(kk')}{\sqrt{k^2}\sqrt{k'^2}} , \quad (8.91)$$

where $k = (k_1 - k_2)/2$ and $k' = (k_1' - k_2')/2$. Then

$$G(s) = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \int d\Phi(k') V(s, s', (kk')) \frac{G(s')}{s' - M^2 - i0} . \quad (8.92)$$

In the centre-of-mass frame $(kk') = -(\mathbf{k}\mathbf{k}')$, $\sqrt{k^2} = \sqrt{-\mathbf{k}^2} = i|\mathbf{k}|$ and $\sqrt{k'^2} = \sqrt{-\mathbf{k}'^2} = i|\mathbf{k}'|$ so $z = (\mathbf{k}\mathbf{k}')/(|\mathbf{k}||\mathbf{k}'|)$; equation (8.90) reads

$$G(s) = \int \frac{d^3\mathbf{k}'}{(2\pi)^3 k_0'} V(s, s', -(\mathbf{k}\mathbf{k}')) \frac{G(s')}{s' - M^2 - i0} . \quad (8.93)$$

Consider now the spectral integral equation for the wave function of a composite system, $\psi(s) = G(s)/(s - M^2)$. To this aim, the identity transformation upon the equation (8.90) should be carried out as follows:

$$(s - M^2) \frac{G(s)}{s - M^2} = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \int d\Phi(k') V(s, s', (kk')) \frac{G(s')}{s' - M^2} . \quad (8.94)$$

Making use of the wave functions, the equation (8.94) can be written in the form:

$$(s - M^2)\psi(s) = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \int d\Phi(k') V(s, s', (kk')) \psi(s') . \quad (8.95)$$

Finally, using $\mathbf{k}'^2$ and $\mathbf{k}^2$ instead of s' and $s - \psi(s) \rightarrow \psi(\mathbf{k}^2)$, we have:

$$(4\mathbf{k}^2 + 4m^2 - M^2)\psi(\mathbf{k}^2) = \int \frac{d^3\mathbf{k}_1'}{(2\pi)^3 k_0'} V(s, s', -(\mathbf{k}\mathbf{k}')) \psi(\mathbf{k}'^2) . \quad (8.96)$$

This is a basic equation for the set of states with $L = 0$. The set is formed by levels with different radial excitations $n = 1, 2, 3, \dots$, and the relevant wave functions are as follows:

$$\psi_1(\mathbf{k}^2), \psi_2(\mathbf{k}^2), \psi_3(\mathbf{k}^2), \dots$$

The wave functions are normalised and orthogonal to each other. The normalisation and orthogonality condition reads:

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3 k_0} \psi_n(\mathbf{k}^2) \psi_{n'}(\mathbf{k}^2) = \delta_{nn'} . \quad (8.97)$$

Here $\delta_{nn'}$ is the Kronecker symbol. The equation (8.97) is due to the consideration of the charge form factors of composite systems with the gauge-invariance requirement imposed, see Chapter 7 for details. This normalisation-orthogonality condition looks the same as in quantum mechanics.

Hence, the spectral integral equation for the S -wave mesons is

$$4(\mathbf{k}^2 + m^2)\psi_n(\mathbf{k}^2) - \int_0^\infty \frac{d\mathbf{k}'^2}{\pi} V_0(\mathbf{k}^2, \mathbf{k}'^2) \phi(\mathbf{k}'^2) \psi_n(\mathbf{k}'^2) = M^2 \psi_n(k^2), \quad (8.98)$$

where $\phi(\mathbf{k}'^2) = |\mathbf{k}'|/(4\pi k'_0)$.

The wave function $\psi_n(\mathbf{k}^2)$ represents a full set of orthogonal and normalised wave functions:

$$\int_0^\infty \frac{d\mathbf{k}^2}{\pi} \psi_a(\mathbf{k}^2) \phi(k) \psi_b(\mathbf{k}^2) = \delta_{ab}. \quad (8.99)$$

The function $V_0(\mathbf{k}^2, \mathbf{k}'^2)$ is the projection of the potential $V(s, s', (k, k'))$ on the S -wave:

$$V_0(k, k') = \int \frac{d\Omega_{\mathbf{k}}}{4\pi} \int \frac{d\Omega_{\mathbf{k}'}}{4\pi} V(s, s', -(\mathbf{k}\mathbf{k}')) . \quad (8.100)$$

Let us expand $V_0(\mathbf{k}^2, \mathbf{k}'^2)$ with respect to a full set of wave functions:

$$V_0(\mathbf{k}^2, \mathbf{k}'^2) = \sum_{a,b} \psi_a(\mathbf{k}^2) v_{ab}^{(0)} \psi_b(\mathbf{k}'^2), \quad (8.101)$$

where the numerical coefficients $v_{ab}^{(0)}$ are defined by the inverse transformation as follows:

$$v_{ab}^{(0)} = \int_0^\infty \frac{d\mathbf{k}^2}{\pi} \frac{d\mathbf{k}'^2}{\pi} \psi_a(\mathbf{k}^2) \phi(\mathbf{k}^2) V_0(\mathbf{k}^2, \mathbf{k}'^2) \phi(\mathbf{k}'^2) \psi_b(\mathbf{k}'^2). \quad (8.102)$$

Taking into account the series (8.101), the equation (8.98) is rewritten as

$$4(\mathbf{k}^2 + m^2)\psi_n(\mathbf{k}^2) - \sum_a \psi_a(\mathbf{k}^2) v_{an}^{(0)} = M^2 \psi_n(k^2). \quad (8.103)$$

Such a transformation should be carried out upon the kinetic energy term, it is also expanded into a series with respect to a full set of wave functions:

$$4(\mathbf{k}^2 + m^2)\psi_n(\mathbf{k}^2) = \sum_a K_{na} \psi_a(\mathbf{k}^2), \quad (8.104)$$

where

$$K_{na} = \int_0^\infty \frac{d\mathbf{k}^2}{\pi} \psi_a(\mathbf{k}^2) \phi(\mathbf{k}^2) 4(\mathbf{k}^2 + m^2) \psi_n(\mathbf{k}^2). \quad (8.105)$$

Finally, the spectral integral equation takes the form:

$$\sum_a K_{na} \psi_a(\mathbf{k}^2) - \sum_a v_{na}^{(0)} \psi_a(\mathbf{k}^2) = M_n^2 \psi_n(\mathbf{k}^2) . \quad (8.106)$$

We take into account that $v_{na}^{(0)} = v_{an}^{(0)}$.

The equation (8.106) is a standard homogeneous equation:

$$\sum_a s_{na} \psi_a(\mathbf{k}^2) = M_n^2 \psi_n(\mathbf{k}^2) , \quad (8.107)$$

with $s_{na} = K_{na} - v_{na}^{(0)T}$. The values M^2 are defined as zeros of the determinant

$$\det|\hat{s} - M^2 I| = 0 , \quad (8.108)$$

where I is the unit matrix.

8.8.1.1 The spectral integral equation for states with angular momentum L

For the wave with an arbitrary angular momentum L , the wave function reads as follows:

$$\psi_{(n)\mu_1, \dots, \mu_L}^{(L)}(s) = X_{\mu_1, \dots, \mu_L}^{(L)}(k) \psi_n^{(L)}(s) . \quad (8.109)$$

Recall that the momentum operator $X_{\mu_1, \dots, \mu_L}^{(L)}(k)$ was introduced in Chapter 3.

The spectral integral equation for the (L, n) -state, presented in the form similar to (8.98), is:

$$\begin{aligned} & 4(\mathbf{k}^2 + m^2) X_{\mu_1, \dots, \mu_L}^{(L)}(k) \psi_n^{(L)}(\mathbf{k}^2) \\ & - X_{\mu_1, \dots, \mu_L}^{(L)}(k) \int_0^\infty \frac{d\mathbf{k}'^2}{\pi} V_L(s, s') X_L^2(k'^2) \phi(\mathbf{k}'^2) \psi_n^{(L)}(\mathbf{k}'^2) \\ & = M^2 X_{\mu_1, \dots, \mu_L}^{(L)}(k) \psi_n^{(L)}(\mathbf{k}^2) , \end{aligned} \quad (8.110)$$

where

$$X_L^2(k'^2) = \int \frac{d\Omega_{\mathbf{k}'}}{4\pi} \left(X_{\nu_1, \dots, \nu_L}^{(L)}(k') \right)^2 = \alpha(L) (k'^2)^L = \alpha(L) (-\mathbf{k}'^2)^L, \quad (8.111)$$

and

$$\alpha(L) = \frac{(2L-1)!!}{L!} \quad (8.112)$$

The potential is expanded into a series with respect to the product of operators $X_{\mu_1, \dots, \mu_L}^{(L)}(k) X_{\mu_1, \dots, \mu_L}^{(L)}(k')$, that is,

$$V(s, s', (kk')) = \sum_{L, \mu_1 \dots \mu_L} X_{\mu_1, \dots, \mu_L}^{(L)}(k) V_L(s, s') X_{\mu_1, \dots, \mu_L}^{(L)}(k') ,$$

$$X_L^2(k^2) V_L(s, s') X_L^2(k'^2) = \int \frac{d\Omega_{\mathbf{k}}}{4\pi} \frac{d\Omega_{\mathbf{k}'}}{4\pi} \quad (8.113)$$

$$\times X_{\nu_1, \dots, \nu_L}^{(L)}(k) V(s, s', (kk')) X_{\nu_1, \dots, \nu_L}^{(L)}(k').$$

Hence, formula (8.110) can be written in the form:

$$4(\mathbf{k}^2 + m^2) \psi_n^{(L)}(\mathbf{k}^2) - \int_0^\infty \frac{d\mathbf{k}'^2}{\pi} V_L(s, s') \alpha(L) (-\mathbf{k}'^2)^L \phi(\mathbf{k}'^2) \psi_n^{(L)}(\mathbf{k}'^2) = M_n^2 \psi_n^{(L)}(\mathbf{k}^2). \quad (8.114)$$

Compared to (8.98) this equation contains an additional factor $X_L^2(k'^2)$; the same factor is present in the normalisation condition, so it would be reasonable to insert it into the phase space. Finally, we have:

$$4(\mathbf{k}^2 + m^2) \psi_n^{(L)}(\mathbf{k}^2) - \int_0^\infty \frac{d\mathbf{k}'^2}{\pi} \tilde{V}_L(s, s') \phi_L(\mathbf{k}'^2) \psi_n^{(L)}(\mathbf{k}'^2) = M_n^2 \psi_n^{(L)}(\mathbf{k}^2) , \quad (8.115)$$

where

$$\phi_L(\mathbf{k}'^2) = \alpha(L) (\mathbf{k}'^2)^L \phi(\mathbf{k}'^2), \quad \tilde{V}_L(s, s') = (-1)^L V_L(s, s'). \quad (8.116)$$

The normalisation condition for a set of wave functions with an orbital momentum L reads:

$$\int_0^\infty \frac{d\mathbf{k}^2}{\pi} \psi_a^{(L)}(\mathbf{k}^2) \phi_L(\mathbf{k}^2) \psi_b^{(L)}(\mathbf{k}^2) = \delta_{ab} . \quad (8.117)$$

One can see that it is similar to the case of $L = 0$, the only difference consists in the redefinition of the phase space $\phi \rightarrow \phi_L$. The spectral integral equation is

$$\sum_a s_{na}^{(L)} \psi_a^{(L)}(\mathbf{k}^2) = M_{n,L}^2 \psi_n^{(L)}(\mathbf{k}^2) , \quad (8.118)$$

with

$$s_{na}^{(L)} = K_{na}^{(L)} - v_{na}^{(L)T} ,$$

$$v_{ab}^{(L)} = \int_0^\infty \frac{d\mathbf{k}^2}{\pi} \frac{d\mathbf{k}'^2}{\pi} \psi_a^{(L)}(\mathbf{k}^2) \phi_L(\mathbf{k}^2) \tilde{V}_L(s, s') \phi_L(\mathbf{k}'^2) \psi_b^{(L)}(\mathbf{k}'^2) ,$$

$$K_{na}^{(L)} = \int_0^\infty \frac{d\mathbf{k}^2}{\pi} \psi_a^{(L)}(\mathbf{k}^2) \phi_L(\mathbf{k}) 4(\mathbf{k}^2 + m^2) \psi_n^{(L)}(\mathbf{k}^2) . \quad (8.119)$$

Using radial excitation levels, one can reconstruct the potential in the L -wave and then, with the help of (8.113), the t -dependent potential.

Miniconclusion

The main point we want to emphasise by presenting the above calculations is the statement that for the restoration of the interaction between constituents the knowledge of levels and their wave functions is equally necessary. Neglecting this, in principle, trivial point leads till now to misleading conclusions about the quark structure of mesons (see, for example, [58] and references therein).

8.9 Appendix 8.E: Wave Functions in the Sector of the Light Quarks

Tables 8.2 – 8.5 give us the $c_i(S, L, J; n)$ coefficients, which determine the wave functions of the $q\bar{q}$ states, $\psi^{(S,L,J)}_{(n)}$, according to the following formula:

$$\psi^{(S,L,J)}_{(n)}(k^2) = e^{-\beta k^2} \sum_{i=1}^{11} c_i(S, L, J; n) k^{i-1}, \quad (8.120)$$

where $k^2 \equiv \mathbf{k}^2$ (recall that $s = 4m^2 + 4\mathbf{k}^2$). The fitting parameter is fixed to be $\beta = 1.2 \text{ GeV}^{-2}$. The normalisation condition for $\psi^{(S,L,J)}_{(n)}(k^2)$ is given in Section 1, Eqs. (8.9) and (8.10). In Figs. 8.5, 8.6, 8.7, 8.8 we demonstrate these wave functions.

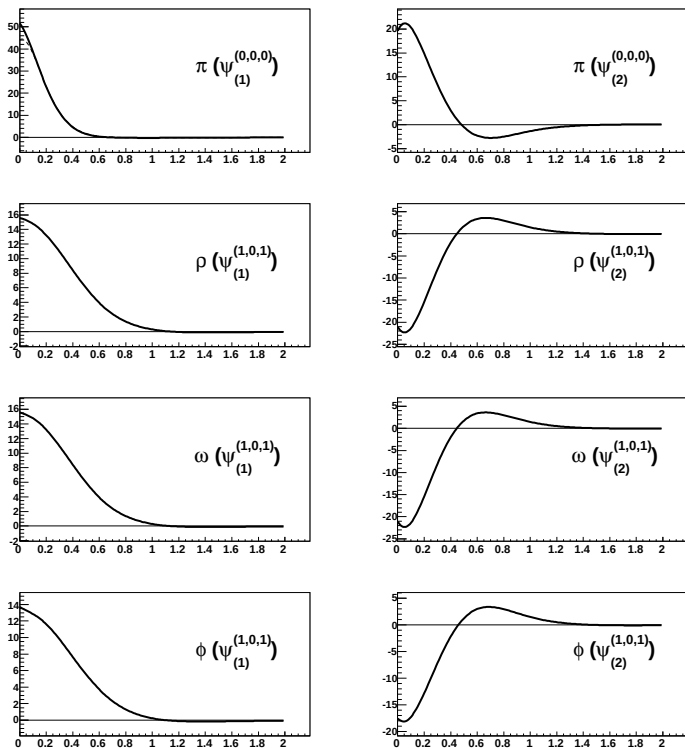


Fig. 8.5 Wave functions (in GeV) of the L=0 group (π , ρ , ω and ϕ mesons). The dotted curve shows the wave function of $\pi(140)$ with instanton-induced forces included.

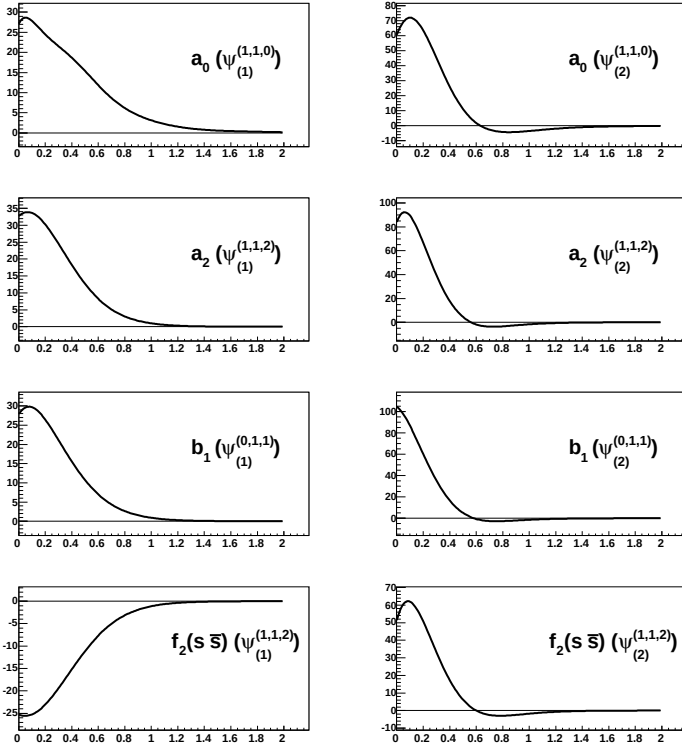


Fig. 8.6 Wave functions of the L=1 group (a_0 , a_1 , a_2 and $f_2(nn)$ mesons).

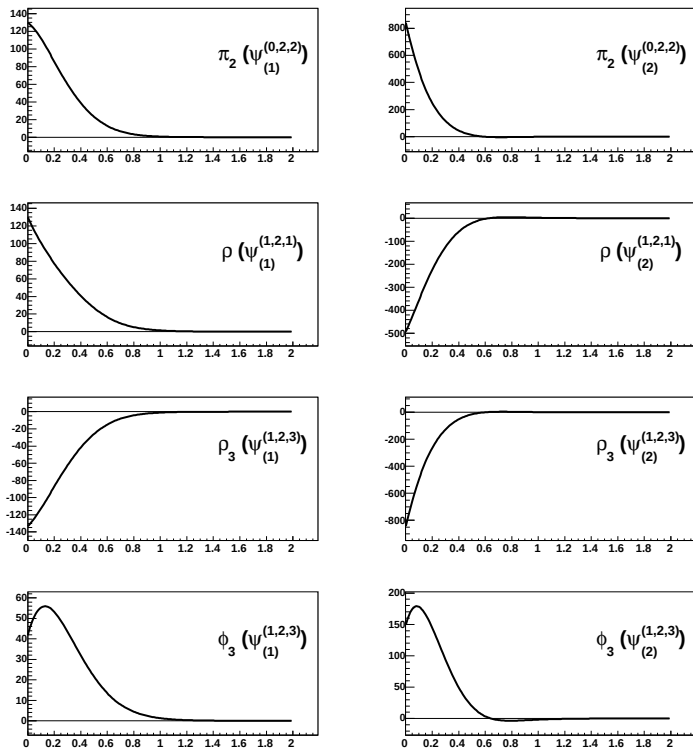


Fig. 8.7 Wave functions of the L=2 group (π_2 , ρ , ρ_3 and ϕ_3 mesons).

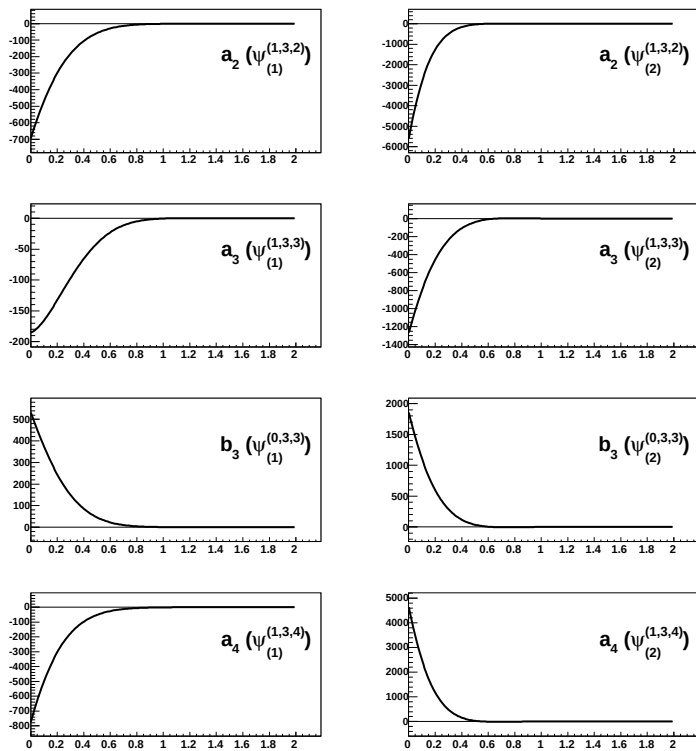


Fig. 8.8 Wave functions of the L=3 group (a_2 , a_3 , a_4 and b_3 mesons).

Table 8.2 Constants $c_i(S, L, J; n)$ (in GeV units, Eq. (8.120)) for mesons with $L = 0$ ($\psi_n^{(S, L, J)}$).

	$\pi(1S)$	$\pi(2S)$	$\pi(3S)$	$\pi(4S)$
i	$\psi_1^{(0,0,0)}$	$\psi_2^{(0,0,0)}$	$\psi_3^{(0,0,0)}$	$\psi_4^{(0,0,0)}$
1	51.6	132.0	-349.9	110.9
2	-75.4	-3416.0	5923.8	-1026.4
3	-786.2	26717.9	-38671.3	2223.1
4	3369.5	-97897.7	130528.4	-2962.3
5	-5983.5	197748.6	-253088.3	18810.8
6	5700.2	-232791.3	291304.4	-31139.4
7	-2952.2	155832.6	-192356.5	24525.2
8	694.6	-49062.0	60017.9	-8448.0
9	12.5	-621.4	758.7	-136.3
10	-48.0	3035.1	-3694.9	640.7
11	21.9	856.9	-1008.6	-102.4

	$\rho(1S)$	$\rho(2S)$	$\rho(3S)$	$\rho(4S)$
i	$\psi_1^{(1,0,1)}$	$\psi_2^{(1,0,1)}$	$\psi_3^{(1,0,1)}$	$\psi_4^{(1,0,1)}$
1	44.2	-47.0	34.4	256.1
2	147.9	96.4	367.3	-3816.4
3	-2576.7	1694.4	-6627.1	21285.8
4	10145.9	-8835.1	31300.6	-61891.6
5	-20331.5	18954.3	-72495.7	106967.9
6	23805.7	-21715.0	95497.7	-115547.6
7	-16569.8	13585.9	-73882.6	77608.2
8	6338.4	-3952.2	31633.5	-29980.2
9	-941.1	119.3	-5588.5	4927.5
10	-59.0	26.4	-333.1	258.1
11	-16.0	88.7	43.2	-25.9

	$\omega(1S)$	$\omega(2S)$	$\omega(3S)$	$\omega(4S)$
i	$\psi_1^{(1,0,1)}$	$\psi_2^{(1,0,1)}$	$\psi_3^{(1,0,1)}$	$\psi_4^{(1,0,1)}$
1	44.2	-47.0	34.4	256.1
2	147.9	96.4	367.3	-3816.4
3	-2576.7	1694.4	-6627.1	21285.8
4	10145.9	-8835.1	31300.6	-61891.6
5	-20331.5	18954.3	-72495.7	106967.9
6	23805.7	-21715.0	95497.7	-115547.6
7	-16569.8	13585.9	-73882.6	77608.2
8	6338.4	-3952.2	31633.5	-29980.2
9	-941.1	119.3	-5588.5	4927.5
10	-59.0	26.4	-333.1	258.1
11	-16.0	88.7	43.2	-25.9

	$\phi(1S)$	$\phi(2S)$	$\phi(3S)$	$\phi(4S)$
i	$\psi_1^{(1,0,1)}$	$\psi_2^{(1,0,1)}$	$\psi_3^{(1,0,1)}$	$\psi_4^{(1,0,1)}$
1	33.8	-26.3	-4.1	35.9
2	163.6	-275.3	-991.5	110.6
3	-2106.0	4223.2	11527.0	-5305.6
4	7358.5	-17247.3	-50406.7	31237.1
5	-13464.0	34846.8	112767.3	-81944.1
6	14678.2	-40203.6	-143333.2	115348.5
7	-9651.7	27042.4	105078.0	-90497.7
8	3537.8	-9630.4	-40848.7	36536.8
9	-561.9	1009.3	5719.6	-4772.9
10	61.0	300.8	191.7	-612.0
11	-83.3	-38.0	579.4	-344.7

Table 8.3 Constants $c_i(S, L, J; n)$ (in GeV units, Eq. (8.120)) for mesons with $L = 1$ ($\psi_n^{(S, L, J)}$).

	$a_0(1P)$	$a_0(2P)$	$a_0(3P)$	$a_0(4P)$
i	$\psi_1^{(1,1,0)}$	$\psi_2^{(1,1,0)}$	$\psi_3^{(1,1,0)}$	$\psi_4^{(1,1,0)}$
1	42.4	79.7	181.7	552.3
2	8.2	174.5	52.8	-3509.0
3	-119.5	-1866.0	-4767.6	6343.4
4	9.0	3990.4	14898.6	302.5
5	205.9	-4036.6	-19963.7	-12748.5
6	-213.0	2137.9	13294.9	14124.0
7	74.1	-506.2	-3795.7	-5290.2
8	-0.0	0.1	0.5	0.9
9	-0.9	8.6	6.4	-28.7
10	-2.2	1.2	131.1	328.4
11	0.0	2.9	-20.0	-66.3

	$a_2(1P)$	$a_2(2P)$	$a_2(3P)$	$a_2(4P)$
i	$\psi_1^{(1,1,2)}$	$\psi_2^{(1,1,2)}$	$\psi_3^{(1,1,2)}$	$\psi_4^{(1,1,2)}$
1	32.2	-77.8	-210.1	647.4
2	20.0	-166.6	408.0	-4983.7
3	-216.6	2089.0	2776.4	14397.9
4	312.8	-5329.3	-11625.2	-20482.8
5	-175.0	6698.4	18318.0	15619.1
6	9.4	-4684.8	-14419.4	-6876.1
7	29.2	1719.0	5260.0	2638.3
8	-8.1	-222.0	-331.7	-1248.4
9	0.1	5.3	10.6	19.1
10	-1.1	-45.6	-345.7	472.1
11	0.5	22.7	171.3	-229.4

	$b_1(1P)$	$b_1(2P)$	$b_1(3P)$	$b_1(4P)$
i	$\psi_1^{(0,1,1)}$	$\psi_2^{(0,1,1)}$	$\psi_3^{(0,1,1)}$	$\psi_4^{(0,1,1)}$
1	39.8	-101.1	289.7	-676.1
2	27.9	59.8	-1349.0	5529.2
3	-436.3	1394.5	1204.7	-17483.5
4	963.5	-4304.2	3319.6	28515.1
5	-1103.7	5943.5	-8934.1	-26716.8
6	766.6	-4579.2	9021.3	15134.0
7	-323.2	1989.9	-4456.2	-5436.3
8	72.5	-418.6	922.3	1319.3
9	-7.9	32.9	-72.1	-124.7
10	5.9	-29.6	128.3	-100.6
11	-3.9	19.7	-78.8	45.7

	$f_2(1P_{ss})$	$f_2(2P_{ss})$	$f_2(3P_{ss})$	$f_2(4P_{ss})$
i	$\psi_1^{(1,1,2)}$	$\psi_2^{(1,1,2)}$	$\psi_3^{(1,1,2)}$	$\psi_4^{(1,1,2)}$
1	32.2	-77.8	-210.1	647.4
2	20.0	-166.6	408.0	-4983.7
3	-216.6	2089.0	2776.4	14397.9
4	312.8	-5329.3	-11625.2	-20482.8
5	-175.0	6698.4	18318.0	15619.1
6	9.4	-4684.8	-14419.4	-6876.1
7	29.2	1719.0	5260.0	2638.3
8	-8.1	-222.0	-331.7	-1248.4
9	0.1	5.3	10.6	19.1
10	-1.1	-45.6	-345.7	472.1
11	0.5	22.7	171.3	-229.4

Table 8.4 Constants $c_i(S, L, J; n)$ (in GeV units, Eq. (8.120)) for mesons with $L = 2$ ($\psi_n^{(S,L,J)}$).

	$\pi_2(1D)$	$\pi_2(2D)$	$\pi_2(3D)$	$\pi_2(4D)$
i	$\psi_1^{(0,2,2)}$	$\psi_2^{(0,2,2)}$	$\psi_3^{(0,2,2)}$	$\psi_4^{(0,2,2)}$
1	1.7	-4.5	-30.4	-25.0
2	-21.7	31.2	317.1	47.5
3	95.0	-63.9	-1269.6	641.8
4	-209.5	-12.3	2466.5	-2875.2
5	232.2	190.4	-2406.5	4478.4
6	-116.6	-192.6	1140.4	-3020.3
7	21.3	53.9	-224.0	738.5
8	-2.8	-4.0	28.6	-70.9
9	3.6	15.0	-23.3	178.3
10	-1.5	-9.1	8.6	-105.1
11	0.2	0.0	-1.8	1.7

	$\rho(1D)$	$\rho(2D)$	$\rho(3D)$	$\rho(4D)$
i	$\psi_1^{(1,2,1)}$	$\psi_2^{(1,2,1)}$	$\psi_3^{(1,2,1)}$	$\psi_4^{(1,2,1)}$
1	32.6	1.9	295.8	1109.3
2	-297.9	-20.8	-2587.2	-9686.9
3	1030.3	85.0	8635.8	32404.0
4	-1720.3	-207.3	-13721.7	-52043.5
5	1257.2	242.8	9530.7	36934.5
6	68.1	4.0	206.3	1219.6
7	-702.1	-203.4	-4305.9	-18749.1
8	419.2	125.4	2314.3	10789.0
9	-113.3	-25.0	-521.0	-2650.0
10	68.2	16.0	378.0	1715.0
11	-58.4	-16.6	-340.7	-1533.5

	$\rho_3(1D)$	$\rho_3(2D)$	$\rho_3(3D)$	$\rho_3(4D)$
i	$\psi_1^{(1,2,3)}$	$\psi_2^{(1,2,3)}$	$\psi_3^{(1,2,3)}$	$\psi_4^{(1,2,3)}$
1	2.7	0.2	-35.8	-51.1
2	-28.9	11.5	345.1	678.5
3	114.9	-100.7	-1263.4	-3288.1
4	-228.6	325.0	2187.8	7495.6
5	228.9	-475.1	-1814.1	-8396.0
6	-101.5	282.1	660.9	4254.9
7	14.1	-36.5	-84.3	-566.5
8	-2.7	6.3	15.4	111.3
9	5.0	-34.1	5.1	-431.2
10	-2.0	17.1	-9.7	213.2
11	0.0	-0.0	-1.4	-0.6

	$\phi_3(1D)$	$\phi_3(2D)$	$\phi_3(3D)$	$\phi_3(4D)$
i	$\psi_1^{(1,2,1)}$	$\psi_2^{(1,2,1)}$	$\psi_3^{(1,2,1)}$	$\psi_4^{(1,2,1)}$
1	-910.2	-2285.4	-2544.6	2193.2
2	3296.7	10036.7	12377.7	-11355.8
3	-4826.6	-17234.9	-23262.5	22679.8
4	3506.8	14308.2	20849.2	-21526.7
5	-1120.1	-5094.6	-7903.5	8590.2
6	-85.4	-442.1	-737.0	859.7
7	197.0	1049.2	1790.4	-2138.3
8	-73.7	-406.7	-710.2	871.4
9	11.0	67.5	122.0	-152.4
10	4.6	27.0	48.2	-60.2
11	-2.5	-18.3	-34.9	45.4

Table 8.5 Constants $c_i(S, L, J; n)$ (in GeV units, Eq. (8.120)) for mesons with $L = 3$ ($\psi_n^{(S, L, J)}$).

	$a_2(1F)$	$a_2(2F)$	$a_2(3F)$	$a_2(4F)$
i	$\psi_1^{(1,3,2)}$	$\psi_2^{(1,3,2)}$	$\psi_3^{(1,3,2)}$	$\psi_4^{(1,3,2)}$
1	302.5	3108.5	-4814.1	-3261.0
2	-143.3	-16363.8	29608.0	22339.5
3	-1820.4	33890.9	-70605.8	-58593.2
4	3544.0	-34294.0	81404.4	73808.0
5	-2486.5	15588.3	-42971.6	-42810.1
6	505.7	-751.9	4532.7	5800.9
7	33.3	-272.7	747.1	788.7
8	224.4	-2230.2	5730.9	5836.8
9	-222.8	1735.8	-4925.5	-5402.9
10	66.4	-422.2	1339.5	1592.2
11	-3.4	1.5	-40.8	-76.1

	$a_3(1F)$	$a_3(2F)$	$a_3(3F)$	$a_3(4F)$
i	$\psi_1^{(1,3,3)}$	$\psi_2^{(1,3,3)}$	$\psi_3^{(1,3,3)}$	$\psi_4^{(1,3,3)}$
1	-185.6	-1273.6	-2824.8	-3304.5
2	100.1	5502.8	15900.4	21342.1
3	997.5	-8945.0	-35307.5	-54519.8
4	-2016.2	6492.3	39328.0	70562.2
5	1587.7	-1474.4	-22363.9	-47827.8
6	-509.7	-457.3	5241.1	14381.9
7	0.9	-5.2	-29.5	-53.1
8	17.7	232.8	263.4	-357.5
9	5.6	-63.5	-203.8	-212.9
10	3.8	7.8	-31.6	-107.2
11	-3.6	-20.5	3.9	94.5

	$b_3(1F)$	$b_3(2F)$	$b_3(3F)$	$b_3(4F)$
i	$\psi_1^{(0,3,3)}$	$\psi_2^{(0,3,3)}$	$\psi_3^{(0,3,3)}$	$\psi_4^{(0,3,3)}$
1	-42.3	-688.7	4871.3	-6800.6
2	-700.1	1416.6	-30922.8	49960.3
3	2996.2	2579.9	80377.0	-148188.4
4	-4886.9	-10605.0	-110657.5	229300.7
5	4029.3	12572.9	84979.6	-194602.3
6	-1569.0	-6072.3	-32063.7	79614.6
7	-0.6	41.7	-23.3	-208.2
8	255.5	1070.7	5175.5	-13213.8
9	-100.3	-333.4	-2066.1	4736.4
10	18.4	56.1	381.9	-847.4
11	-2.9	-54.2	-45.2	373.8

	$a_4(1F)$	$a_4(2F)$	$a_4(3F)$	$a_4(4F)$
i	$\psi_1^{(1,3,4)}$	$\psi_2^{(1,3,4)}$	$\psi_3^{(1,3,4)}$	$\psi_4^{(1,3,4)}$
1	61.3146	-279.4	-6335.7	3793.3
2	-805.5228	-494.1	38516.0	-26710.9
3	2125.7747	4617.2	-92101.7	71500.9
4	-2401.6045	-7997.8	107642.9	-91068.5
5	1213.7027	5255.0	-58196.3	52536.8
6	-155.4700	-853.4	8074.5	-7758.0
7	54.5187	438.3	-3773.4	4071.3
8	-214.3060	-1528.9	13678.2	-14276.3
9	136.2202	1018.8	-8962.3	9476.4
10	-1.9588	-13.6	127.4	-133.2
11	-29.9559	-242.3	2084.9	-2255.0

8.10 Appendix 8.F: How Quarks Escape from the Confinement Trap?

Till now, it was not discussed how the decay processes can be taken into account in the spectral integral equation. Apparently, it can be done directly: in the framework of the spectral integration technique we have to include for $q\bar{q}$ a second, two-meson channel (making use of the dispersion relation method, this is easy) and solve the problem within additional transitions $q\bar{q} \rightarrow \text{meson} + \text{meson}$ (Fig. 8.9). The price we have to pay is that a new t -channel interaction appears with quantum numbers of the coloured quark, Fig. 8.9a. The described way of acting, though a direct one, is by far not easy. It requires the investigation of the blocks in Figs. 8.9b and 8.9c: the blocks in Fig. 8.9b have to contain meson singularities coming from the intermediate two-meson states, while in the blocks of Fig. 8.9c there are no quark singularities. These properties should be realised by the interaction shown in Fig. 8.9a.

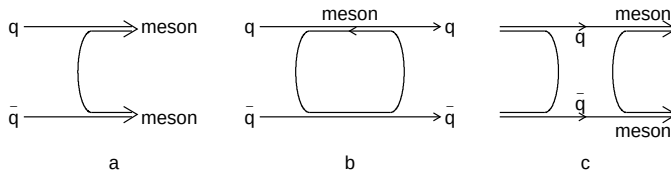


Fig. 8.9 a) Diagram for quark escape from the confinement trap; b,c) the blocks which appear in the spectral integral equation diagrams due to the process of the quark escaping from the confinement trap.

There may be another approach suggested by the radiative processes. In these processes (see Chapter 7) we have calculated reactions where the quarks leave the confinement trap via their annihilation (two-photon annihilation $q\bar{q} \rightarrow \gamma\gamma$) or fly away creating a pair with another quark ($q\bar{q}_{in} \rightarrow \gamma + q\bar{q}_{out}$).

Such processes can take place without the participation of photons on the hadronic level, taking into account that the pion mass is small and the $m_\pi \rightarrow 0$ approximation can be used. In this case the escape from the confinement trap happens following analogous scenarios: (i) annihilation into two pions $q\bar{q}_{in} \rightarrow \pi\pi$, see Fig. 8.10a,b, and (ii) cascade pion emission $q\bar{q}_{in} \rightarrow \pi + q\bar{q}_{out-1} \rightarrow \pi + (\pi + q\bar{q}_{out-2})$, and so on, see Fig. 8.10c,d.

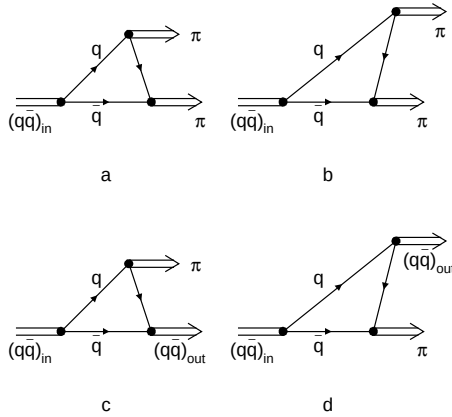


Fig. 8.10 a,b) Annihilation of quark-antiquark state into two pions $q\bar{q}_{in} \rightarrow \pi\pi$. c,d) Element of the cascade with pion emission $q\bar{q}_{in} \rightarrow \pi + q\bar{q}_{out-1}$: the subsequent decays $q_{out-1} \rightarrow \pi + q\bar{q}_{out-2}$ create a cascade (pion comb).

These processes realise the quark deconfinement in the chiral limit ($m_\pi \rightarrow 0$). Using the technique given in Chapter 7, they can be calculated without problems. The introduction of decay channels in the spectral equation (which can be done on a perturbative level only) seems to make it possible to give a phenomenological description of the quark escape from the confinement trap.

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Chapter 9

Outlook

The region of soft quark and gluon interactions can and has to be considered in different ways. One of the approaches is the introduction of effective particles – constituent quarks and the investigation of effective interactions between these constituents. This is just the approach we are applying to the soft QCD region, and the effective particles and interactions are the instrument with the help of which we hope to understand the QCD mechanisms for strong interactions. In a way, this approach is based on a conception used in condensed matter physics, where effective particles and effective interactions were introduced.

In this chapter we try to summarise what is known about strong interactions in the framework of this approach, and discuss problems which would substantially add to this knowledge.

9.1 Quark Structure of Mesons and Baryons

Let us consider an object which was introduced long ago and the properties of which are, as we think, quite well known – the constituent quark. Contrary to the quark corresponding to the perturbative QCD, the constituent quark is a massive particle. In soft processes the masses of the light u and d quarks are of the order of 300–400 MeV. Do the masses of the light quarks remain unchanged in all soft processes?

Let us begin with an extreme example. We know from high energy experiments (in which mesons and baryons collide with TeV-energies) that the quark size is growing as $\sim \ln s$ (see Chapter 1, Fig. 1.9). Does this mean that the mass of the effective (constituent) quark is also increasing? At the first sight, the answer seems to be obvious: indeed, the quarks shown in Fig. 1.9 as black discs are characterised by their growing masses. The

increase of the mass is owing to the fact that the effective mass is provided by the self-energy part of the quark propagator; in different processes, at different energies different components of the self-energy parts are essential. At the same time, on the basis of the Regge approach (or of the parton model) we understand that the mass of the constituent quark can be kept around 300–400 MeV, while the growth of the size of the black discs is due to the interaction which form the reggeon combs (see Chapter 1, Subsection 1.7.2, and references therein). Therefore, the notion “constituent quark” (and, correspondingly, its characteristics) depends on the type of the model we have used.

Now turn to the structure of mesons and baryons considered in terms of spectral integral or Bethe–Salpeter equations. Will the mass of light quarks in these objects remain unchanged? Or, on the contrary, are the masses of constituent quarks in low-lying hadrons (*e.g.* in the basic ones, $n = 1$) less than in high-energy excited states? In other words:

Does the constituent quark mass change as the hadron becomes more and more excited, does the mass of the constituent in a hadron depend on the radial quantum number n or the orbital quantum number L ?

The answer is not obvious at all. In spite of the fact that the effective mass is formed by the self-energy part of the quark propagator and can change, we may face the effect of mass “freezing” in a broad interval of low-energy physics. Besides, there exists always the possibility of forcing this freezing by introducing an additional interaction (similarly to the consideration of reggeon amplitudes at high energies). So the problem has to be handled especially carefully.

There is another problem concerning the constituent quarks, also related to the behaviour of total cross sections with the increase of energy. We know that the cross sections $\sigma_{tot}(p\bar{p})$ grow with the increase of energy, and we are almost sure that they will continue to grow as $\ln^2 s$. How does this affect the phenomenon of confinement? Does the confinement radius also increase, or does the hadron, being a black disc (Fig. 9.1a) from the point of hadronic interactions, break up into a number of white domains at superhigh energies (Fig. 9.1b)?

The question of the hadron content at such high excitations is related just to these different possible versions of behaviour for the hadron disc (Fig. 9.1) at superhigh energies. Indeed, to what extent is the standard quark content of a hadron fixed? (For example, does a low energy meson consist of a quark and an antiquark?) Or: can it be seen experimentally if a black disc breaks up into white domains in the space of colour quantum

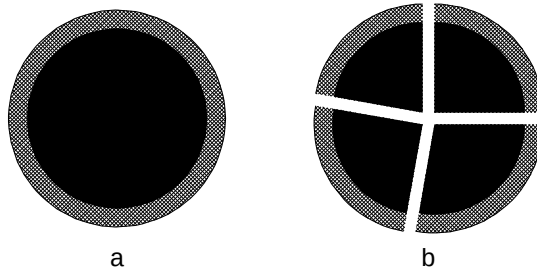


Fig. 9.1 Superhigh energies: (a) the hadron (black disc) grows with the growth of the confinement radius, (b) the black disc increases ($r_{hadron} \sim \ln s$), but the growth of the confinement radius is slower and the hadron dissipates into several white domains (due to the conventions, we have separated the white domains of the disc by white strips).

numbers? This problem has been started to be discussed long ago and is discussed up to now (see refs. [1, 2, 3] and references therein).

The standard quark structure of hadrons is realised by numbers which we are used to for a long time already: a meson is consisting of two quarks ($q\bar{q}$), a baryon of three quarks (qqq). Two quarks, or rather a quark and an antiquark, is indeed that pair of constituents which gave the possibility to construct a large amount of observed mesons, both basic and excited ones. The number “three”, however, is apparently too large for highly excited baryons. Recent experiments indicate that the latter consist of two constituents, a quark and a diquark (qd). Strictly speaking, there are by far not enough highly excited baryons to cover all the possible excitations of a three-body system.

Does this mean that the predominant number for highly excited baryons is the same “two” as for mesons? We shall return to this question when discussing the glueballs which, as we can now state with certainty, are observed experimentally.

9.2 Systematics of the ($q\bar{q}$)-Mesons and Baryons

The meson systematisation in the *radial excitation/mass squared* plane, (n, M^2) , provided us with an essentially new level in understanding hadron physics. It turned out to be possible to locate almost all “light mesons” (with a few exceptions discussed later on) on linear trajectories [4] $M^2 = M_0^2 + \mu^2(n - 1)$ with the universal parameter $\mu^2 \simeq 1.2 \text{ GeV}^2$. It looks

like that a similar systematisation, but with a different slope parameter μ^2 , works for mesons containing heavy quarks [5, 6].

Let us underline: virtually all mesons consisting of light quarks lie on the linear trajectories; this is the case up to masses of the order of 2400 MeV (for higher masses there are no reliable experimental data). This fact raises not only interesting questions, but leads — depending on the answers — also to important conclusions. Where are all those resonances which could be consisting of four quarks (two quarks and two antiquarks, $qq\bar{q}\bar{q}$) or of a quark, an antiquark and an effective gluon (hybrid $q\bar{q}g$)? In the region higher than 1500 MeV there should be a large amount of such mesons — but we do not see any. In the last decades several mesons which can be considered as exotic, *e.g.* $qq\bar{q}\bar{q}$ and $q\bar{q}g$, were detected, but the existence of these mesons is questionable. Thus, we have the following possibilities:

(i) Four-quark meson states and hybrids did exist, but melted in the process of the accumulation of widths by the neighbouring states having simpler structures (see Chapter 3). If so, we have to concentrate on the observation of broad resonances and resonances with exotic quantum numbers which cannot be $q\bar{q}$ systems.

In principle, there is another solution:

(ii) The confinement forces are not able to retain more than two coloured objects.

But can this be the case? We know with certainty that low-lying baryons consist of three quarks. As to highly excited baryons, we underlined it many times that they are, most probably, consisting of a quark and a diquark. Let us discuss the question on a simple qualitative level.

Consider first a meson. The wave functions of S -wave $q\bar{q}$ -states (of pions or ρ -mesons, for example) are presented in Chapter 8 for basic and excited states. These wave functions provide the probability density of the quark matter; they are shown (in the coordinate space) in Fig. 9.2a (for the basic state) and in Fig. 9.2b (for an excited state with $n = 3$). From the point of view of an observer placed on the antiquark of the excited state (*i.e.* with $n > 1$), the antiquark is encircled by spheres of the quark matter.

Let us now turn to the baryons considered in the framework of the quark–diquark picture (recall that a diquark is a bound system of two quarks). The suggested quark–diquark picture of a baryon reminds a meson. The only difference is that the antiquark of the meson has to be substituted by a diquark (the colour quantum numbers of an antiquark and a diquark coincide), and, naturally, the symmetrisation of the quark variables has to be carried out (see Chapters 1 and 7). We handle a low-

lying S -wave baryon ($N_{1/2}^+$ or $\Lambda_{1/2}^+$) in the quark–diquark picture virtually in the same way as that in the classical three-quark scheme: the three-quark system is considered as a superposition of a quark and an S -wave basic (and not radially excited) diquark, *i.e.* $d_{S_{qq}=1}^{I_{qq}=1}(n_{qq}=1, L_{qq}=0) \equiv d_1^1(1,0)$ or $d_{S_{qq}=0}^{I_{qq}=0}(n_{qq}=1, L_{qq}=0) \equiv d_0^0(1,0)$.

Contrary to this, for a highly excited baryon the quark–diquark and the classical three-quark pictures differ in principle. In the first case the basic diquark ($d_1^1(1,0)$ or $d_0^0(1,0)$) is encircled by spheres of the quark matter; the equality of the colour charges in the $q\bar{q}$ and qd systems and the similar quark matter distribution lead to similar (n, M^2) -trajectories in the meson and baryon sectors (see Chapter 2).

In the classical three-quark picture a quark–diquark reexpansion of the wave functions can also be carried out. In this case, however, we obtain diquarks of different sorts $d_{S_{qq}}^{I_{qq}}(n_{qq}, L_{qq})$ with various $I_{qq}, S_{qq}, n_{qq}, L_{qq}$ values. It is just this variety of possible $I_{qq}, S_{qq}, n_{qq}, L_{qq}$ values which lead to a large number of baryon states in the classical three-quarks models.

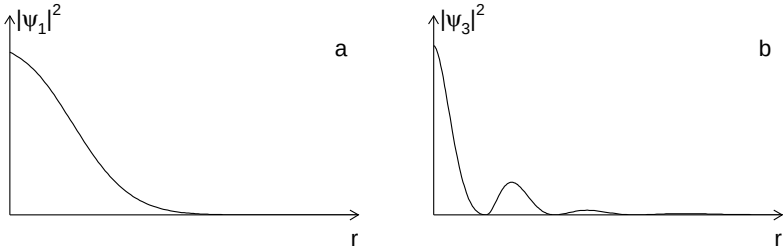


Fig. 9.2 Quark–antiquark $|\Psi_n(r)|^2$ in coordinate representation for $n=1$ (a) and $n=3$ (b). We suppose that there is an analogous quark–diquark structure for baryons where the diquark plays the role of the antiquark.

To prevent the excitation of the diquark, what forces should exist between the quarks? At the first sight this seems to be obvious: it should be a three-body confinement interaction. However, this possibility raises a lot of questions. Let us put forward just the simplest one:

(i) What type of three-body confinement interactions can be suggested?

9.3 Additive Quark Model, Radiative Decays and Spectral Integral Equation

The additive quark model works well in radiative processes at low energies — we have a lot of arguments in favour of that, see Chapters 1 and 7. Moreover, from the investigations of high energy hadron collisions (see Chapter 1 and references therein) we know with certainty that additivity exists in hadron collisions at least up to the total energy squared $s \sim 200 \text{ GeV}^2$. Bearing in mind that there are two particles participating in the hadron collisions, we have good reason to suppose that the additive model can be applied to hadrons with masses $M \lesssim 10 \text{ GeV}$.

The additive model was used successfully for light nuclei in nuclear physics (long before the notion of quarks came into existence) but deviations from additivity were also observed. In electromagnetic processes with deuteron these deviations are owing, first of all, to exchange currents (the photon interacts with a charged particle, forming forces for the proton and the neutron, *e.g.* with a t -channel pion). However, the exchange forces appear in the deuteron in a rather specific way: they depend essentially on the type of the considered process. The exchange forces turned out to be suppressed in the form factors. The vertices of the $d \rightarrow np$ transitions were reconstructed in the framework of the dispersion relation analysis of the np -scattering in the energy region below the $\Delta\Delta$ threshold. In the additive model, the deuteron form factors calculated with these vertices provided a good description of the data at $Q^2 \lesssim 2 \text{ GeV}^2$. In the deuteron electro-disintegration reactions $\gamma d \rightarrow np$, however, the additive model is successful only up to $E_{mc} \sim 100 \text{ MeV}$ (here $\sqrt{s} = 2m_N + E_{mc}$), at higher energies the predictions differ from the measured data. Hence, the region of applicability of the additive quark model may depend radically on the type of reactions (see Chapter 4).

We have no universal answer about the additivity in meson and baryon physics, and it would not be reasonable to guess and make any definite predictions: our knowledge about the structure of forces in hadrons is insufficient. They can either be due to gluonic interactions, *i.e.* be electrically neutral, or to quark exchanges (both types of interactions we discuss briefly in Chapter 7). Actually, we need facts: calculations and comparison of data to the calculated results.

Spectral integral calculations are performed in a gauge invariant way so that they enable us to come to reliable conclusions about applicability or failure of the additive model approach to the considered electromagnetic

process and, in the case of applicability, to give a preliminary estimation of the quark wave functions or quark vertices (examples of such calculations for low-lying hadrons are presented in Chapter 7).

To determine the interactions of constituents in terms of the spectral integral equation technics, we have to know both the levels and the wave functions of the bound states (apparently, this is true not only for the spectral integral method). The lack of information about the wave functions makes it difficult to restore precisely the structure of light quark interactions. Indeed, to learn more about the wave functions, we would need data on form factors of mesons with different quantum numbers. Nevertheless, even existing data allow us to see some characteristic features of the interactions at large and small distances:

- (1) At small quark distances Coulomb-like forces α_s/r are important with the QCD coupling frozen at $\alpha_s \simeq 0.4$, *i.e.* in the region of values which look rather natural from the point of view of strong QCD (see Chapter 8).
- (2) At large distances the confinement interaction dominates (it is singular, $\sim 1/t^2$, that corresponds in the coordinate representation to the behaviour of the potential $\sim br$) – we observe two types of universal t -channel interactions: scalar and vector exchanges with equal couplings, $(I \otimes I - \gamma_\mu \otimes \gamma_\mu)$, see Chapter 8.

The scalar exchange, $I \otimes I$, has been discussed for a long time in connection with the estimate of confinement forces in lattice calculations. But the reconstruction of linear trajectories in the (n, M^2) planes requires also the vector-type exchange, $\gamma_\mu \otimes \gamma_\mu$. Although this statement needs additional testing, we do not think that it would be reasonable to rely completely on lattice results. As was already emphasised in the Preface, the lattice uses countable sets, while integral equations work in continuum space: corresponding results may be not sewn with one another — the fractal theory tells us about that unambiguously (we return to this point below when discussing the glueballs.)

The spectral integral equations reproduce rather well the linear meson trajectories in the (n, M^2) plane and allow us to calculate the $q\bar{q}$ system wave functions which, in their turn, describe satisfactorily the available radiative decay data set. Unfortunately, it is not rich enough, so the measurement of radiative decays and, even better for checking the scheme, of the transitions $\gamma^*(Q_1^2)\gamma^*(Q_2^2) \rightarrow \text{meson}$ is an absolute necessity. The information on such transitions can be obtained from the reactions $e^+e^- \rightarrow e^+e^- + \text{hadrons}$.

Both the spectral integral equations and phenomenologically con-

structed (n, M^2) trajectories provide us with masses and quantum numbers of resonances which are not seen yet in experiments. Considering the (n, M^2) planes (Chapter 2) we see that there is a number of states (open circles in Figs. 2.1–2.3) waiting to be discovered. In Tables 2.1 and 2.2 these states are marked by bold numbers. These states must exist if the developed scheme is correct. Still, the questions are: (i) Do they really exist? Another, closely related question, the answer to which leads to far-reaching consequences is: (ii) Are there other states which do not lie on the trajectories? If yes, how many and what kind of states are they?

Strange as it might be, the answers to the last two questions depend on the way we define the notions of a resonance, the method of calculation of its characteristics. We discuss these problems in the next section.

Let us now turn our attention to a very important fact which does not allow us to compare directly the results of calculations carried out with the help of spectral integration technique and in the framework of the Bethe–Salpeter equation, respectively. The Bethe–Salpeter equation includes “animal-type” diagrams (see Chapters 3 and 8) which appear due to the cancelation of the intermediate state quark propagator $((m^2 - k_i^2)^{-1}$ with factors $\sim k_i^2$ in the numerator of the Bethe–Salpeter equation (these factors always appear in the calculation of the fermion loop diagrams). Consequently, in the spectral integration technique we are dealing with pure $q\bar{q}$ states, while in the Bethe–Salpeter equation this is lost: the meson acquires additional, definitely not quark–antiquark type components.

9.4 Resonances and Their Characteristics

In the investigation of meson states one should not forget about the existing “stumbling stones”.

The standard – traditional – way of observing resonances does not raise any doubts: it means to notice in the hadron spectrum a peak against a smooth background. The position of the peak provides the mass of the resonance, the width at its half-height is the width of the resonance. (This was, *e.g.*, the way the $\phi(1020)$ resonance was found in the $K\bar{K}$ spectrum). In this case the peak itself is described by the Breit–Wigner formula, and the background by a smooth polynomial. We understand now, however, that such a standard method can be applied only in rare cases. We know that the resonance can reveal itself not only as a peak but also as a dip in the spectrum (this is the destructive interference of the resonance and

the background), and, moreover, it can appear also as a shoulder. The $f_0(980)$ resonance shows all these versions of behaviour (see Chapter 6): in the $\pi^-p \rightarrow \pi\pi n$ reaction at small squared momenta t transferred to the nucleon we see a sharp dip in the $\pi\pi$ spectrum at $M_{\pi\pi} \sim 1$ GeV, while in the region $|t| \sim 1.5$ GeV² the experiment gives a clear peak. Intermediate values of momenta transferred demonstrate a variety of forms of the $\pi\pi$ spectra and may serve as an illustration for the different manifestations of the resonance when there is a strong interference with the background.

An analogous problem appears when the resonance decays in different channels. Namely, determining the position of a resonance by making use of the position of the pole in a spectrum, one may ask which hadron spectrum should be considered? Indeed, the positions of the peaks are rather different in different hadron spectra.

The prevailing characteristic feature of an unstable bound state is the position of the amplitude pole in the complex- M^2 plane (see discussions in Chapter 2 and 3 for more detail): $M^2 = M_{Resonance}^2 = M_R^2 - iM_R\Gamma_R$. Its real part, M_R^2 , can be called the resonance mass squared, while Γ_R is its total width. The quantities M_R and Γ_R are invariant, *i.e.* they do not depend on the type of the process in which the resonance is observed. Because of that, precisely these values should be given in various compilations. Unfortunately, this is not the case. The residue in the poles of the amplitudes determine the invariant couplings. In other words, in the complex plane we have: $A \simeq g_{in}(M^2 - M_R^2 + iM_R\Gamma_R)^{-1}g_{out} + \text{smooth term}$, where the product $g_{in}g_{out}$, up to factor $(2\pi i)$, is the pole residue. This leads to the universal and factorised complex-valued couplings g_{in} and g_{out} . For example, in the case when the $(IJ^{PC} = 00^{++})$ -resonance is coupled with two channels (to be definite, with $\pi\pi$ and $K\bar{K}$), in the reactions $\pi\pi \rightarrow \pi\pi$, $\pi\pi \rightarrow K\bar{K}$, $K\bar{K} \rightarrow K\bar{K}$ the residues divided by $(2\pi i)$ are equal to $g_{\pi\pi}^2$, $g_{\pi\pi}g_{K\bar{K}}$ and $g_{K\bar{K}}^2$, respectively. Let us underline once more that the couplings are complex $g_a = |g_a|\exp(i\varphi_a)$ (here $a = \pi\pi, K\bar{K}$). It is just these couplings, not the bumps we see in the spectra, what characterise the connections of the resonances with channels $\pi\pi$ and $K\bar{K}$.

The number of poles increases if the resonance has more than one decay channel. The situation becomes especially complicated if the threshold of one of the decay channels is close to the position of a pole. This happens quite often: we have discussed such cases when we considered the resonances $f_0(980)$ (double poles owing to the $K\bar{K}$ threshold, see Chapter 3) and $f_0(1570)$ (double poles due to the $\omega\omega$ threshold, see Chapter 6). A similar splitting of the poles can be seen also for other resonances: $a_0(980)$ (the

$K\bar{K}$ threshold) and $a_2(1730)$ (the $\rho\omega$ threshold), and so on. The presence of two poles in the amplitudes of the states discussed above tells us that two components are visible in these states: $q\bar{q}$ and $meson_1 + meson_2$. However, we should keep in mind that a definite separation of the components is hardly possible: at small distances the two-meson component may turn into $q\bar{q}$ owing to quark–hadron duality; such a separation needs to introduce some type of bag model, so in many aspects it may be considered as “hand-made”. The only meson–meson (or multi-meson) components which are determined uniquely are components of real mesons – the K -matrix procedure singles out just these ones.

Hence, the only way to obtain a complete and reliable information about the resonances is to restore the analytic amplitude in the physical region, on the real axis of the complex- M^2 plane, and then to continue it into the region of negative $\text{Im } M^2$. The restoration of the analytic amplitude requires the correct account of singularities on the real axis (the threshold singularities) and, if possible, constraints owing to the unitarity.

So, the program of determination of resonances consists in a simultaneous fit to a possibly large number of data in different reactions, with the requirement of fulfilling the analyticity and unitarity. The fitting to separate reactions with the subsequent averaging of the results leads to much larger errors, since all the fitting procedures contain their own systematic errors, and systematic errors are not to be averaged.

One more phenomenon which can occur in the physics of resonances has to be taken into account: the accumulation of widths by one of the resonances if they overlap. As a result, we have one broad resonance and a group of narrow ones. The systematisation of the $q\bar{q}$ mesons and the search for exotic states requires the knowledge of all states, among others those which dived rather deeply into the complex- M^2 plane; it is impossible to find the broad resonances without the analysis of a large amount of reactions covering a broad region of physical masses.

9.5 Exotic States — Glueballs

The systematisation of the $q\bar{q}$ states allowed us to fix two exotic states – the scalar and tensor glueballs which in the standard terminology are the broad resonances $f_0(1200 - 1600)$ and $f_2(2000)$ (see Chapters 2, 3 and 6). The arguments in favour of the glueball character of these resonances are, as follows:

(i) They are superfluous from the point of view of $q\bar{q}$ systematics, *i.e.* there is no room for these states on the linear (n, M^2) trajectories.

(ii) From the point of view of the decays, these resonances are rather close to states which can be considered as flavour blind (singlets in the flavour space). Strictly speaking, this is not quite true: the strange quark is heavier which leads to a suppression of the $s\bar{s}$ pair production by gluons. Hence the “quasi-flavour-blind state” is what corresponds to our expectations of a glueball.

(iii) There is one more characteristic property indicating the glueball character of $f_0(1200 - 1600)$ and $f_2(2000)$: their large width. Indeed, $f_0(1200 - 1600)$ and $f_2(2000)$ accumulated a considerable part of widths of their neighbours-resonances. It seems to be natural that the gluonium states which occurred near the $q\bar{q}$ mesons having the same quantum numbers became the centres of accumulation of widths. Mixing the gluonium and quarkonium states, the admixture of the quarkonium component in the gluonium is of the order of N_f/N_c (where N_f and N_c are the numbers of light flavours and colours), while the admixture of the gluonium in the quarkonium is of the order of $1/N_c$. Consequently, when the decay channels enter, the first to dive into the complex- M^2 plane is the gluonium states. In the course of subsequent mixing the states get away from each other (since the mixing of the resonances is strong owing to decay processes $resonance_1 \rightarrow real\ mesons \rightarrow resonance_2$). As a result, the gluonium (or, better to call it the glueball descendant) occurs deep in the complex plane, thus turning out to be a broad resonance.

The effect of accumulation of widths by one of the resonances which is close to its neighbouring resonances was first observed in nuclear physics nearly forty years ago. As we see now, it reveals itself also in the physics of mesons.

All the presented arguments are sufficiently serious, so we are entitled to state that $f_0(1200 - 1600)$ and $f_2(2000)$ are of glueball nature. There are also additional considerations in favour of this idea. Indeed, it is not surprising that the lowest scalar glueball is located in the region of ~ 1400 MeV which is the mass region $\sim 2m_g$. The effective mass of the soft gluon ($m_g \simeq 700 - 800$ MeV) was first estimated in the reaction $J/\psi \rightarrow \gamma + M_X$. In this reaction (which in the quark–gluon language may be deciphered as $J/\psi \rightarrow \gamma + gg \rightarrow \gamma + hadrons$) the spectrum of the missing mass M_X is strongly suppressed at $M_X < 1400$ MeV. It looks also natural that the tensor glueball is found in the 2000 MeV region: the pomeron trajectory which is determined at moderately high energies (see Chapter 1) is linear

in the region of the diffractive cones in the elastic pp , $p\bar{p}$ and πp cross sections. Continuing this linear trajectory into the region of positive t values, we obtain the mass of the tensor glueball (the first physical state of the pomeron trajectory) to be just around 2000 MeV.

We have already mentioned that the predictions given by lattice QCD for the meson characteristics should be handled with great care. Many lattice QCD calculations have predicted for scalar state $f_0(1710)$ as a glueball. In the K -matrix analysis this resonance (denoted in Chapter 2 as $f_0(1755)$ in accordance with the results of the data fit) is a relatively narrow state, far from being flavour blind; moreover, $f_0(1755)$ lies comfortably on the (n, M^2) trajectory. As to the tensor glueball, the lattice QCD prediction has been 2350 MeV for a long time. Only recently, introducing the linearity condition for the trajectories in the (J, M^2) plane, the predicted place of the first tensor glueball became the region of 2000 MeV.

We have, definitely, two glueballs. We do not know, however, anything about them except that they exist and are mixtures of gluonia gg , quarkonia $(q\bar{q})_{glueball}$ (here $(q\bar{q})_{glueball} = \sqrt{\frac{2}{2+\lambda}}(u\bar{u} + d\bar{d}) + \sqrt{\frac{\lambda}{2+\lambda}}s\bar{s}$ with the strange quark suppression parameter $\lambda \sim 0.5 - 0.8$) and a hadron “coat” as a result of width accumulation of neighbouring resonances. New experiments are necessary: it is essential to find new glueball states, first of all, the pseudoscalar glueball.

9.6 White Remnants of the Confinement Singularities

We have serious reasons to suspect that the confinement singularities (the t -channel singularities in the scalar and vector states) have a complicated structure: they contain quark–antiquark, gluon and hadron constituents. In the colour space these are octet states but, maybe, they contain white components too – see the discussions in Chapters 2 and 3.

If the confinement singularities have, indeed, white constituents, this raises immediately the following questions:

- (i) How do these constituents reveal themselves in white channels?
- (ii) Can they be identified?

In the scalar channel we face the problem of the σ meson ($IJ^{PC} = 00^{++}$): its existence is quite plausible, although there are no reliable data for it. If the white scalar confinement singularity exists, it would be reasonable to consider it as the σ meson revealing itself: because of the transitions into the $\pi\pi$ state, the confinement singularity could move to the second sheet.

If so, the σ meson can certainly not reveal itself as a lonely amplitude singularity $1/t^2$ but a group of poles (see Chapter 8, Eq. (8.14)).

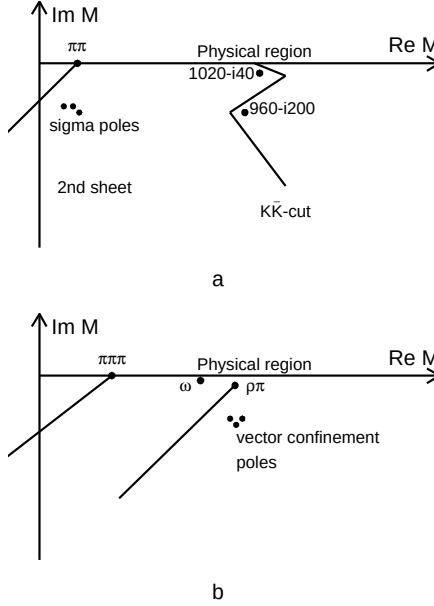


Fig. 9.3 Complex- M planes for (a) $IJ^{PC} = 00^{++}$ and (b) $IJ^{PC} = 01^{--}$: singularities related to thresholds ($\pi\pi$, $K\bar{K}$, $\pi\pi\pi$, and $\rho\pi$), composite states (poles corresponding to $f_0(980)$ and $\omega(780)$) and confinement singularities. The confinement singularities in white channels may split into several poles.

Indeed, the $1/t^2$ singularity corresponds to the idealised case when the confinement appears as an impenetrable wall ($V_{confinement}(r) \sim br$ in the coordinate representation). However, decay channels also exists. In terms of potentials, this means that the confinement is in fact a barrier, and the singularity $1/t^2$ splits into a number of close pole singularities.

The possible position of the confinement singularities in the 00^{++} -channel is presented for this case in Fig. 9.3a: they are on the second sheet, under the physical region (*i.e.* the real axis at $\text{Re } M > 2m_\pi$). In this picture the sigma singularities are represented by the group of poles; for the sake of completeness, we show here also the $\pi\pi$ and $K\bar{K}$ cuts and the poles corresponding to $f_0(980)$.

A similar scenario may be valid also for the vector confinement singularity in the $\pi\pi\pi$ ($IJ^{PC} = 01^{--}$) channel. In this case the picture of poles

related to the confinement may be as shown in Fig. 9.3b. It is natural to assume that the strong channel $\rho\pi$ “attracts” the white confinement singularities.

All these statements are, however, nothing but hypotheses. As we already mentioned, the problem of the σ meson is widely discussed. But the existence of a left cut in the $\pi\pi$ amplitude, or the presence of other channels when searching for the σ meson in multiparticle processes makes it impossible to come to a conclusion. That is why in Chapter 3 where the σ meson is discussed (in the framework of the dispersion relation analysis of the partial $00^{++} - \pi\pi \rightarrow \pi\pi$ amplitude), we do not even try to investigate whether the sigma singularities can be described by several poles (as shown in Fig. 9.3a). In order to minimise the number of parameters, in Chapter 3 we approximate it by one pole.

To understand the problem of the σ meson we need very good experimental data in which the left singularities are suppressed.

9.7 Quark Escape from Confinement Trap

The mechanism of quark confinement is much more complicated than that used in the spectral integral equations of Chapter 8. Having sufficient energy, the quarks can fly away from the confinement trap, producing a new quark–antiquark pair and forming a white state by joining one of them.

Can this deconfinement process be included in the consideration of spectra? We came close to raise this problem, trying to solve it within the developed approach. Now it is a serious challenge for physicists, and we think the ideas pushed forward in [7] will be helpful.

In this book numerous ideas analogous (or partly analogous) to those developed here were not touched, as well as alternative ones, — they may be found in many works [8–43]. In our opinion, to be acquainted with these works would significantly complement the substance of this book.

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